Interpolation by subdivision

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Abstract

These notes provide an introduction to the interpolation of data and functions by recursive subdivision.

1 Introduction

Given a sequence of values $f_k \in \mathbb{R}$, for $k = 0, 1, 2, \ldots, n$, we want to find an interpolant, i.e., a function $g : [0, n] \to \mathbb{R}$ such that $g(k) = f_k$, for all k, with good smoothness and approximation properties. One way of doing this is to use interpolatory subdivision. One of the earliest and best known examples of interpolatory subdivision is the four-point scheme, studied by Dubuc and Dyn, Gregory, and Levin.

We start by adding two data at each end, so that we have data f_k for $-2 \le k \le n + 2$. We can think of the extra data as 'boundary conditions' that will influence the interpolant g. We now initialize the scheme by setting $g_{0,k} = f_k$, $k = -2, \ldots, n + 2$, and, then for each $j = 0, 1, 2, \ldots$ we generate data by the rules

$$g_{j+1,2k} = g_{j,k}, \tag{1}$$

$$g_{j+1,2k+1} = -\frac{1}{16} g_{j,k-1} + \frac{9}{16} g_{j,k} + \frac{9}{16} g_{j,k+1} - \frac{1}{16} g_{j,k+2}, \tag{2}$$

We will compute the interpolant g as the limit of polygons through these data. We define the polygon g_j as the piecewise linear interpolant to the data $(x_{j,k}, g_{j,k})$, $k = -2, \ldots, 2^j n + 2$, where

$$x_{j,k} := 2^{-j} k.$$

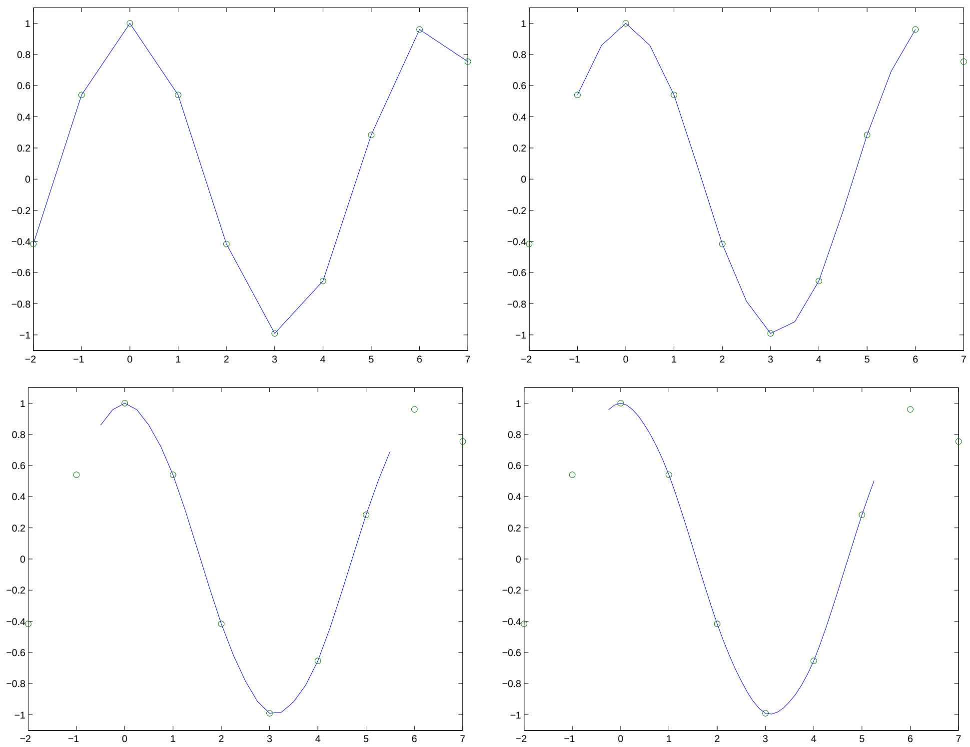

Figure 1: Polygons g_j, top row: $j = 0, 1$, bottom row: $j = 2, 3$.

The points $x_{0,k}$ are integers, the points $x_{1,k}$ half-integers, $x_{2,k}$ quarter integers, and so on. The points $x_{j,k}$ are sometimes referred to as *dyadic points*. Figure 1 shows the first four polygons g_0, g_1, g_2, g_3 of an example data set.

