Generalized barycentric coordinates

Michael S. Floater

October 20, 2009

In this lecture, we review the definitions and properties of barycentric coordinates on triangles, and study generalizations to convex polygons. These generalized coordinates have various applications in computer graphics, including curve, surface, and image deformation, and parameterization of triangular meshes.

1 Triangular coordinates

Let T be a triangle in $\mathbb{R}^2$ with vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. For convenience we will assume that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are in anti-clockwise order around the boundary of T, as in Figure 1. It has been known for a long time, and was at least known by Mobius, that any point $\mathbf{x}$ in T can be expressed uniquely as a convex combination of the three vertices. In other words, there are unique real values $\lambda_1, \lambda_2, \lambda_3 \geq 0$ such that

$$\lambda_1 + \lambda_2 + \lambda_3 = 1, \qquad (1)$$

and

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \lambda_3 \mathbf{v}_3 = \mathbf{x}. \qquad (2)$$

To see this, observe that the scalar equation (1) and the vector equation (2) together form a linear system of three equations,

$$\begin{pmatrix} 1 & 1 & 1 \\ v_1^1 & v_2^1 & v_3^1 \\ v_1^2 & v_2^2 & v_3^2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 \\ x^1 \\ x^2 \end{pmatrix}, \qquad (3)$$

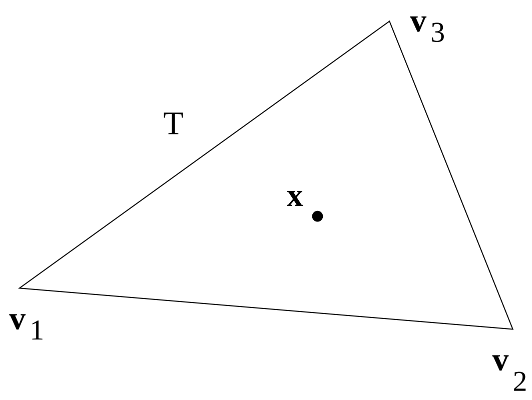

Figure 1: Point in a triangle.

where $\mathbf{v}_j = (v_j^1, v_j^2)$, $j = 1, 2, 3$, and $\mathbf{x} = (x^1, x^2)$. Since the signed area of the triangle T is given by

$$A(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ v_1^1 & v_2^1 & v_3^1 \\ v_1^2 & v_2^2 & v_3^2 \end{vmatrix}, \tag{4}$$

the assumption that T is non-degenerate implies that the matrix in (3) is non-singular, and Cramer's rule gives the unique solution

$$\lambda_1 = \frac{A(\mathbf{x}, \mathbf{v}_2, \mathbf{v}_3)}{A(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)}, \quad \lambda_2 = \frac{A(\mathbf{v}_1, \mathbf{x}, \mathbf{v}_3)}{A(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)}, \quad \lambda_3 = \frac{A(\mathbf{v}_1, \mathbf{v}_2, \mathbf{x})}{A(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)}. \tag{5}$$

The values λ_i are called the *barycentric coordinates* of the point $\mathbf{x}$. Observe that due to the anti-clockwise ordering of the vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, all four areas in (5) are non-negative and therefore $\lambda_1, \lambda_2, \lambda_3$ have the important property of being *non-negative*. Had the vertices been ordered clockwise, all the areas in (5) would have had the opposite sign but we would again have $\lambda_1, \lambda_2, \lambda_3 \geq 0$.

The three areas in the numerators in (5) are shown in Figure 2 where A_i denotes the triangle area $A(\mathbf{x}, \mathbf{v}_{i+1}, \mathbf{v}_{i+2})$ with indices regarded cyclically: if $j = k + 3m$, with $k \in \{1, 2, 3\}$ and $m \in \mathbb{Z}$, then $\mathbf{v}_j := \mathbf{v}_k$.

Viewed as functions of $\mathbf{x}$, we see from (5) that the λ_i are linear polynomials, and from now on we treat them as functions of $\mathbf{x}$. Using either (2) or (5) we see that they have the *Lagrange property*,

$$\lambda_i(\mathbf{v}_j) = \delta_{ij}. \tag{6}$$

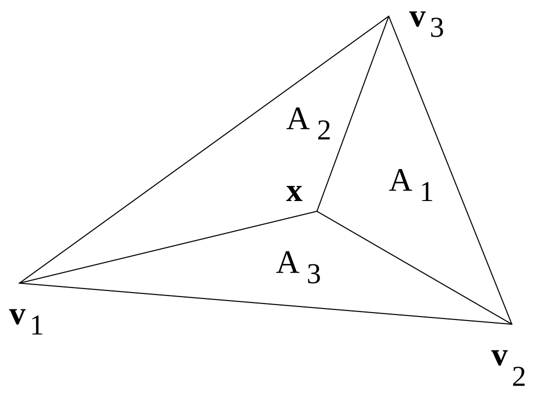

Figure 2: Areas of triangles formed by $\mathbf{x}$.

