# Polar forms

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These notes study the polar form of a polynomial and use it to develop further aspects of Bézier curves and surfaces such as subdivision and smoothness conditions.

## 1 The polar form

Every polynomial of degree $\le n$,

$$p(x) = \sum_{i=0}^{n} a_i x^i, \qquad a_i \in \mathbb{R}, \tag{1}$$

has a unique $n$-variate *polar form* or *blossom* $P$. The polar form $P$ is the unique polynomial $P(x_1, x_2, \ldots, x_n)$ that is

(i) symmetric,

(ii) multi-affine, and

(iii) agrees with $p$ on its diagonal.

By symmetric we mean that $P$ has the same value if we swap any two variables $x_i$ and $x_j$:

$$P(\ldots, x_i, \ldots, x_j, \ldots) = P(\ldots, x_j, \ldots, x_i, \ldots).$$

By multi-affine we mean that $P$ is affine with respect to each variable: if

$$x_1 = (1 - \lambda)x + \lambda y,$$

then

$$P(x_1, x_2, \ldots, x_n) = (1-\lambda)P(x, x_2, \ldots, x_n) + \lambda P(y, x_2, \ldots, x_n),$$

and similarly for the other variables. By the diagonal property we understand that

$$P(\underbrace{x, x, \ldots, x}_{n}) = p(x).$$

We will see in the next section that the polar form is unique, but to see that $p$ has such a polar form, just replace each term $x^i$ in (1) by the expression

$$\sum_{1 \le k_1 < k_2 < \cdots < k_i \le n} x_{k_1} x_{k_2} \cdots x_{k_i} \Big/ \binom{n}{i}. \tag{2}$$

For example, the quadratic

$$p(x) = a_0 + a_1 x + a_2 x^2$$

has the bivariate polar form

$$P(x_1, x_2) = a_0 + a_1 \frac{x_1 + x_2}{2} + a_2 x_1 x_2,$$

and the cubic

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

has the trivariate polar form

$$P(x_1, x_2, x_3) = a_0 + a_1 \frac{x_1 + x_2 + x_3}{3} + a_2 \frac{x_1 x_2 + x_1 x_3 + x_2 x_3}{3} + a_3 x_1 x_2 x_3.$$

Using (2), the symmetry and multi-affine properties of $P$ are easily verified. The diagonal property holds because $\binom{n}{i}$ is the number of terms in the sum in (2).

We will use the polar form to derive various properties of Bézier curves without needing the explicit formula (2). It is the defining properties (i), (ii), (iii) that are important.

## 2 de Casteljau's algorithm

We now view the control points of Bézier curves and the points in de Casteljau's algorithm in a new light, in terms of the polar form. Let $\mathbf{p}$ be a Bézier curve on the interval $[a, b]$, i.e.,

$$\mathbf{p}(t) = \sum_{i=0}^{n} \mathbf{c}_i B_i^n(u), \qquad \mathbf{c}_i \in \mathbb{R}^d,$$

where $u = (t-a)/(b-a)$. We will now establish the remarkable fact:

**Theorem 1** *If* $\mathbf{P}$ *is any* $n$*-variate polar form of the polynomial* $\mathbf{p}$ *then*

$$\mathbf{c}_i = \mathbf{P}(\underbrace{a, \ldots, a}_{n-i}, \underbrace{b, \ldots, b}_{i}).$$

*Proof.* For each $r = 0, 1, \ldots, n$ and $i = 0, 1, \ldots, n-r$, let

$$\mathbf{c}_i^r = \mathbf{P}(\underbrace{a, \ldots, a}_{n-r-i}, \underbrace{b, \ldots, b}_{i}, \underbrace{t, \ldots, t}_{r}).$$

Then by the symmetry and multi-affine properties of $\mathbf{P}$ and the fact that

$$t = (1-u)a + ub,$$

it follows that

$$\mathbf{c}_i^r = (1-u)\mathbf{c}_i^{r-1} + u\mathbf{c}_{i+1}^{r-1}. \tag{3}$$

Therefore the points $\mathbf{c}_i^r$ satisfy de Casteljau's algorithm and

$$\mathbf{c}_0^n = \sum_{i=0}^{n} \mathbf{c}_i^0 B_i^n(u).$$

So by the diagonal property of $\mathbf{P}$,

$$\mathbf{p}(t) = \sum_{i=0}^{n} \mathbf{c}_i^0 B_i^n(u).$$

Since this holds for all $t$ and the $B_i^n$ are linearly independent, $\mathbf{c}_i^0 = \mathbf{c}_i$. □