The coefficients appearing in (2),

$$-\frac{1}{16}, \frac{9}{16}, \frac{9}{16}, -\frac{1}{16},$$

are the values at $x = 1/2$ of the four cubic Lagrange polynomials that have value 1 at one of the points $x = -1, 0, 1, 2$, and value zero at the other three. Because of this the subdivision scheme *reproduces cubic polynomials*: if $f_k = f(k)$ for some cubic polynomial f, then $g_{j,k} = f(x_{j,k})$ for all $j = 0, 1, 2, \ldots$ and all k. We say that the scheme has *cubic precision*. Dyn, Gregory, and Levin considered the more general coefficients,

$$-w, \frac{1}{2} + w, \frac{1}{2} + w, -w,$$

which includes the former ones, when $w = 1/16$. For general values of w, one can check that the scheme reproduces linear polynomials, but not cubic ones.

2 Convergence

We hope that the sequence of polygons $g_0, g_1, g_2, \ldots$ has a limit function, and that it is in some sense smooth. In order to establish this we will use a well known result from analysis that says that a sufficient condition for such convergence is that the functions g_j form a Cauchy sequence in the max norm

$$\|\phi\| := \sup_{x \in [0,n]} |\phi(x)|.$$

Thus we need to show that for any $\epsilon > 0$ there is some N such that for all $i, j \geq N$,

$$\|g_i - g_j\| \leq \epsilon. \tag{3}$$

To this end we will use the following lemma.

Lemma 1 *If there are positive constants C and $\lambda < 1$ such that*

$$\|g_{j+1} - g_j\| \leq C\lambda^j, \qquad j = 0, 1, 2, \ldots, \tag{4}$$

then $(g_j)_{j=0,1,2,\ldots}$ is a Cauchy sequence.

Proof. Observe that under condition (4), if $i > j \geq N$,

$$\begin{aligned} \|g_i - g_j\| &\leq \|g_{j+1} - g_j\| + \|g_{j+2} - g_{j+1}\| + \cdots + \|g_i - g_{i-1}\| \\ &\leq C\lambda^j(1 + \lambda + \lambda^2 + \cdots + \lambda^{i-1-j}) \\ &\leq C\lambda^j/(1-\lambda) \leq C\lambda^N/(1-\lambda). \end{aligned}$$

Thus (3) holds if we take N large enough that $C\lambda^N/(1-\lambda) \leq \epsilon$. □

We now use this lemma to prove convergence of the scheme (1–2).

Theorem 1 *The sequence $g_0, g_1, g_2, \ldots,$ has a continuous limit*

$$g(x) = \lim_{j \to \infty} g_j(x), \qquad x \in [0, n].$$

Proof. Observe that the difference $g_{j+1} - g_j$ on $[0, n]$ is itself a polygon at level $j + 1$, and since it is zero at every even point $x = x_{j+1,2k}$, it attains its maximum absolute value at an odd point, $x_{j+1,2k+1}$, i.e.,

$$\|g_{j+1} - g_j\| = \max_k |g_{j+1,2k+1} - (g_{j,k} + g_{j,k+1})/2|.$$

But from (2),

$$g_{j+1,2k+1} - \frac{1}{2}(g_{j,k} + g_{j,k+1}) = \frac{1}{16}\Delta g_{j,k-1} - \frac{1}{16}\Delta g_{j,k+1},$$

where

$$\Delta g_{j,r} := g_{j,r+1} - g_{j,r}.$$

Therefore,

$$\|g_{j+1} - g_j\| \leq \frac{1}{8} \max_k |\Delta g_{j,k}|.$$

Thus if we can show that there are constants K and $\lambda < 1$ such that

$$\max_k |\Delta g_{j,k}| \leq K\lambda^j, \qquad j = 0, 1, 2, \ldots, \tag{5}$$

we can apply Lemma 1 with $C = K/8$. To this end observe that, from (1–2),

$$\Delta g_{j+1,2k} = \frac{1}{16}\Delta g_{j,k-1} + \frac{1}{2}\Delta g_{j,k} - \frac{1}{16}\Delta g_{j,k+1}, \tag{6}$$

$$\Delta g_{j+1,2k+1} = -\frac{1}{16}\Delta g_{j,k-1} + \frac{1}{2}\Delta g_{j,k} + \frac{1}{16}\Delta g_{j,k+1}, \tag{7}$$

and it follows that

$$\max_k |\Delta g_{j+1,k}| \leq \frac{5}{8} \max_k |\Delta g_{j,k}|, \tag{8}$$

and therefore that (5) holds with $\lambda = 5/8 < 1$ and $K = \max_k |\Delta g_{0,k}|$. □

3 Smoothness

We next consider the smoothness of the limit function g, by considering the divided differences,

$$g^{[1]}_{j,k} := \frac{\Delta g_{j,k}}{x_{j,k+1} - x_{j,k}} = 2^j \Delta g_{j,k}.$$

We let $g^{[1]}_j$ be the piecewise linear interpolant to the data $(x_{j,k}, g^{[1]}_{j,k})$. Figure 2 shows a plot of $g^{[1]}_5$.