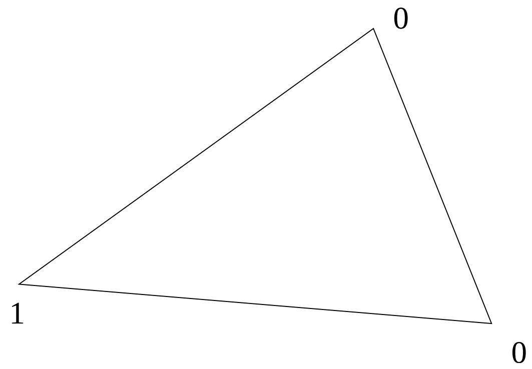

Figure 3: Values of λ_1 at the vertices.

Here, δ_{ij} denotes the Kronecker delta function that has value 1 when $i = j$ and value 0 when $i \neq j$. The values of λ_1 at the vertices are shown in Figure 3.

The linear polynomials $\lambda_1, \lambda_2, \lambda_3$ are clearly well-defined for all $\mathbf{x} \in \mathbb{R}^2$. However, they are are not all positive outside T. Their signs are shown in Figure 4.

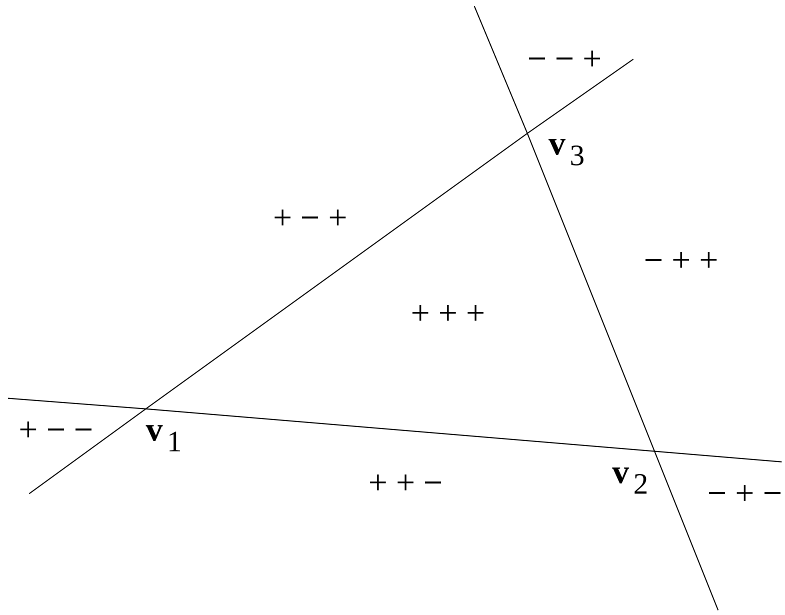

Figure 4: Signs of $\lambda_1, \lambda_2, \lambda_3$.

1.1 Interpolation

Barycentric coordinates are useful for interpolation on triangles. Given a function $f : T \to \mathbb{R}$, we can define the linear polynomial

$$g(\mathbf{x}) = If(\mathbf{x}) = \sum_{i=1}^{3} \lambda_i(\mathbf{x}) f(\mathbf{v}_i), \qquad \mathbf{x} \in T.$$

Due to the Lagrange property (6), it is easy to verify that $g(\mathbf{v}_i) = f(\mathbf{v}_i)$, $i = 1, 2, 3$. Thus g is the linear interpolant to the data. The interpolation operator I has linear precision. This comes from the barycentric property (2) combined with (1): if

$$f(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x} + b,$$

then

$$g(\mathbf{x}) = \sum_{i=1}^{3} \lambda_i(\mathbf{x})(\mathbf{a} \cdot \mathbf{v}_i + b) = \mathbf{a} \cdot \sum_{i=1}^{3} \lambda_i(\mathbf{x}) \mathbf{v}_i + b \sum_{i=1}^{3} \lambda_i(\mathbf{x}) = f(\mathbf{x}).$$

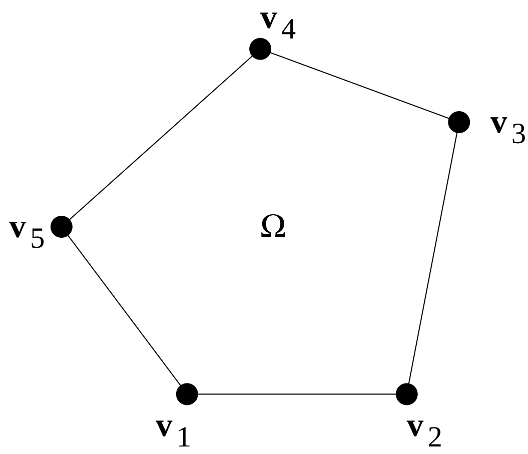

Figure 5: Example of a convex polygon.

