Generalized barycentric coordinates

Michael S. Floater

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In this lecture, we review the definitions and properties of barycentric coordinates on triangles, and study generalizations to convex, and non-convex polygons. These generalized coordinates have various applications in computer graphics, including curve, surface, and image deformation, and parameterization of triangular meshes.

1 Triangular coordinates

Let T be a triangle in $\mathbb{R}^2$ with vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. For convenience we will assume that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are in anti-clockwise order around the boundary of T, as in Figure 1. It has been known for a long time, and was at least known by Mobius, that any point $\mathbf{x}$ in T can be expressed uniquely as a convex combination of the three vertices. In other words, there are unique real values $\lambda_1, \lambda_2, \lambda_3 \geq 0$ such that

$$\lambda_1 + \lambda_2 + \lambda_3 = 1, \tag{1}$$

and

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \lambda_3 \mathbf{v}_3 = \mathbf{x}. \tag{2}$$

To see this, observe that the scalar equation (1) and the vector equation (2) together form a linear system of three equations,

$$\begin{pmatrix} 1 & 1 & 1 \\ v_1^1 & v_2^1 & v_3^1 \\ v_1^2 & v_2^2 & v_3^2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 \\ x^1 \\ x^2 \end{pmatrix}, \tag{3}$$

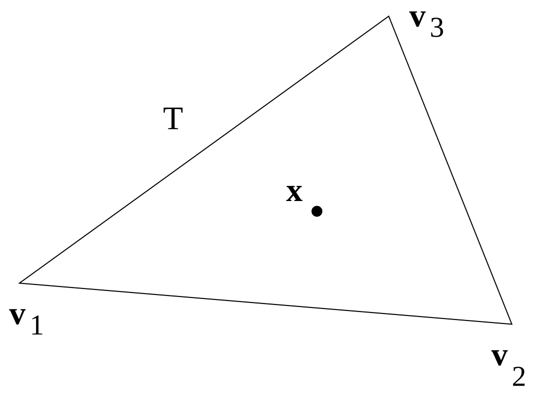

Figure 1: Point in a triangle.

where $\mathbf{v}_j = (v_j^1, v_j^2)$, $j = 1, 2, 3$, and $\mathbf{x} = (x^1, x^2)$. Since the signed area of the triangle T is given by

$$A(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ v_1^1 & v_2^1 & v_3^1 \\ v_1^2 & v_2^2 & v_3^2 \end{vmatrix}, \tag{4}$$

the assumption that T is non-degenerate implies that the matrix in (3) is non-singular, and Cramer's rule gives the unique solution

$$\lambda_1 = \frac{A(\mathbf{x}, \mathbf{v}_2, \mathbf{v}_3)}{A(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)}, \quad \lambda_2 = \frac{A(\mathbf{v}_1, \mathbf{x}, \mathbf{v}_3)}{A(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)}, \quad \lambda_3 = \frac{A(\mathbf{v}_1, \mathbf{v}_2, \mathbf{x})}{A(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)}. \tag{5}$$

The values λ_i are called the *barycentric coordinates* of the point $\mathbf{x}$. Observe that due to the anti-clockwise ordering of the vertices $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, all four areas in (5) are non-negative and therefore $\lambda_1, \lambda_2, \lambda_3$ have the important property of being *non-negative*. Had the vertices been ordered clockwise, all the areas in (5) would have had the opposite sign but we would again have $\lambda_1, \lambda_2, \lambda_3 \geq 0$.

The three areas in the numerators in (5) are shown in Figure 2 where A_i denotes the triangle area $A(\mathbf{x}, \mathbf{v}_{i+1}, \mathbf{v}_{i+2})$ with indices regarded cyclically: if $j = k + 3m$, with $k \in \{1, 2, 3\}$ and $m \in \mathbb{Z}$, then $\mathbf{v}_j := \mathbf{v}_k$.