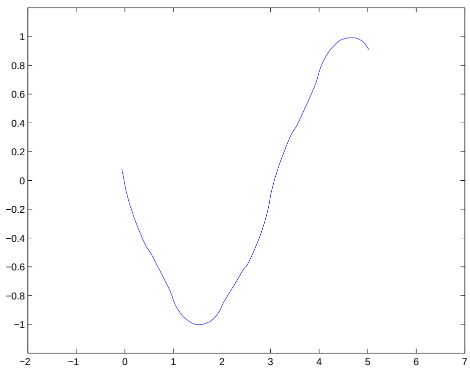

Figure 2: Polygon $g_5^{[1]}$

Theorem 2 *The limit function g of Theorem 1 is C^1.*

Proof. We show

(i) that the sequence of polygons $g_j^{[1]}$ has a continuous limit

$$g^{[1]}(x) := \lim_{j\to\infty} g_j^{[1]}(x), \qquad x \in [0,n], \tag{9}$$

and

(ii) that

$$g(x) - g(0) = \int_0^x g^{[1]}(y)\, dy, \qquad x \in [0,n], \tag{10}$$

which implies that g is differentiable with $g'(x) = g^{[1]}(x)$.

Starting with (i), we show that $g_j^{[1]}$ is a Cauchy sequence. From (6-7), there is a scheme for the $g_{j,k}^{[1]}$,

$$g_{j+1,2k}^{[1]} = \frac{1}{8} g_{j,k-1}^{[1]} + g_{j,k}^{[1]} - \frac{1}{8} g_{j,k+1}^{[1]}, \tag{11}$$

$$g_{j+1,2k+1}^{[1]} = -\frac{1}{8} g_{j,k-1}^{[1]} + g_{j,k}^{[1]} + \frac{1}{8} g_{j,k+1}^{[1]}. \tag{12}$$

Since the difference $g_{j+1}^{[1]} - g_j^{[1]}$ on $[0,n]$ takes on its maximum absolute value either at a point $x_{j+1,2k}$ or $x_{j+1,2k+1}$,

$$\|g_{j+1}^{[1]} - g_j^{[1]}\| \leq \max\{A_0, A_1\},$$

where

$$A_0 = \max_k |g^{[1]}_{j+1,2k} - g^{[1]}_{j,k}|, \qquad A_1 = \max_k |g^{[1]}_{j+1,2k+1} - (g^{[1]}_{j,k} + g^{[1]}_{j,k+1})/2|.$$

From (11–12),

$$\begin{aligned} g^{[1]}_{j+1,2k} - g^{[1]}_{j,k} &= -\frac{1}{8}\Delta g^{[1]}_{j,k-1} - \frac{1}{8}\Delta g^{[1]}_{j,k}, \\ g^{[1]}_{j+1,2k+1} - \frac{1}{2}(g^{[1]}_{j,k} + g^{[1]}_{j,k+1}) &= \frac{1}{8}\Delta g^{[1]}_{j,k-1} - \frac{3}{8}\Delta g^{[1]}_{j,k}, \end{aligned}$$

and therefore,

$$\|g^{[1]}_{j+1} - g^{[1]}_{j}\| \le \frac{1}{2}\max_k |\Delta g^{[1]}_{j,k}|.$$

Thus, similar to the proof of Theorem 1, we can use Lemma 1 if we can show that there are constants C_2 and $\lambda < 1$ such that

$$\max_k |\Delta g^{[1]}_{j,k}| \le C_2 \lambda^j, \qquad j = 0, 1, 2, \ldots. \tag{13}$$

Taking differences of the $g^{[1]}_{j,k}$ in (11–12) gives

$$\Delta g^{[1]}_{j+1,2k} = \frac{1}{4}\Delta g^{[1]}_{j,k-1} + \frac{1}{4}\Delta g^{[1]}_{j,k}, \tag{14}$$

$$\Delta g^{[1]}_{j+1,2k+1} = -\frac{1}{8}\Delta g^{[1]}_{j,k-1} + \frac{3}{4}\Delta g^{[1]}_{j,k} - \frac{1}{8}\Delta g^{[1]}_{j,k+1}. \tag{15}$$