1.2 Derivatives

It is sometimes useful to know the derivatives of $\lambda_1, \lambda_2, \lambda_3$. Using the notation

$$D_k f = \frac{\partial \lambda}{\partial x^k}, \quad k = 1, 2,$$

for partial derivatives of a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$, we can can write the gradient of f as $\nabla f = (D_1 f, D_2 f)$. Differentiating the expression in (5), using (4), gives

$$\nabla \lambda_i = \frac{\text{rot}(\mathbf{v}_{i+2} - \mathbf{v}_{i+1})}{2A(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)}, \quad i = 1, 2, 3,$$

where $\text{rot}(\mathbf{a}) = \text{rot}(a^1, a^2) := (-a^2, a^1)$, which can be interpreted as a rotation of the vector $\mathbf{a}$ by $\pi/2$. Thus, for example the gradient of λ_1 points in the inward direction perpendicular to the edge $[\mathbf{v}_2, \mathbf{v}_3]$.

2 Polygonal coordinates

Let Ω be a convex polygon in the plane, regarded as a closed set, with vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, $n \geq 3$, in an anticlockwise ordering. Figure 5 shows an example with $n = 5$. We will call any set of functions $\lambda_i : \Omega \to \mathbb{R}$, $i = 1, \ldots, n$, *barycentric coordinates* if they satisfy, for all $\mathbf{x} \in \Omega$, the three properties

$$\lambda_i(\mathbf{x}) \geq 0, \qquad i = 1, 2, \ldots, n, \tag{7}$$

$$\sum_{i=1}^{n} \lambda_i(\mathbf{x}) = 1 \tag{8}$$

and

$$\sum_{i=1}^{n} \lambda_i(\mathbf{x})\mathbf{v}_i = \mathbf{x}. \tag{9}$$

In the special case that $n = 3$, the λ_i are the unique triangular coordinates of (5). For $n \geq 4$, and for general $\mathbf{x} \in \Omega$, there is no unique choice of the n values $\lambda_1(\mathbf{x}), \ldots, \lambda_n(\mathbf{x})$ that satisfies the three conditions. In most applications we would like functions λ_i that are as smooth as possible.

2.1 General properties

Even though barycentric coordinates are not unique for $n \geq 4$, they share some general properties that follow from the three defining axioms (7), (8) and (9). First, they have the Lagrange property $\lambda_i(\mathbf{v}_j) = \delta_{ij}$ and are linear along each edge of Ω. To see this, observe that the axioms (8) and (9) imply linear precision, i.e., for any linear function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$\sum_{i=1}^{n} \lambda_i(\mathbf{x})f(\mathbf{v}_i) = f(\mathbf{x}). \tag{10}$$

Therefore, since the area $A(\mathbf{x}, \mathbf{v}_j, \mathbf{v}_{j+1})$ is linear in $\mathbf{x}$, we have

$$\sum_{i=1}^{n} \lambda_i(\mathbf{x})A(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_{j+1}) = A(\mathbf{x}, \mathbf{v}_j, \mathbf{v}_{j+1}),$$

and so if $\mathbf{x}$ belongs to the edge $[\mathbf{v}_j, \mathbf{v}_{j+1}]$,

$$\sum_{i \neq j, j+1} \lambda_i(\mathbf{x})A(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_{j+1}) = 0.$$

Since $A(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_{j+1}) > 0$ for $i \neq j, j+1$ by the convexity of Ω, and since $\lambda_i(\mathbf{x}) \geq 0$ by (7) it follows that $\lambda_i(\mathbf{x}) = 0$ for all $i \neq j, j+1$. This implies that from (9),

$$\lambda_j(\mathbf{x})\mathbf{v}_j + \lambda_{j+1}(\mathbf{x})\mathbf{v}_{j+1} = \mathbf{x}.$$

We have thus shown that all barycentric coordinates share the same values on the boundary of Ω. We can also use the defining axioms to obtain some information about the coordinates in the interior of Ω. For each

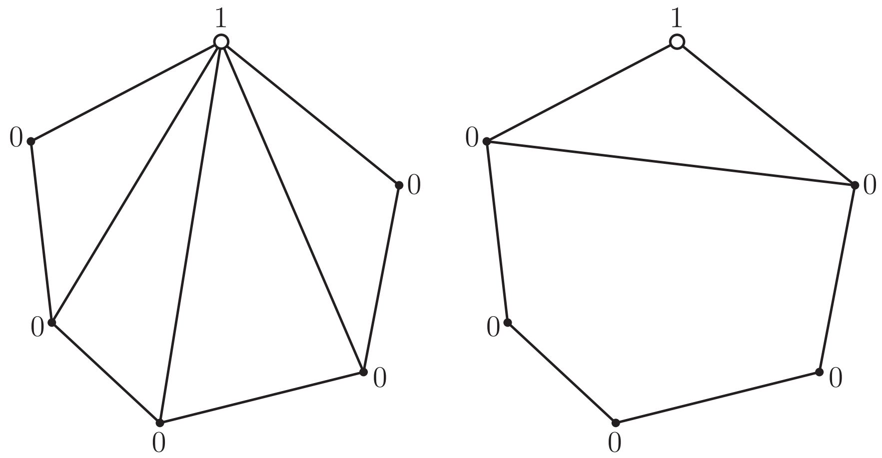

Figure 6: Upper and lower bounds.