In fact the $n$-variate polar form is unique. To see this suppose that $\mathbf{P}$ and $\mathbf{Q}$ are $n$-variate polar forms of $\mathbf{p}$, and let $t_1, \ldots, t_n \in \mathbb{R}$. For each $r = 0, 1, \ldots, n$ and $i = 0, 1, \ldots, n - r$, let

$$\mathbf{c}_i^r = \mathbf{P}(\underbrace{a, \ldots, a}_{n-r-i}, \underbrace{b, \ldots, b}_{i}, t_1, \ldots, t_r),$$

and

$$\mathbf{d}_i^r = \mathbf{Q}(\underbrace{a, \ldots, a}_{n-r-i}, \underbrace{b, \ldots, b}_{i}, t_1, \ldots, t_r).$$

By the symmetry and multi-affine properties of $\mathbf{P}$ and $\mathbf{Q}$ and since

$$t_r = (1 - u_r)a + u_r b,$$

where $u_r = (t_r - a)/(b - a)$, we have the recursions

$$\mathbf{c}_i^r = (1 - u_r)\mathbf{c}_i^{r-1} + u_r \mathbf{c}_{i+1}^{r-1},$$

and

$$\mathbf{d}_i^r = (1 - u_r)\mathbf{d}_i^{r-1} + u_r \mathbf{d}_{i+1}^{r-1}.$$

By the theorem, $\mathbf{c}_i^0 = \mathbf{d}_i^0 = \mathbf{c}_i$, and so by induction on $r$, $\mathbf{c}_i^r = \mathbf{d}_i^r$. Hence, $\mathbf{c}_0^n = \mathbf{d}_0^n$, or equivalently,

$$\mathbf{P}(t_1, \ldots, t_n) = \mathbf{Q}(t_1, \ldots, t_n).$$

# 3 Subdivision

One application of the polar form is to subdivision. Consider the two outer diagonal rows of points in the de Casteljau algorithm (3),

$$\mathbf{c}_0^0, \mathbf{c}_0^1, \ldots, \mathbf{c}_0^n, \qquad \text{and} \qquad \mathbf{c}_0^n, \mathbf{c}_1^{n-1}, \ldots, \mathbf{c}_n^0.$$

By the theorem, these can be expressed in terms of $\mathbf{P}$ as

$$\mathbf{c}_0^i = \mathbf{P}(\underbrace{a, \ldots, a}_{n-i}, \underbrace{t, \ldots, t}_{i}), \qquad \text{and} \qquad \mathbf{c}_i^{n-i} = \mathbf{P}(\underbrace{b, \ldots, b}_{i}, \underbrace{t, \ldots, t}_{n-i}).$$

Therefore, again by the theorem, but applied to the sub-interval $[a, t]$, we see that the $\mathbf{c}_0^i$ are the control points of $\mathbf{p}$ when represented as a Bézier curve on $[a, t]$, i.e.,

$$\mathbf{p}(s) = \sum_{i=0}^{n} \mathbf{c}_0^i B_i^n(v),$$

where $v = (s-a)/(t-a)$, and similarly, the $\mathbf{c}_i^{n-i}$ are the control points of $\mathbf{p}$ represented as a Bézier curve on the sub-interval $[t,b]$,

$$\mathbf{p}(s) = \sum_{i=0}^{n} \mathbf{c}_i^{n-i} B_i^n(w),$$

where $w = (s-t)/(b-t)$.

The computation of the control points over the two intervals $[a,t]$ and $[t,b]$ is known as *subdivision*. On subdividing $\mathbf{p}$ repeatedly, one can obtain the Bézier control points of $\mathbf{p}$ on any number of adjacent sub-intervals $[a_0,a_1]$, $[a_1,a_2]$, …, $[a_{k-1},a_k]$, with $a_0 = a$ and $a_k = b$. Together, these polygons form a composite polygon of $\mathbf{p}$ with $kn+1$ vertices. As the sub-intervals get smaller, the composite polygon tends to get closer to $\mathbf{p}$. This offers an alternative way of *plotting* the Bézier curve $\mathbf{p}$. The obvious way of plotting $\mathbf{p}$ is to sample it at several parameter values $t_j \in [a,b]$ and plot the polygon formed by the points $\mathbf{p}(t_j)$. We can instead subdivide $\mathbf{p}$ a few times and plot its composite control polygon.

# 4 Joining curves together

Another application of the polar form is the derivation of the continuity conditions for joining Bézier curves together smoothly. Earlier we derived the conditions for $C^r$ continuity of orders $r = 0,1,2$ from the formulas for endpoint derivatives. We now use the polar form to obtain the general $C^r$ continuity conditions.