Viewed as functions of $\mathbf{x}$, we see from (5) that the λ_i are linear polynomials, and from now on we treat them as functions of $\mathbf{x}$. Using either (2) or (5) we see that they have the *Lagrange property*,

$$\lambda_i(\mathbf{v}_j) = \delta_{ij}. \tag{6}$$

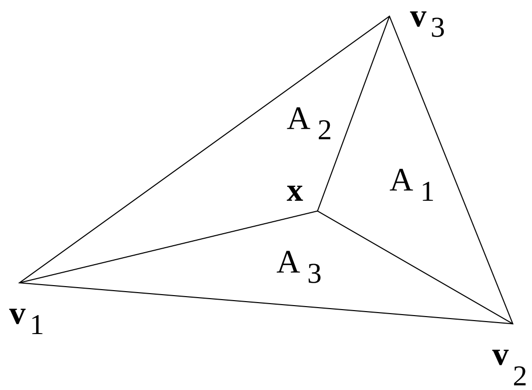

Figure 2: Areas of triangles formed by $\mathbf{x}$.

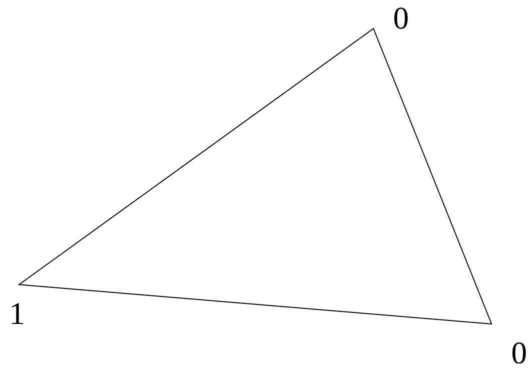

Figure 3: Values of λ_1 at the vertices.

Here, δ_{ij} denotes the Kronecker delta function that has value 1 when $i = j$ and value 0 when $i \neq j$. The values of λ_1 at the vertices are shown in Figure 3.

The linear polynomials $\lambda_1, \lambda_2, \lambda_3$ are clearly well-defined for all $\mathbf{x} \in \mathbb{R}^2$. However, they are are not all positive outside T. Their signs are shown in Figure 4.

2 Polygonal coordinates

Let $\Omega \subset \mathbb{R}^2$ be a convex polygon in the plane, regarded as a closed set, with vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, $n \geq 3$, in an anticlockwise ordering. Figure 5 shows an example with $n = 5$. We will call any set of functions $\lambda_i : \Omega \to \mathbb{R}$,

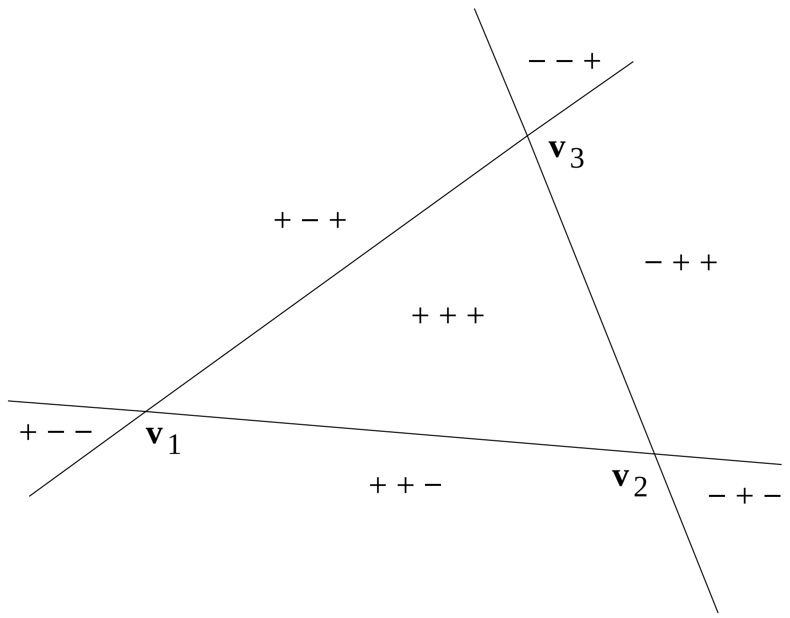

Figure 4: Signs of $\lambda_1, \lambda_2, \lambda_3$.