$i \in \{1, \ldots, n\}$, let $L_i : \Omega \to \mathbb{R}$ be the function that is linear in each triangle of the form $[\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_{j+1}]$, $j \neq i-1, i$, and has the values $L_i(\mathbf{v}_j) = \delta_{ij}$ at the vertices of Ω, as illustrated in Figure 6a. Conversely, let $\ell_i : \Omega \to \mathbb{R}$ be the function that is linear on the triangle $[\mathbf{v}_{i-1}, \mathbf{v}_i, \mathbf{v}_{i+1}]$, and on the polygon $[\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n]$, and also has the values $L_i(\mathbf{v}_j) = \delta_{ij}$ at the vertices, as illustrated in Figure 6b.

Theorem 1 *For* $\mathbf{x} \in \Omega$ *and* $i = 1, \ldots, n$,

$$\ell_i(\mathbf{x}) \leq \lambda_i(\mathbf{x}) \leq L_i(\mathbf{x}). \tag{11}$$

Proof. The point $\mathbf{x}$ belongs to at least one triangle of the form $[\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_{j+1}]$, $j \neq i-1, i$. Since $A(\mathbf{x}, \mathbf{v}_j, \mathbf{v}_{j+1})$ is linear in $\mathbf{x}$, the two properties (8) and (9) imply

$$A(\mathbf{x}, \mathbf{v}_j, \mathbf{v}_{j+1}) = \sum_{k=1}^{n} \lambda_k(\mathbf{x}) A(\mathbf{v}_k, \mathbf{v}_j, \mathbf{v}_{j+1}) \geq \lambda_i(\mathbf{x}) A(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_{j+1}),$$

the latter inequality due to the positivity property (7). This implies that

$$\lambda_i(\mathbf{x}) \leq A(\mathbf{x}, \mathbf{v}_j, \mathbf{v}_{j+1}) / A(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_{j+1}) = L_i(\mathbf{x}).$$

The opposite inequality, $\ell_i(\mathbf{x}) \leq \lambda_i(\mathbf{x})$, is trivial if $\mathbf{x}$ is outside the triangle $[\mathbf{v}_{i-1}, \mathbf{v}_i, \mathbf{v}_{i+1}]$, for then $\ell_i(\mathbf{x}) = 0$. So suppose $\mathbf{x} \in [\mathbf{v}_{i-1}, \mathbf{v}_i, \mathbf{v}_{i+1}]$. Then, since

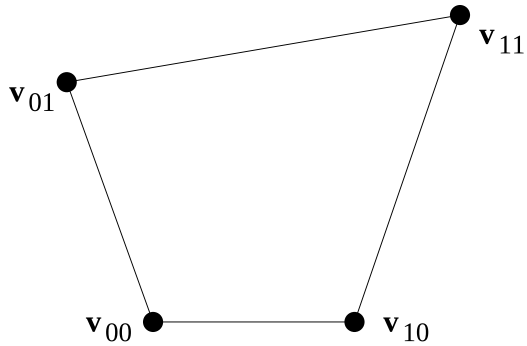

Figure 7: Convex quadrilateral

$A(\mathbf{x}, \mathbf{v}_{i+1}, \mathbf{v}_{i-1})$ is linear in $\mathbf{x}$,

$$\begin{aligned} A(\mathbf{x}, \mathbf{v}_{i+1}, \mathbf{v}_{i-1}) &= \sum_{k=1}^{n} \lambda_k(\mathbf{x}) A(\mathbf{v}_k, \mathbf{v}_{i+1}, \mathbf{v}_{i-1}) \\ &= \lambda_i(\mathbf{x}) A(\mathbf{v}_i, \mathbf{v}_{i+1}, \mathbf{v}_{i-1}) - \sum_{k \neq i-1, i, i+1} \lambda_k(\mathbf{x}) A(\mathbf{v}_k, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}) \\ &\leq \lambda_i(\mathbf{x}) A(\mathbf{v}_i, \mathbf{v}_{i+1}, \mathbf{v}_{i-1}) \end{aligned}$$

which implies that

$$\lambda_i(\mathbf{x}) \geq A(\mathbf{x}, \mathbf{v}_{i+1}, \mathbf{v}_{i-1}) / A(\mathbf{v}_i, \mathbf{v}_{i+1}, \mathbf{v}_{i-1}) = \ell_i(\mathbf{x}).$$

□

2.2 Quadrilateral coordinates

A common approach to building smooth barycentric coordinates in the special case that Ω is a quadrilateral is to view the quadrilateral as the image of a bilinear map from the unit square $[0, 1] \times [0, 1]$. This is often used in the PDE literature on finite elements.