Consider first the situation that both $\mathbf{p}$ and $\mathbf{q}$ are Bézier curves defined on the *same* interval $[a,b]$;

$$\mathbf{p}(t) = \sum_{i=0}^{n} \mathbf{c}_i B_i^n(u), \qquad \mathbf{q}(t) = \sum_{i=0}^{n} \mathbf{d}_i B_i^n(u),$$

where $u = (t-a)/(b-a)$. We showed earlier that the $k$-th derivative of $\mathbf{p}$ at the end point $t = a$ is

$$\mathbf{p}^{(k)}(a) = \frac{n!}{(n-k)!(b-a)^k} \Delta^k \mathbf{c}_0.$$

It follows from this that $\mathbf{p}$ and $\mathbf{q}$ have the same derivatives at $t = a$ of orders $k = 0,1,\ldots,r$ if and only if

$$\mathbf{d}_i = \mathbf{c}_i, \qquad i = 0,1,\ldots,r.$$

Suppose then that $\mathbf{p}$ is as above but that $\mathbf{q}$ is a Bézier curve defined on an *adjacent* interval $[b, c]$,

$$\mathbf{q}(t) = \sum_{i=0}^{n} \mathbf{d}_i B_i^n(v),$$

where $v = (t-b)/(c-b)$. The condition for $C^r$ continuity between $\mathbf{p}$ and $\mathbf{q}$ at $t = b$ is therefore

$$\mathbf{d}_i = \mathbf{e}_i, \qquad i = 0, 1, \ldots, r,$$

where $\mathbf{e}_0, \ldots, \mathbf{e}_n$ are the control points of $\mathbf{p}$ with respect to the parameter interval $[b, c]$, i.e., such that

$$\mathbf{p}(t) = \sum_{i=0}^{n} \mathbf{e}_i B_i^n(v).$$

The points $\mathbf{e}_i$ can be computed from de Casteljau's algorithm applied to $\mathbf{p}$ on $[a, b]$ with $t = c$, so the condition reduces to

$$\mathbf{d}_i = \sum_{j=0}^{i} \mathbf{c}_{n-i+j} B_j^i(\mu), \qquad i = 0, 1, \ldots, r,$$

where $\mu = (c-a)/(b-a)$. This agrees with what we derived earlier for $r \leq 2$.

# 5 Triangular Bézier surfaces

As for polynomials of one variable, there is a polar form for polynomials of several variables. Consider the bivariate polynomial,

$$p(x, y) = \sum_{i+j \leq n} a_{i,j} x^i y^j. \tag{4}$$

We can view it as a function of the single variable $\mathbf{x} = (x, y) \in \mathbb{R}^2$. There is a unique $n$-variate polar form $P$ for $p$ in which each variable is also a point in $\mathbb{R}^2$. The polar form, $P(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$, $\mathbf{x}_k \in \mathbb{R}^2$, is, as before, uniquely defined by three properties: that it is (i) symmetric, (ii) multi-affine, and (iii) agrees with $p(\mathbf{x})$ on its diagonal. The polar form can be obtained from $p$ by replacing the term $x^i y^j$ in (4) by the expression

$$\sum x_{k_1} x_{k_2} \cdots x_{k_i} y_{\ell_1} y_{\ell_2} \cdots y_{\ell_j} \Big/ \left( \frac{n!}{i!j!(n-i-j)!} \right). \tag{5}$$

Here the sum is over all pairs of sequences

$$1 \le k_1 < k_2 < \cdots < k_i \le n, \qquad 1 \le \ell_1 < \ell_2 < \cdots < \ell_j \le n,$$

whose elements are pairwise distinct, i.e., such that $\ell_\beta \ne k_\alpha$. For example, the quadratic

$$p(\mathbf{x}) = a_0 + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2$$

has the bivariate polar form

$$P(\mathbf{x}_1, \mathbf{x}_2) = a_0 + a_{10}\frac{x_1 + x_2}{2} + a_{01}\frac{y_1 + y_2}{2} + a_{20}x_1x_2 + a_{11}\frac{x_1y_2 + x_2y_1}{2} + a_{02}y_1y_2.$$

The monomials $p(\mathbf{x}) = xy$ and $p(\mathbf{x}) = x^2y$ have the trivariate polar forms

$$P(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \frac{1}{6}(x_1y_2 + x_1y_3 + x_2y_1 + x_2y_3 + x_3y_1 + x_3y_2),$$

and

$$P(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \frac{1}{3}(x_1x_2y_3 + x_1x_3y_2 + x_2x_3y_1),$$

respectively. As in the univariate case, the symmetry and multi-affine properties of $P$ are easily verified from (5), and the diagonal property holds because $n!/(i!j!(n-i-j)!)$ is the number of terms in the sum.