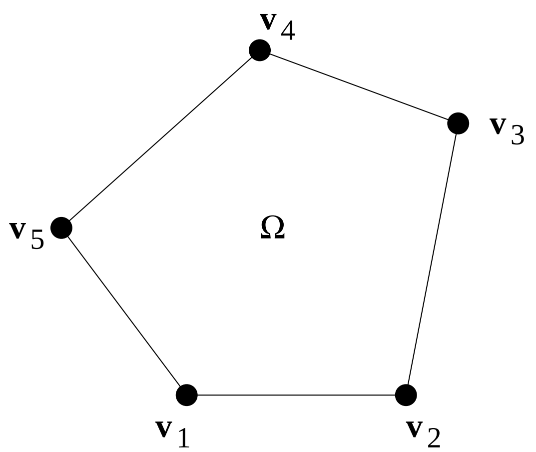

Figure 5: Example of a convex polygon.

$i = 1, \ldots, n$, *barycentric coordinates* if they satisfy, for all $\mathbf{x} \in \Omega$, the three properties

$$\lambda_i(\mathbf{x}) \geq 0, \qquad i = 1, 2, \ldots, n, \tag{7}$$

$$\sum_{i=1}^{n} \lambda_i(\mathbf{x}) = 1 \tag{8}$$

and

$$\sum_{i=1}^{n} \lambda_i(\mathbf{x})\mathbf{v}_i = \mathbf{x}. \tag{9}$$

In the special case that $n = 3$, the λ_i are the unique triangular coordinates of (5). For $n \geq 4$, and for general $\mathbf{x} \in \Omega$, there is no unique choice of the n values $\lambda_1(\mathbf{x}), \ldots, \lambda_n(\mathbf{x})$ that satisfies the three conditions. In most applications we would like functions λ_i that are as smooth as possible.

Even though barycentric coordinates are not unique for $n \geq 4$, they share some general properties that follow from the three defining axioms (7), (8) and (9). First, given a function $f : \Omega \to \mathbb{R}$, we can define an approximation $g = I(f)$, by the formula

$$g(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i(\mathbf{x}) f(\mathbf{v}_i), \qquad \mathbf{x} \in \Omega.$$

The operator I has *linear precision* in the sense that if f is a linear polynomial then $g = f$. This comes from the barycentric property (9) combined with (8): if

$$f(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x} + b,$$

then

$$g(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i(\mathbf{x})(\mathbf{a} \cdot \mathbf{v}_i + b) = \mathbf{a} \cdot \sum_{i=1}^{n} \lambda_i(\mathbf{x})\mathbf{v}_i + b \sum_{i=1}^{n} \lambda_i(\mathbf{x}) = f(\mathbf{x}).$$

Another property is the Lagrange property, $\lambda_i(\mathbf{v}_j) = \delta_{ij}$, and the λ_i are linear along each edge of Ω. To see this, observe that since the triangle area $A(\mathbf{x}, \mathbf{v}_j, \mathbf{v}_{j+1})$ is linear in $\mathbf{x}$, the linear preicsion property implies

$$\sum_{i=1}^{n} \lambda_i(\mathbf{x}) A(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_{j+1}) = A(\mathbf{x}, \mathbf{v}_j, \mathbf{v}_{j+1}),$$

and so if $\mathbf{x}$ belongs to the edge $[\mathbf{v}_j, \mathbf{v}_{j+1}]$,

$$\sum_{i \neq j, j+1} \lambda_i(\mathbf{x}) A(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_{j+1}) = 0.$$

Since $A(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_{j+1}) > 0$ for $i \neq j, j+1$ by the convexity of Ω, and since $\lambda_i(\mathbf{x}) \geq 0$ by (7) it follows that $\lambda_i(\mathbf{x}) = 0$ for all $i \neq j, j+1$. This implies that from (9),

$$\lambda_j(\mathbf{x})\mathbf{v}_j + \lambda_{j+1}(\mathbf{x})\mathbf{v}_{j+1} = \mathbf{x}.$$