It follows that

$$\max_k |\Delta g^{[1]}_{j+1,k}| \le \max_k |\Delta g^{[1]}_{j,k}|, \tag{16}$$

but this merely shows (13) holds with $\lambda = 1$. One way to fix this is to use a double step: applying a second iteration of the scheme (14–15) we find

$$\begin{bmatrix} \Delta g^{[1]}_{j+2,4k} \\ \Delta g^{[1]}_{j+2,4k+1} \\ \Delta g^{[1]}_{j+2,4k+2} \\ \Delta g^{[1]}_{j+2,4k+3} \end{bmatrix} = \frac{1}{64}\begin{bmatrix} -2 & 16 & 2 & 0 \\ 1 & 7 & 7 & 1 \\ 0 & 2 & 16 & -2 \\ 0 & -8 & 32 & -8 \end{bmatrix} \begin{bmatrix} \Delta g^{[1]}_{j,k-2} \\ \Delta g^{[1]}_{j,k-1} \\ \Delta g^{[1]}_{j,k} \\ \Delta g^{[1]}_{j,k+1} \end{bmatrix},$$

from which it follows that

$$\max_k |\Delta g^{[1]}_{j+2,k}| \le \frac{3}{4}\max_k |\Delta g^{[1]}_{j,k}|, \tag{17}$$

and therefore, by a similar analysis to that of Lemma 1, the sequence $(g_j)_j$ is Cauchy, which establishes (9).

Considering (ii), since both sides of the equation (10) are continuous in x, it is sufficient to show that it holds for all dyadic points $x = x_{J,K}$. Then for any $j \geq J$, we have $x = x_{j,k}$, where $k = 2^{J-j}K$, and so

$$g(x) - g(0) = \sum_{i=0}^{k-1} (g_{j,i+1} - g_{j,i}) = \sum_{i=0}^{k-1} (x_{j,i+1} - x_{j,i}) g_{j,i}^{[1]} = A + B,$$

where

$$A = \sum_{i=0}^{k-1} (x_{j,i+1} - x_{j,i}) g^{[1]}(x_{j,i}),$$

and

$$B = \sum_{i=0}^{k-1} (x_{j,i+1} - x_{j,i}) (g_j^{[1]}(x_{j,i}) - g^{[1]}(x_{j,i})).$$

Now, as $j \to \infty$, since $g^{[1]}$ is a continuous function, A converges to the integral in (10) and

$$|B| \leq \|g_j^{[1]} - g^{[1]}\| \sum_{i=0}^{k-1} (x_{j,i+1} - x_{j,i}) = \|g^{[1]} - g_j^{[1]}\| (x - 0) \to 0,$$

and this establishes (10). □

3.1 Hölder regularity

It can be shown that the limit function g is not in general twice differentiable, although of course it will be in the special case that the initial data are drawn from a cubic polynomial, i.e., if $f_k = f(k)$ for some cubic polynomial f.

However, for any initial data, the limit function is close to C^2 in the following sense. First we need to define what we mean by Hölder continuity. A function $\phi : [a, b] \to \mathbb{R}$ is said to be Hölder continuous with exponent α, $0 < \alpha < 1$, if there is a constant $C > 0$ such that

$$\frac{|\phi(y) - \phi(x)|}{|y - x|^\alpha} \leq C, \qquad \text{for } a \leq x < y \leq b,$$

in which case we write $\phi \in C^\alpha[a, b]$, or just $\phi \in C^\alpha$. Hölder continuity in the limiting case $\alpha = 1$ is the same as Lipschitz continuity. We also write $\phi \in C^{k+\alpha}$ for $k = 1, 2, \ldots$ and $\alpha \in (0, 1)$ if $\phi^{(k)} \in C^\alpha$.

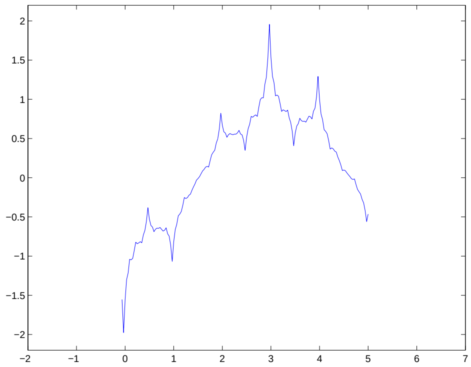

Figure 3: Polygon $g_5^{[2]}$

Theorem 3 *The limit function g of the scheme (1–2) is in $C^{1+\alpha}[0,n]$ for all $\alpha \in (0,1)$.*

In proving this theorem we will also see a new way of showing that $g \in C^1$, which avoids the need for a double step estimate such as (17). We work with the second order divided differences,

$$g_{j,k}^{[2]} := \frac{\Delta g_{j,k}^{[1]}}{x_{j,k+2} - x_{j,k}} = 2^{j-1} \Delta g_{j,k}^{[1]},$$

and let $g_j^{[2]}$ be the piecewise linear interpolant to the data $(x_{j,k}, g_{j,k}^{[2]})$. Figure 3 shows a plot of $g_5^{[2]}$.