Suppose now that $\mathbf{p}$ is a triangular Bézier surface in $\mathbb{R}^d$,

$$\mathbf{p}(\mathbf{t}) = \sum_{|\mathbf{i}|=n} \mathbf{c}_\mathbf{i} B^n_\mathbf{i}(\mathbf{u}), \qquad \mathbf{c}_\mathbf{i} \in \mathbb{R}^d,$$

over the triangular parameter domain $T = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3] \subset \mathbb{R}^2$, where $\mathbf{i} = (i, j, k)$, $|\mathbf{i}| = i+j+k$, and $\mathbf{u} = (u_1, u_2, u_3)$ is the set of barycentric coordinates of $\mathbf{t}$ with respect to $T$.

In analogy to the previous theorem, one can show that if $\mathbf{P}$ is the polar form of $\mathbf{p}$, then

$$\mathbf{c}_\mathbf{i} = \mathbf{P}(\underbrace{\mathbf{a}_1, \ldots, \mathbf{a}_1}_{i}, \underbrace{\mathbf{a}_2, \ldots, \mathbf{a}_2}_{j}, \underbrace{\mathbf{a}_3, \ldots, \mathbf{a}_3}_{k}).$$

This can be done by defining the points

$$\mathbf{c}^r_\mathbf{i} = \mathbf{P}(\underbrace{\mathbf{a}_1, \ldots, \mathbf{a}_1}_{i} \underbrace{\mathbf{a}_2, \ldots, \mathbf{a}_2}_{j}, \underbrace{\mathbf{a}_3, \ldots, \mathbf{a}_3}_{k}, \underbrace{\mathbf{t}, \ldots, \mathbf{t}}_{r}), \qquad |\mathbf{i}| = n - r, \qquad (6)$$

and showing that these are the points in the de Casteljau algorithm for finding $\mathbf{p}(t)$. One can further demonstrate the uniqueness of $\mathbf{P}$ by replacing the $\mathbf{t}$'s in (6) by $\mathbf{t}_1, \ldots, \mathbf{t}_r$.

In analogy to the univariate case, since

$$\mathbf{c}^k_{i,j,0} = \mathbf{P}(\underbrace{\mathbf{a}_1, \ldots, \mathbf{a}_1}_{i} \underbrace{\mathbf{a}_2, \ldots, \mathbf{a}_2}_{j}, \underbrace{\mathbf{t}, \ldots, \mathbf{t}}_{k}), \qquad i + j + k = n,$$

these are the control points of the representation of $\mathbf{p}$ on the sub-triangle $[\mathbf{a}_1, \mathbf{a}_2, \mathbf{t}]$.

Finally, consider the conditions for two triangular Bézier surfaces to join with $C^r$ continuity. Suppose that $\mathbf{a}_4 \in \mathbb{R}^2$ is any point on the side of the edge $[\mathbf{a}_1, \mathbf{a}_2]$ opposite to $\mathbf{a}_3$, and let $\tilde{\mathbf{p}}$ be a Bézier surface on the adjacent triangle $U = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4]$,

$$\tilde{\mathbf{p}}(\mathbf{t}) = \sum_{|\mathbf{i}|=n} \tilde{\mathbf{c}}_{\mathbf{i}} B^n_{\mathbf{i}}(\tilde{\mathbf{u}}),$$

where $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_4)$ and the values $\tilde{u}_1, \tilde{u}_2, \tilde{u}_4$ are the barycentric coordinates of $\mathbf{t}$ with respect to $U$.

In analogy to the univariate case, the condition for $\mathbf{p}$ and $\tilde{\mathbf{p}}$ to join with $C^r$ continuity on the common edge $[\mathbf{a}_1, \mathbf{a}_2]$ is that

$$\tilde{\mathbf{c}}_{i,j,k} = \mathbf{e}_{i,j,k}, \qquad k = 0, 1, \ldots, r, \quad i + j = n - k,$$

where the points $\mathbf{e}_{i,j,k}$ are the control points of $\mathbf{p}$ when represented as a Bézier surface over $U$. These points can be computed from the de Casteljau algorithm for $\mathbf{p}$ on $T$ with $\mathbf{t} = \mathbf{a}_4$, and the $C^r$ condition reduces to

$$\tilde{\mathbf{c}}_{i,j,k} = \sum_{\alpha+\beta+\gamma=k} \mathbf{c}_{i+\alpha,j+\beta,\gamma} B^k_{\alpha,\beta,\gamma}(\boldsymbol{\mu}), \qquad k = 0, 1, \ldots, r, \quad i + j = n - k,$$

where $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$ is the set of barycentric coordinates of $\mathbf{a}_4$ with respect to $T$.