Spline curves

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In this lecture we introduce spline curves and study some of their basic properties.

1 Spline curves

For any integers $d \geq 0$ and $n \geq 1$, we call a sequence $(t_1, t_2, \ldots, t_{n+d+1})$, $t_i \in \mathbb{R}$, a *knot vector* if $t_i \leq t_{i+1}$ and $t_i < t_{i+d+1}$. Such a sequence of knots together with a sequence of *control points* $\mathbf{c}_i \in \mathbb{R}^m$, $i = 1, \ldots, n$, define a *spline curve*

$$\mathbf{s}(t) = \sum_{i=1}^{n} \mathbf{c}_i N_i^d(t), \qquad t \in \mathbb{R}, \tag{1}$$

where the functions N_i^d are *B-splines*. These B-splines can be defined recursively:

$$N_i^0(t) = \begin{cases} 1 & t \in [t_i, t_{i+1}); \\ 0 & \text{otherwise}, \end{cases} \tag{2}$$

and for $d \geq 1$,

$$N_i^d(t) = \frac{t - t_i}{t_{i+d} - t_i} N_i^{d-1}(t) + \frac{t_{i+d+1} - t}{t_{i+d+1} - t_{i+1}} N_{i+1}^{d-1}(t). \tag{3}$$

We use the convention here that

$$N_i^{r-1} = \frac{N_i^{r-1}}{t_{i+r} - t_i} = 0, \qquad \text{if } t_{i+r} = t_i.$$

From this recursion it follows that N_i^d is a piecewise polynomial of degree d, which is positive in (t_i, t_{i+d+1}) and zero outside $[t_i, t_{i+d+1}]$.

2 Evaluation

Similar to Bezier curves, there are two ways of evaluating a spline curve. One way is to use the recursion (3) and then the formula (1). Suppose $t \in [t_k, t_{k+1})$. Then,

$$\mathbf{s}(t) = \sum_{i=k-d}^{k} \mathbf{c}_i N_i^d(t),$$

and we only need to compute $N_{k-d}^d(t), \ldots, N_k^d(t)$, for all the other B-splines are zero in $[t_k, t_{k+1})$. The recursion (3) can then be carried out in a triangular scheme,

$$\begin{array}{ccccccc}
1 & = & N_k^0 & N_{k-1}^1 & N_{k-2}^2 & \cdots & N_{k-d}^d \\
 & & & N_k^1 & N_{k-1}^2 & \cdots & N_{k-d+1}^d \\
 & & & & N_k^2 & \cdots & N_{k-d+2}^d \\
 & & & & & \ddots & \vdots \\
 & & & & & & N_k^d
\end{array}$$

Alternatively, one can use a more direct recursion algorithm. Let $\mathbf{c}_i^0 = \mathbf{c}_i$, $i = k-d, \ldots, k$. Then for $r = 1, \ldots, d$, and $i = k-d+r, \ldots, k$, let

$$\mathbf{c}_i^r = \frac{t_{i+d+1-r} - t}{t_{i+d+1-r} - t_i} \mathbf{c}_{i-1}^{r-1} + \frac{t - t_i}{t_{i+d+1-r} - t_i} \mathbf{c}_i^{r-1}. \qquad (4)$$

One can show that the last point computed is the point on the curve, $\mathbf{c}_k^d = \mathbf{s}(t)$. Similar to the de Casteljau algorithm, this can be shown by showing, more generally, by induction on r, that

$$\mathbf{s}(t) = \sum_{i=k-d+r}^{k} \mathbf{c}_i^r N_i^{d-r}(t). \qquad (5)$$

This algorithm can also be arranged in a triangular scheme, here row-wise,

$$\begin{array}{ccccccccc}
\mathbf{c}_{k-d}^0 & & \mathbf{c}_{k-d+1}^0 & & \mathbf{c}_{k-d+2}^0 & & \cdots & & \mathbf{c}_k^0 \\
 & \mathbf{c}_{k-d+1}^1 & & \mathbf{c}_{k-d+2}^1 & & \cdots & & \mathbf{c}_k^1 & \\
 & & \ddots & & & & \iddots & & \\
 & & & \mathbf{c}_{k-1}^{d-1} & & \mathbf{c}_k^{d-1} & & & \\
 & & & & \mathbf{c}_k^d & & & &
\end{array}$$

3 Control points from the polar form

In analogy to Bezier curves we can express control points of spline curves in terms of polar forms. Recall that the d-variate polar form $\mathcal{P}[p](x_1, \dots, x_d)$ of the polynomial

$$p(x) = \sum_{i=0}^{d} a_i x^i, \qquad a_i \in \mathbb{R}, \tag{6}$$

is

$$\mathcal{P}[p](x_1, \dots, x_d) = \sum_{i=0}^{d} a_i S_i(x_1, \dots, x_d),$$

where S_i is the symmetric polynomial

$$S_i(x_1, \dots, x_d) = \sum_{1 \le k_1 < k_2 < \cdots < k_i \le d} x_{k_1} x_{k_2} \cdots x_{k_i} \Big/ \binom{d}{i}. \tag{7}$$

Consider again the spline curve $\mathbf{s}$ restricted to some non-empty interval $[t_k, t_{k+1})$. In this interval $\mathbf{s}$ is a polynomial which we can denote by $\mathbf{s}_k$,

$$\mathbf{s}_k(t) = \sum_{i=k-d}^{k} \mathbf{c}_i N_i^d(t), \qquad t \in [t_k, t_{k+1}).$$