Proof. By multiplying the coefficients in the scheme (14–15) by 2 we obtain the scheme

$$\begin{aligned} g_{j+1,2k}^{[2]} &= \frac{1}{2} g_{j,k-1}^{[2]} + \frac{1}{2} g_{j,k}^{[2]} \\ g_{j+1,2k+1}^{[2]} &= -\frac{1}{4} g_{j,k-1}^{[2]} + \frac{3}{2} g_{j,k}^{[2]} - \frac{1}{4} g_{j,k+1}^{[2]}. \end{aligned}$$

Taking differences of this scheme gives

$$\begin{aligned} \Delta g_{j+1,2k}^{[2]} &= \frac{3}{4} \Delta g_{j,k-1}^{[2]} - \frac{1}{4} \Delta g_{j,k}^{[2]} \\ \Delta g_{j+1,2k+1}^{[2]} &= -\frac{1}{4} \Delta g_{j,k-1}^{[2]} + \frac{3}{4} \Delta g_{j,k}^{[2]}. \end{aligned}$$

From this it follows that

$$\max_k |\Delta g^{[2]}_{j+1,k}| \le \max_k |\Delta g^{[2]}_{j,k}|,$$

and therefore that

$$\max_k |\Delta g^{[2]}_{j,k}| \le C,$$

for some constant C independent of j. From the scheme for the $g^{[2]}_{j,k}$, we deduce that there is some new constant C such that both

$$\|g^{[2]}_{j+1} - g^{[2]}_j\| \le C.$$

Then

$$\|g^{[2]}_j - g^{[2]}_0\| \le \|g^{[2]}_j - g^{[2]}_{j-1}\| + \cdots + \|g^{[2]}_1 - g^{[2]}_0\| \le Cj,$$

and so

$$\|g^{[2]}_j\| \le K + Cj.$$

By the definition of $g^{[2]}_{j,k}$,

$$|\Delta g^{[1]}_{j,k}| \le 2^{-j}(K + Cj), \tag{18}$$

for new constants C and K. Note that since the term 2^{-j} dominates j as $j \to \infty$, this is a better estimate than (16) and could have been used to show that g is C^1.

From the scheme for the $g^{[1]}_{j,k}$, we deduce that there are new constants K and C such that for

$$\|g^{[1]}_{j+1} - g^{[1]}_j\| \le 2^{-j}(K + Cj), \tag{19}$$

and therefore there are new constants such that

$$\|g^{[1]} - g^{[1]}_j\| \le 2^{-j}(K + Cj). \tag{20}$$

Choose any x and y with $0 \le x < y \le n$, and suppose $t := y - x < 1$. Let j be the unique integer such that

$$2^{-j} > t \ge 2^{-(j+1)}.$$

Then we use the inequality

$$|g^{[1]}(y) - g^{[1]}(x)| \le |g^{[1]}(y) - g^{[1]}_j(y)| + |g^{[1]}_j(y) - g^{[1]}_j(x)| + |g^{[1]}_j(x) - g^{[1]}(x)|.$$

Since $t < 2^{-j}$,

$$|g_j^{[1]}(y) - g_j^{[1]}(x)| \leq \max_k |\Delta g_{j,k}^{[1]}|,$$

and so from (18) and (20), there are constants such that

$$|g^{[1]}(y) - g^{[1]}(x)| \leq 2^{-j}(K + Cj).$$

Therefore, since

$$2^{-j} \leq 2t, \qquad \text{and} \qquad j < \frac{\log(1/t)}{\log(2)},$$

it follows that

$$\left|g^{[1]}(y) - g^{[1]}(x)\right| \leq t(K + C\log(1/t)),$$

for further constants K and C. This shows that for all $\alpha < 1$,

$$\frac{\left|g^{[1]}(y) - g^{[1]}(x)\right|}{t^\alpha} \leq t^{1-\alpha}(K + C\log(1/t)),$$

which is bounded for $t \in (0, 1)$, and so $g^{[1]} \in C^\alpha$ as claimed. □