Denote the vertices of Ω by $\mathbf{v}_{00}, \mathbf{v}_{10}, \mathbf{v}_{01}, \mathbf{v}_{11}$, as in Figure 7. Then, for each $\mathbf{x} \in \Omega$, there exist unique λ and μ in $[0, 1]$ such that

$$\mathbf{x} = (1 - \lambda)(1 - \mu)\mathbf{v}_{00} + \lambda(1 - \mu)\mathbf{v}_{10} + (1 - \lambda)\mu\mathbf{v}_{01} + \lambda\mu\mathbf{v}_{11}. \tag{12}$$

This gives four barycentric coordinates λ_{ij}, $i, j = 0, 1$. If a function f is given this gives an interpolant

$$g(\mathbf{x}) = \sum_{i=0}^{1} \sum_{j=0}^{1} \lambda_{ij} f(\mathbf{v}_{ij}),$$

with linear precision.

It remains to find λ and μ, given $\mathbf{x} \in \Omega$. Letting

$$\mathbf{a} = \mathbf{v}_{00} - \mathbf{x}, \quad \mathbf{b} = \mathbf{v}_{10} - \mathbf{v}_{00}, \quad \mathbf{c} = \mathbf{v}_{01} - \mathbf{v}_{00}, \quad \mathbf{d} = \mathbf{v}_{00} - \mathbf{v}_{10} - \mathbf{v}_{01} + \mathbf{v}_{11},$$

we can re-write (12) as

$$\mathbf{a} + \mathbf{b}\lambda + \mathbf{c}\mu + \mathbf{d}\lambda\mu = 0. \tag{13}$$

Taking the cross product of both sides of this equation with $(\mathbf{a} + \mathbf{c}\mu)$ gives

$$(\mathbf{a} + \mathbf{c}\mu) \times (\mathbf{b} + \mathbf{d}\mu) = 0,$$

or

$$(\mathbf{c} \times \mathbf{d})\mu^2 + (\mathbf{c} \times \mathbf{b} + \mathbf{a} \times \mathbf{d})\mu + \mathbf{a} \times \mathbf{b} = 0$$

which is a quadratic in μ and can be solved explicitly for μ. Note though that the special case $\mathbf{c} \times \mathbf{d} = 0$ must be treated separately. The other unknown, λ, can be found in a smilar way.

3 Wachspress coordinates

In this chapter we study Wachspress coordinates. These are barycentric coordinates over convex polygons that are rational functions, and are defined by

$$\lambda_i(\mathbf{x}) = \frac{w_i(\mathbf{x})}{\sum_{j=1}^{n} w_j(\mathbf{x})}, \qquad \mathbf{x} \in \Omega, \tag{14}$$

where

$$w_i(\mathbf{x}) = B_i \prod_{j \neq i-1, i} A_j(\mathbf{x}), \tag{15}$$

and

$$A_i(\mathbf{x}) := A(\mathbf{x}, \mathbf{v}_i, \mathbf{v}_{i+1}) \quad \text{and} \quad B_i := A(\mathbf{v}_{i-1}, \mathbf{v}_i, \mathbf{v}_{i+1}).$$

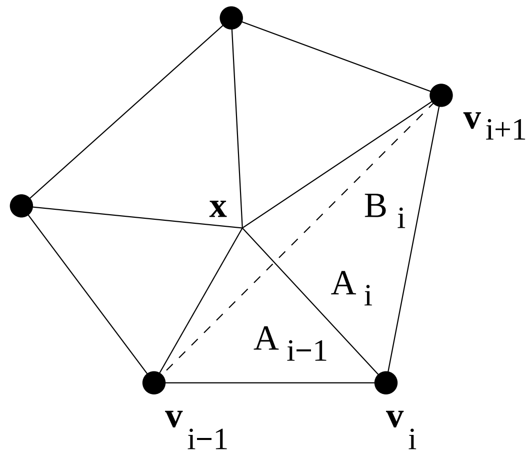

Figure 8: Triangles areas

The triangle areas A_i and B_i are shown in Figure 8. Since A_j is linear in $\mathbf{x}$ we see that w_i is a polynomial of degree $n-2$. It follows that the numerator of λ_i has degree $n-2$. Thus the denominator,

$$W(\mathbf{x}) = \sum_{j=1}^{n} w_j(\mathbf{x}), \tag{16}$$

has degree at most $n-2$, but in fact its degree is $n-3$. This follows from the barycentric property λ_i which we prove in the next section.