Theorem 1 *For $i = k-d, \dots, k$,*

$$\mathbf{c}_i = \mathcal{P}[\mathbf{s}_k](t_{i+1}, \dots, t_{i+d}).$$

To prove this let

$$\mathbf{c}_i^r = \mathcal{P}[\mathbf{s}_k](t_{i+1}, \dots, t_{i+d-r}, \underbrace{t, \dots, t}_{r}).$$

Since $\mathcal{P}[\mathbf{s}_k]$ is multi-affine and symmetric, and since

$$t = (1-\alpha) t_i + \alpha t_{i+d-r+1},$$

where

$$\alpha = \frac{t - t_i}{t_{i+d-r+1} - t_i},$$

it follows that $\mathbf{c}_i^r$ satisfies the recursion (4). Therefore,

$$\mathbf{c}_k^d = \sum_{i=k-d}^{k} \mathbf{c}_i^0 N_i^d(t),$$

and so, by the diagonal property of $\mathcal{P}[\mathbf{s}_k]$,

$$\mathbf{s}_k(t) = \sum_{i=k-d}^{k} \mathbf{c}_i^0 N_i^d(t) = \sum_{i=k-d}^{k} \mathcal{P}[\mathbf{s}_k](t_{i+1}, \ldots, t_{i+d}) N_i^d(t).$$

Moreover, this equation also shows that any polynomial of degree $\leq d$ in the interval $[t_k, t_{k+1})$ can be expressed as a linear combination of the polynomials $N_{k-d}^d, \ldots, N_k^d$, and since there are $d+1$ of these, it follows that $N_{k-d}^d, \ldots, N_k^d$, when restricted to $[t_k, t_{k+1})$, form a basis for π_d. Hence the theorem follows.

4 Derivatives and smoothness

Some simple calculations show that from (7),

$$S_i(x_1, \ldots, x_{d-1}, b) - S_i(x_1, \ldots, x_{d-1}, a) = (b-a)\frac{i}{d} S_{i-1}(x_1, \ldots, x_{d-1}),$$

and therefore, since

$$p'(x) = \sum_{i=1}^{d} i a_i x^{i-1}, \tag{8}$$

we deduce that

$$\mathcal{P}[p](x_1, \ldots, x_{d-1}, b) - \mathcal{P}[p](x_1, \ldots, x_{d-1}, a) = (b-a)\frac{1}{d} \mathcal{P}[p'](x_1, \ldots, x_{d-1}),$$

which gives a formula for the polar form of the first derivative p' in terms of the polar form of p, for any $b \neq a$.

Consider again the first derivative of the spline segment $\mathbf{s}_k$. Since it is a polynomial of degree $\leq d-1$, there must be coefficients $\mathbf{d}_{k-d+1}, \ldots, \mathbf{d}_k$ such that

$$\mathbf{s}_k'(t) = \sum_{i=k-d+1}^{k} \mathbf{d}_i N_i^{d-1}(t).$$

We can now use the polar form to determine these coefficients,

$$\mathbf{d}_i = \mathcal{P}[\mathbf{s}'_k](t_{i+1}, \ldots, t_{i+d-1}).$$

Using the formula above with $a = t_i$ and $b = t_{i+d}$, it follows that

$$\mathbf{d}_i = \frac{d}{t_{i+d} - t_i}(\mathbf{c}_i - \mathbf{c}_{i-1}).$$

Since these coefficients are independent of k, it follows that

$$\mathbf{s}'(t) = \sum_{i=k-d+1}^{k} \mathbf{d}_i N_i^{d-1}(t), \qquad t \in \mathbb{R}.$$

We can continue to differentiate in this way, and thus express the higher derivatives of $\mathbf{s}$ as splines of lower degree.

Consider now the smoothness of $\mathbf{s}$. Suppose first that

$$t_i < t_{i+1} = \cdots = t_{i+d} < t_{i+d+1},$$

in which case we say that

$$z := t_{i+1} = \cdots = t_{i+d}$$

is a *d-fold knot*, or that the knot z has *multiplicity d*. We can use polar forms to show that $\mathbf{s}$ is continuous at z. Consider the control point $\mathbf{c}_i$. From the theorem, we can express it in terms of the polar form of either the segment $\mathbf{s}_i$ or the adjacent segment $\mathbf{s}_{i+d}$, and we find

$$\mathbf{c}_i = \mathcal{P}[\mathbf{s}_i](t_{i+1}, \ldots, t_{i+d}) = \mathcal{P}[\mathbf{s}_i](\underbrace{z, \ldots, z}_{d}) = \mathbf{s}_i(z),$$

and

$$\mathbf{c}_i = \mathcal{P}[\mathbf{s}_{i+d}](t_{i+1}, \ldots, t_{i+d}) = \mathcal{P}[\mathbf{s}_{i+d}](\underbrace{z, \ldots, z}_{d}) = \mathbf{s}_{i+d}(z),$$

and therefore $\mathbf{s}$ is indeed continuous at z.

It follows that the $(d-r)$-th derivative of $\mathbf{s}$, being a linear combination of the B-splines N_i^r, is continuous at an r-fold knot. Thus, $\mathbf{s}$ has smoothness $d - r$ at a knot of multiplicity r. In particular, at a simple knot, i.e., at a knot with multiplicity 1, $\mathbf{s}$ has smoothness C^{d-1}.