3 Wachspress coordinates

These are barycentric coordinates over convex polygons that are rational functions, and are defined by

$$\lambda_i(\mathbf{x}) = \frac{w_i(\mathbf{x})}{\sum_{j=1}^{n} w_j(\mathbf{x})}, \qquad \mathbf{x} \in \Omega, \tag{10}$$

where

$$w_i(\mathbf{x}) = B_i \prod_{j \neq i-1, i} A_j(\mathbf{x}), \tag{11}$$

and

$$A_i(\mathbf{x}) := A(\mathbf{x}, \mathbf{v}_i, \mathbf{v}_{i+1}) \quad \text{and} \quad B_i := A(\mathbf{v}_{i-1}, \mathbf{v}_i, \mathbf{v}_{i+1}).$$

The triangle areas A_i and B_i are shown in Figure 6. Since A_j is linear in $\mathbf{x}$ we see that w_i is a polynomial of degree $n-2$. It follows that the numerator of λ_i has degree $n-2$. Thus the denominator,

$$W(\mathbf{x}) = \sum_{j=1}^{n} w_j(\mathbf{x}), \tag{12}$$

has degree at most $n-2$, but in fact its degree is $n-3$. This follows from the barycentric property λ_i which we prove in the next section.

Wachspress coordinates are clearly smooth (C^∞) and also rational polynomials in the coordinates x^1 and x^2 of the point $\mathbf{x} = (x^1, x^2)$ with degree at most $n-2$.

We note that in the case that Ω is a regular polygon, the areas $B_1, \ldots, B_n$ are equal in which case they can be removed from the formula and we have simply

$$w_i(\mathbf{x}) = \prod_{j \neq i-1, i} A_j(\mathbf{x}). \tag{13}$$

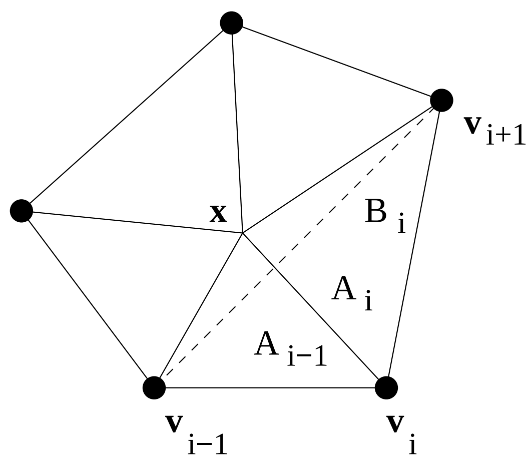

Figure 6: Triangles areas

To establish the barycentric property (9) for the Wachspress coordinates (10), it helps first to express the coordinates in a different form. Multiplying w_i by a constant in i does not change λ_i, and so dividing w_i by $\prod_{j=1}^n A_j$ gives the alternative form

$$\lambda_i(\mathbf{x}) = \frac{\tilde{w}_i(\mathbf{x})}{\sum_{j=1}^n \tilde{w}_j(\mathbf{x})}, \qquad \tilde{w}_i(\mathbf{x}) = \frac{B_i}{A_{i-1}(\mathbf{x})A_i(\mathbf{x})}, \tag{14}$$

which is valid for $\mathbf{x} \in \text{Int}(\Omega)$. In contrast to w_i, the rational function $\tilde{w}_i$ depends only on the three local vertices $\mathbf{v}_{i-1}$, $\mathbf{v}_i$, and $\mathbf{v}_{i+1}$.

Theorem 1 *With $\tilde{w}_i$ as in (14),*

$$\sum_{i=1}^n \tilde{w}_i(\mathbf{x})(\mathbf{v}_i - \mathbf{x}) = 0, \qquad \mathbf{x} \in \text{Int}(\Omega). \tag{15}$$

Proof. We express $\mathbf{x}$ as a barycentric combination of $\mathbf{v}_{i-1}$, $\mathbf{v}_i$, and $\mathbf{v}_{i+1}$,

$$\mathbf{x} = \frac{A_i(\mathbf{x})}{B_i}\mathbf{v}_{i-1} + \frac{(B_i - A_{i-1}(\mathbf{x}) - A_i(\mathbf{x}))}{B_i}\mathbf{v}_i + \frac{A_{i-1}(\mathbf{x})}{B_i}\mathbf{v}_{i+1},$$

and rearrange this equation in the form

$$\frac{B_i}{A_{i-1}(\mathbf{x})A_i(\mathbf{x})}(\mathbf{v}_i - \mathbf{x}) = \frac{1}{A_{i-1}(\mathbf{x})}(\mathbf{v}_i - \mathbf{v}_{i-1}) - \frac{1}{A_i(\mathbf{x})}(\mathbf{v}_{i+1} - \mathbf{v}_i).$$