Wachspress coordinates are clearly smooth (C^∞) and also rational polynomials in the coordinates x^1 and x^2 of the point $\mathbf{x} = (x^1, x^2)$ with degree at most $n-2$.

We note that in the case that Ω is a regular polygon, the areas $B_1, \ldots, B_n$ are equal in which case they can be removed from the formula and we have simply

$$w_i(\mathbf{x}) = \prod_{j \neq i-1, i} A_j(\mathbf{x}). \tag{17}$$

We now establish the barycentric property (9) for the Wachspress coordinates (14). It helps first to express the coordinates in a different form. Multiplying w_i by a constant in i does not change λ_i, and so dividing w_i by $\prod_{j=1}^{n} A_j$ gives the alternative form

$$\lambda_i(\mathbf{x}) = \frac{\tilde{w}_i(\mathbf{x})}{\sum_{j=1}^{n} \tilde{w}_j(\mathbf{x})}, \qquad \tilde{w}_i(\mathbf{x}) = \frac{B_i}{A_{i-1}(\mathbf{x}) A_i(\mathbf{x})}, \tag{18}$$

which is valid for $\mathbf{x} \in \mathrm{Int}(\Omega)$. In contrast to w_i, the rational function $\tilde{w}_i$ depends only on the three local vertices $\mathbf{v}_{i-1}$, $\mathbf{v}_i$, and $\mathbf{v}_{i+1}$.

Theorem 2 *With $\tilde{w}_i$ as in (18),*

$$\sum_{i=1}^{n} \tilde{w}_i(\mathbf{x})(\mathbf{v}_i - \mathbf{x}) = 0, \qquad \mathbf{x} \in \mathrm{Int}(\Omega). \tag{19}$$

Proof. We express $\mathbf{x}$ as a barycentric combination of $\mathbf{v}_{i-1}$, $\mathbf{v}_i$, and $\mathbf{v}_{i+1}$,

$$\mathbf{x} = \frac{A_i(\mathbf{x})}{B_i}\mathbf{v}_{i-1} + \frac{(B_i - A_{i-1}(\mathbf{x}) - A_i(\mathbf{x}))}{B_i}\mathbf{v}_i + \frac{A_{i-1}(\mathbf{x})}{B_i}\mathbf{v}_{i+1},$$

and rearrange this equation in the form

$$\frac{B_i}{A_{i-1}(\mathbf{x})A_i(\mathbf{x})}(\mathbf{v}_i - \mathbf{x}) = \frac{1}{A_{i-1}(\mathbf{x})}(\mathbf{v}_i - \mathbf{v}_{i-1}) - \frac{1}{A_i(\mathbf{x})}(\mathbf{v}_{i+1} - \mathbf{v}_i).$$

Summing both sides of this equation over $i = 1, \ldots, n$ gives the result. □

The barycentric property (9) immediately follows from this. Another consequence is that

$$\sum_{i=1}^{n} \tilde{w}_i(\mathbf{x})\mathbf{v}_i = \mathbf{x}\sum_{i=1}^{n} \tilde{w}_i(\mathbf{x}),$$

and so

$$\sum_{i=1}^{n} w_i(\mathbf{x})\mathbf{v}_i = \mathbf{x}\sum_{i=1}^{n} w_i(\mathbf{x}).$$

Then, since the left hand side is a (vector) polynomial of degree $\leq n-2$, the sum $W(\mathbf{x}) = \sum_{i=1}^{n} w_i(\mathbf{x})$ must be a polynomial of degree $\leq n-3$.

3.1 Wachspress coordinates outside the polygon

Do Wachspress coordinates extend outside the polygon? We saw that barycentric coordinates over a triangle continue to be well-defined outside the triangle, even though they are no longer all positive there.

Wachspress coordinates, however, may not be well-defined outside the polygon. Consider the example of Fig 9, where $n = 4$, and Ω is a quadrilateral. The lines through the fours sides of the quadrilateral intersect in

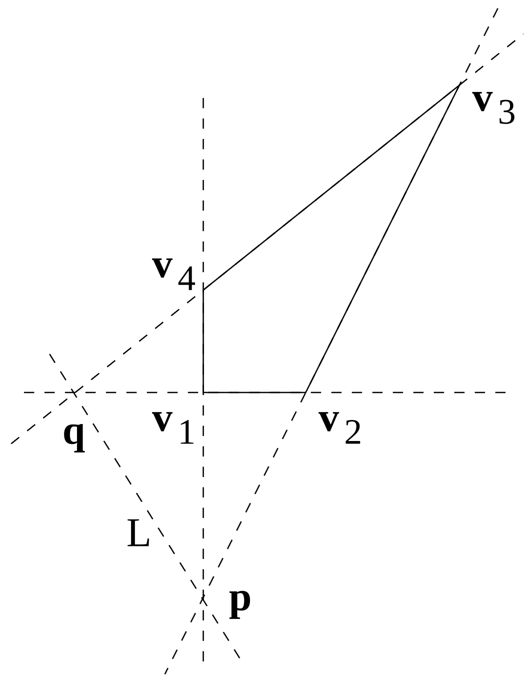

Figure 9: Singularities of Wachspress coordinates

the two points $\mathbf{p}$ and $\mathbf{q}$. By definition, we see that $A_2(\mathbf{p}) = A_4(\mathbf{p}) = 0$. Therefore, $w_i(\mathbf{p}) = 0$ for all $i = 1, 2, 3, 4$, and so $W(\mathbf{p}) = 0$. Similarly, $A_1(\mathbf{q}) = A_3(\mathbf{q}) = 0$, and so $W(\mathbf{q}) = 0$ We showed earlier that the polynomial W has degree $\leq n-2$, which in this quadrilateral case is degree 1. Thus, W is zero along the line L through $\mathbf{p}$ and $\mathbf{q}$ and we conclude that the Wachspress coordinates $\lambda_1, \ldots, \lambda_4$ are not well-defined on the line L.

4 Mean value coordinates

Another set of barycentric coordinates for convex polygons are the so-called mean value coordinates,

$$\lambda_i(\mathbf{x}) = \frac{w_i(\mathbf{x})}{\sum_{j=1}^n w_j(\mathbf{x})}, \qquad \mathbf{x} \in \Omega, \tag{20}$$

where

$$w_i(\mathbf{x}) = \frac{\tan(\alpha_{i-1}(\mathbf{x})/2) + \tan(\alpha_i(\mathbf{x})/2)}{\|\mathbf{v}_i - \mathbf{x}\|}, \tag{21}$$

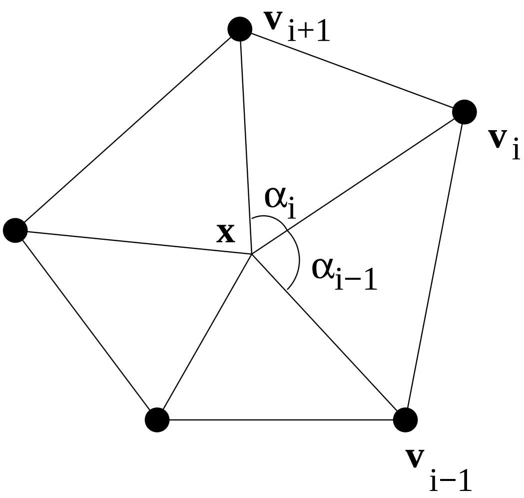

Figure 10: Angles in mean value formula

and $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^2$ and $\alpha_i(\mathbf{x})$ is the angle at $\mathbf{x}$ in the triangle $[\mathbf{x}, \mathbf{v}_i, \mathbf{v}_{i+1}]$, illustrated in Figure 10.

Note that the tangents in the formula do not need to be evaluated directly. Instead, using the notation $\mathbf{d}_i = \mathbf{v}_i - \mathbf{x}$, $r_i = \|\mathbf{d}_i\|$, and $\mathbf{e}_i = \mathbf{d}_i / r_i$, we can write

$$\sin \alpha_i = \mathbf{e}_i \times \mathbf{e}_{i+1}, \qquad \text{and} \qquad \cos \alpha_i = \mathbf{e}_i \cdot \mathbf{e}_{i+1},$$

where $\times$ and $\cdot$ denote cross and dot products of vectors in $\mathbb{R}^2$, and then use either of the two formulas

$$\tan(\alpha/2) = (1 - \cos\alpha)/\sin\alpha = \sin\alpha/(1 + \cos\alpha).$$

Nevertheless, the mean value weight (21) requires computing square roots in order to evaluate r_{i-1}, r_i, and r_{i+1}.

4.1 Barycentric property

We now show that the mean value coordinates satisfy the barycentric property (9).