Summing both sides of this equation over $i = 1, \ldots, n$ gives the result. □

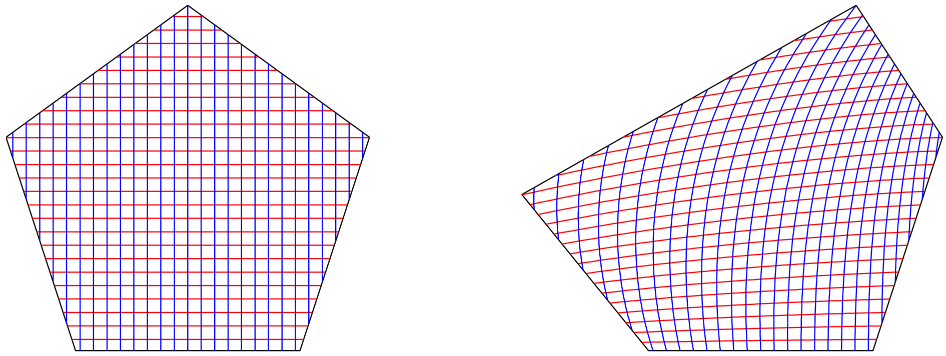

Figure 7: A Wachspress barycentric mapping

The barycentric property (9) follows from this. Another consequence is that

$$\sum_{i=1}^{n} \tilde{w}_i(\mathbf{x})\mathbf{v}_i = \mathbf{x} \sum_{i=1}^{n} \tilde{w}_i(\mathbf{x}),$$

and so

$$\sum_{i=1}^{n} w_i(\mathbf{x})\mathbf{v}_i = \mathbf{x} \sum_{i=1}^{n} w_i(\mathbf{x}).$$

Then, since the left hand side is a (vector) polynomial of degree $\leq n-2$, the sum $W(\mathbf{x}) = \sum_{i=1}^{n} w_i(\mathbf{x})$ must be a polynomial of degree $\leq n-3$.

4 Barycentric mapping

Barycentric coordinates such as Wachspress coordinates can be used to deform one polygon into another. We can map the 'source' polygon Ω into the 'target' polygon Ω', with vertives $\mathbf{v}'_1, \ldots, \mathbf{v}'_n$ by the mapping

$$\phi(\mathbf{x}) = \sum_{i} \lambda_i(\mathbf{x})\mathbf{v}'_i.$$

Figure 7 shows such a Wachspress mapping. Such a mapping can be used to deform a smooth curve. One can enclose the curve in a polygon, and then deform the curve by moving the vertices of the polygon, in a similar way to modelling with Bezier and spline curves. Figure 8 shows the result of applying the mapping above to a circle.

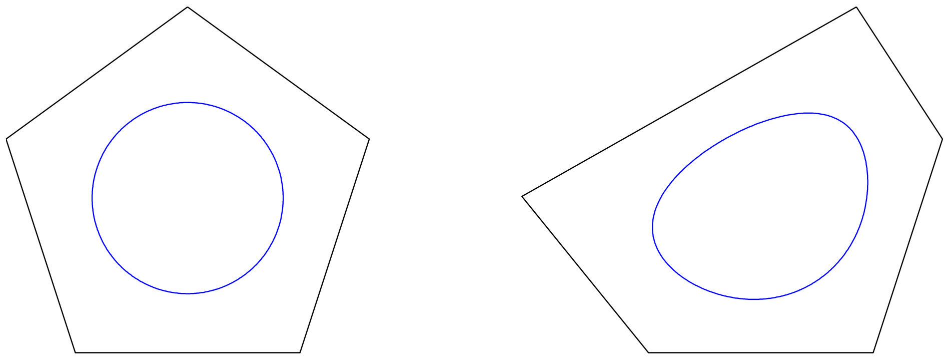

Figure 8: Curve deformation

4.1 Wachspress coordinates outside the polygon

Do Wachspress coordinates extend outside the polygon? We saw that barycentric coordinates over a triangle continue to be well-defined outside the triangle, even though they are no longer all positive there.