Theorem 3 *With w_i as in (21),*

$$\sum_{i=1}^{n} w_i(\mathbf{x})(\mathbf{v}_i - \mathbf{x}) = 0, \qquad \mathbf{x} \in \mathrm{Int}(\Omega). \tag{22}$$

Proof. With $\mathbf{x} \in \mathrm{Int}(\Omega)$ fixed, equation (22) is equivalent to

$$\sum_{i=1}^{n} \big(\tan(\alpha_{i-1}/2) + \tan(\alpha_i/2) \big) \mathbf{e}_i = 0,$$

which can be rewritten as

$$\sum_{i=1}^{n} \tan(\alpha_i/2)(\mathbf{e}_i + \mathbf{e}_{i+1}) = 0. \tag{23}$$

To show that this equation holds, express the unit vector $\mathbf{e}_i$ as $\mathbf{e}_i = (\cos\theta_i, \sin\theta_i)$. Then $\alpha_i = \theta_{i+1} - \theta_i$, and so

$$\begin{aligned} \tan\left(\frac{\alpha_i}{2}\right)(\mathbf{e}_i + \mathbf{e}_{i+1}) &= \tan\left(\frac{\theta_{i+1} - \theta_i}{2}\right)(\cos\theta_i + \cos\theta_{i+1}, \sin\theta_i + \sin\theta_{i+1}) \\ &= (\sin\theta_{i+1} - \sin\theta_i, \cos\theta_i - \cos\theta_{i+1}), \end{aligned}$$

the last line following from the addition and subtraction formulas for sines and cosines. Summing this expression over $i = 1, \ldots, n$ gives equation (23) as required. □

4.2 Alternative expression

The formula (21) is only valid for $\mathbf{x}$ in the interior of Ω. However, since the upper and lower bounds in Theorem 1 apply to the mean value coordinates at every interior point, and because the bounds are equal at the boundary of Ω we see that that the coordinates uniquely continue to the boundary in the usual way. An alternative way to see this is the following alternative formula, which is clearly valid all $\mathbf{x}$ in Ω.

Theorem 4 *The mean value coordinates λ_i of (20–21) can be expressed as*

$$\lambda_i(\mathbf{x}) = \frac{\tilde{w}_i(\mathbf{x})}{\sum_{j=1}^{n} \tilde{w}_j(\mathbf{x})}, \tag{24}$$

where

$$\tilde{w}_i(\mathbf{x}) = \left((r_{i-1}r_{i+1} - \mathbf{d}_{i-1} \cdot \mathbf{d}_{i+1}) \prod_{j \neq i-1, i} (r_j r_{j+1} + \mathbf{d}_j \cdot \mathbf{d}_{j+1}) \right)^{1/2}. \tag{25}$$

Proof. From the addition formula for sines, we have

$$w_i = \frac{1}{r_i}\left(\frac{\sin(\alpha_{i-1}/2)}{\cos(\alpha_{i-1}/2)} + \frac{\sin(\alpha_i/2)}{\cos(\alpha_i/2)}\right) = \frac{\sin((\alpha_{i-1}+\alpha_i)/2)}{r_i\cos(\alpha_{i-1}/2)\cos(\alpha_i/2)}.$$

Then, to get rid of the half-angles we use the identities

$$\sin(A/2) = \sqrt{(1-\cos A)/2},$$

$$\cos(A/2) = \sqrt{(1+\cos A)/2},$$

to obtain

$$w_i = \frac{\sqrt{2}}{r_i}\left(\frac{1-\cos(\alpha_{i-1}+\alpha_i)}{(1+\cos\alpha_{i-1})(1+\cos\alpha_i)}\right)^{1/2}.$$

Now we substitute in the scalar product formula,

$$\cos(\alpha_{i-1}+\alpha_i) = \frac{\mathbf{d}_{i-1}\cdot\mathbf{d}_{i+1}}{r_{i-1}r_{i+1}},$$

and similarly for $\cos(\alpha_{i-1})$ and $\cos(\alpha_i)$, and the $1/r_i$ term cancels out:

$$w_i = \sqrt{2}\left(\frac{r_{i-1}r_{i+1} - \mathbf{d}_{i-1}\cdot\mathbf{d}_{i+1}}{(r_{i-1}r_i + \mathbf{d}_{i-1}\cdot\mathbf{d}_i)(r_i r_{i+1} + \mathbf{d}_i\cdot\mathbf{d}_{i+1})}\right)^{1/2}.$$

Finally, we set $\tilde{w}_i = Cw_i$, where C is the constant

$$C = \frac{1}{\sqrt{2}}\prod_{j=1}^{n}(r_j r_{j+1} + \mathbf{d}_j\cdot\mathbf{d}_{j+1})^{1/2},$$

giving (25). □

Now, if $\mathbf{x}$ lies on the edge $[v_j, v_{j+1}]$, then we see that $\tilde{w}_i(\mathbf{x}) = 0$ for all $i \neq j, j+1$ and $\tilde{w}_j(\mathbf{x}) > 0$ and $\tilde{w}_{j+1}(\mathbf{x}) > 0$, and so the λ_i are well-defined at $\mathbf{x}$ with $\lambda_i(\mathbf{x}) = 0$ for all $i \neq j, j+1$ and

$$\lambda_j(\mathbf{x})\mathbf{v}_j + \lambda_{j+1}(\mathbf{x})\mathbf{v}_{j+1} = \mathbf{x}.$$

While the formula is numerically valid at the boundary it requires more square root computations.