Wachspress coordinates, however, may not even be well-defined outside the polygon. Consider the example of Fig 9, where $n = 4$, and Ω is a quadrilateral. The lines through the fours sides of the quadrilateral intersect in the two points $\mathbf{p}$ and $\mathbf{q}$. By definition, we see that $A_2(\mathbf{p}) = A_4(\mathbf{p}) = 0$. Therefore, $w_i(\mathbf{p}) = 0$ for all $i = 1, 2, 3, 4$, and so $W(\mathbf{p}) = 0$. Similarly, $A_1(\mathbf{q}) = A_3(\mathbf{q}) = 0$, and so $W(\mathbf{q}) = 0$ We showed earlier that the polynomial W has degree $\leq n-2$, which in this quadrilateral case is degree 1. Thus, W is zero along the line L through $\mathbf{p}$ and $\mathbf{q}$ and we conclude that the Wachspress coordinates $\lambda_1, \ldots, \lambda_4$ are not well-defined on the line L.

5 Mean value coordinates

Can we find coordinates that are well-defined for points both inside and outside the polygon, and ideally, also for a non-convex polygon? One answer is *mean value* coordinates. We consider initially the case that Ω is again convex and $\mathbf{x}$ is inside. The mean value coordinates of $\mathbf{x}$ are then

$$\lambda_i(\mathbf{x}) = \frac{w_i(\mathbf{x})}{\sum_{j=1}^{n} w_j(\mathbf{x})}, \qquad \mathbf{x} \in \Omega, \tag{16}$$

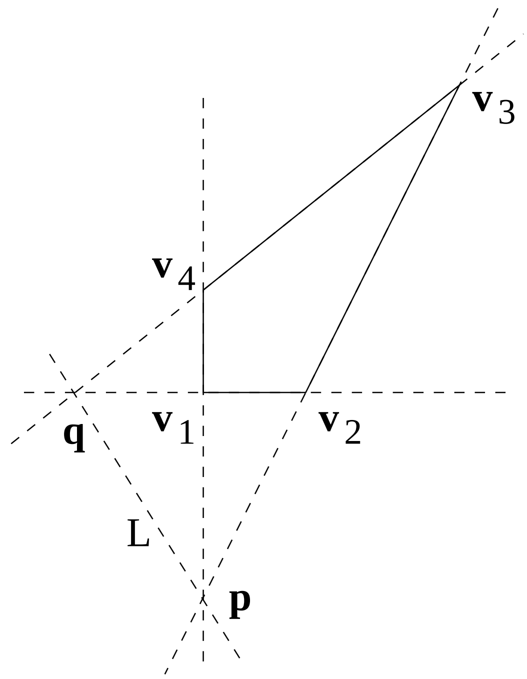

Figure 9: Singularities of Wachspress coordinates

where

$$w_i(\mathbf{x}) = \frac{\tan(\alpha_{i-1}(\mathbf{x})/2) + \tan(\alpha_i(\mathbf{x})/2)}{\|\mathbf{v}_i - \mathbf{x}\|}, \tag{17}$$

and $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^2$ and $\alpha_i(\mathbf{x})$ is the angle at $\mathbf{x}$ in the triangle $[\mathbf{x}, \mathbf{v}_i, \mathbf{v}_{i+1}]$, illustrated in Figure 10.

Note that the tangents in the formula do not need to be evaluated directly. Instead, using the notation $\mathbf{d}_i = \mathbf{v}_i - \mathbf{x}$, $r_i = \|\mathbf{d}_i\|$, and $\mathbf{e}_i = \mathbf{d}_i/r_i$, we can write

$$\sin\alpha_i = \mathbf{e}_i \times \mathbf{e}_{i+1}, \qquad \text{and} \qquad \cos\alpha_i = \mathbf{e}_i \cdot \mathbf{e}_{i+1},$$

where $\times$ and $\cdot$ denote cross and dot products of vectors in $\mathbb{R}^2$, and then use either of the two formulas

$$\tan(\alpha/2) = (1 - \cos\alpha)/\sin\alpha = \sin\alpha/(1 + \cos\alpha).$$

Nevertheless, the mean value weight (17) requires computing square roots in order to evaluate r_{i-1}, r_i, and r_{i+1}.

To show that mean value coordinates satisfy the barycentric property (9) it is enough to show

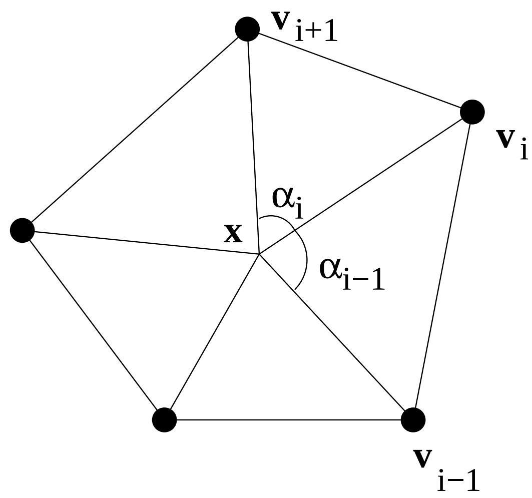

Figure 10: Angles in mean value formula

Theorem 2 *With w_i as in (17),*

$$\sum_{i=1}^{n} w_i(\mathbf{x})(\mathbf{v}_i - \mathbf{x}) = 0, \qquad \mathbf{x} \in \mathrm{Int}(\Omega). \tag{18}$$

Proof. With $\mathbf{x} \in \mathrm{Int}(\Omega)$ fixed, equation (18) is equivalent to

$$\sum_{i=1}^{n} \big(\tan(\alpha_{i-1}/2) + \tan(\alpha_i/2)\big)\mathbf{e}_i = 0,$$

which can be rewritten as

$$\sum_{i=1}^{n} \tan(\alpha_i/2)(\mathbf{e}_i + \mathbf{e}_{i+1}) = 0. \tag{19}$$

To show that this equation holds, express the unit vector $\mathbf{e}_i$ as $\mathbf{e}_i = (\cos\theta_i, \sin\theta_i)$. Then $\alpha_i = \theta_{i+1} - \theta_i$, and so

$$\begin{aligned} \tan\left(\frac{\alpha_i}{2}\right)(\mathbf{e}_i + \mathbf{e}_{i+1}) &= \tan\left(\frac{\theta_{i+1} - \theta_i}{2}\right)(\cos\theta_i + \cos\theta_{i+1}, \sin\theta_i + \sin\theta_{i+1}) \\ &= (\sin\theta_{i+1} - \sin\theta_i, \cos\theta_i - \cos\theta_{i+1}), \end{aligned}$$

the last line following from the addition and subtraction formulas for sines and cosines. Summing this expression over $i = 1, \ldots, n$ gives equation (19) as required. □

An advantage of mean value coordinates over Wachspress coordinates is that they are well defined (and positive) even when Ω is a star-shaped polygon, as long as $\mathbf{x}$ is in its kernel. This means that mean value coordinates can be used to express a vertex in a (planar) triangular mesh as a convex combination of its neighbouring vertices. This was the original motivation for mean value coordinates and they have used for this purpose in order to parameterize (non-planar) triangular meshes.

Another advantage of mean value coordinates, is that they are still well-defined, though not in general positive, for *arbitrary* polygons, and in fact for any $\mathbf{x}$ in $\mathbb{R}^2$, provided that the angle α_i in the formula is taken to be the *signed* angle formed by the two vectors $\mathbf{v}_i - \mathbf{x}$ and $\mathbf{v}_{i+1} - \mathbf{x}$. The coordinates are not in general positive, but they still sum to one and have the Lagrange property on the boundary of the polygon. This means that they can be used for deforming polygons, even when the source polygon is not convex.

Figure 11 show the use of mean value coordinates to deform a digital image (to 'straighten' the leaning tower of pisa).

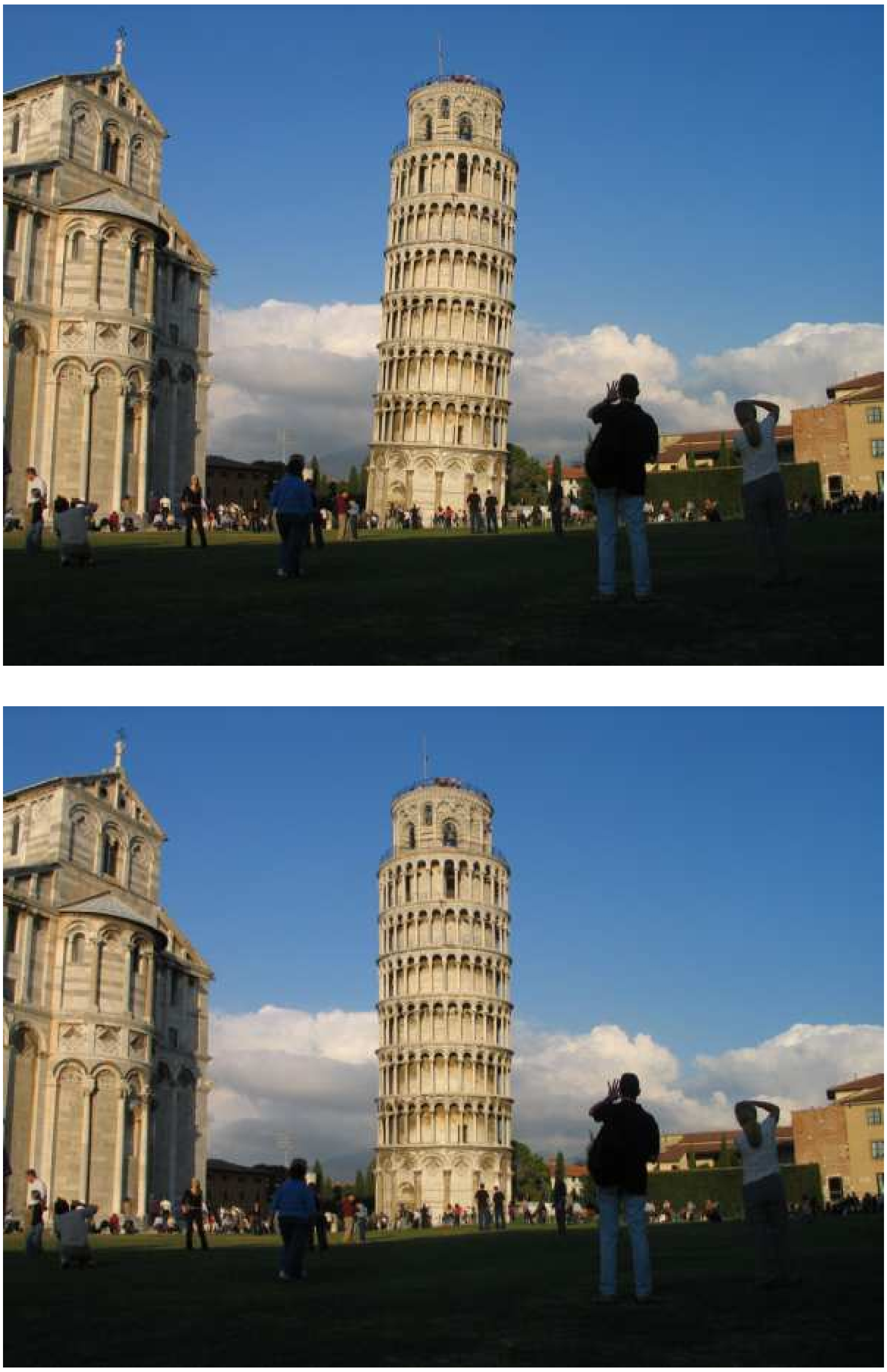

Figure 11: Image warping using MV coordinates.