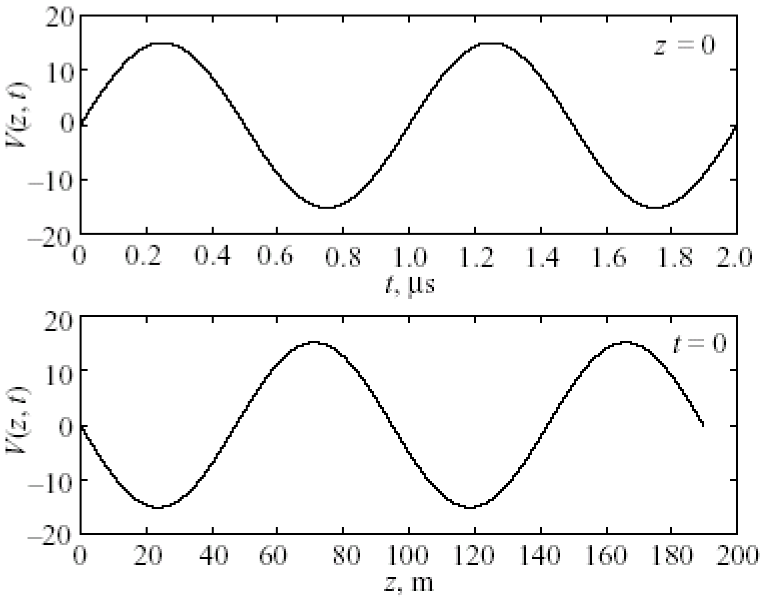
Ch. 2. Transmission Line Analysis



Traveling voltage wave

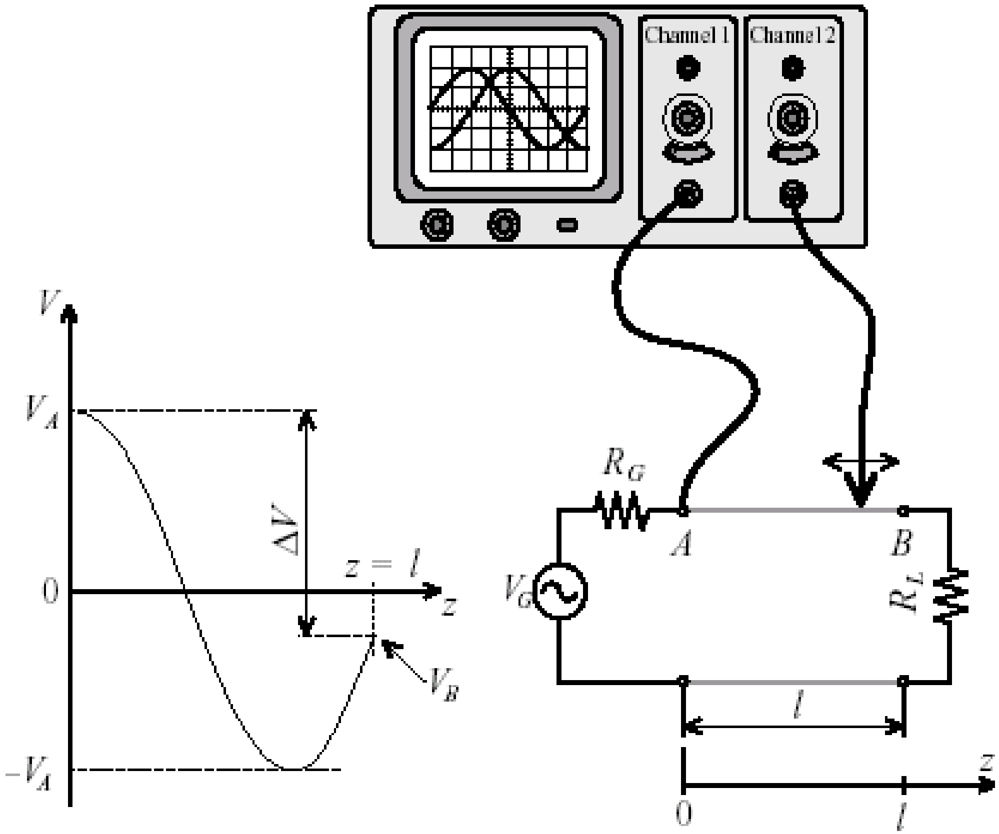
$$V(z,t) = \frac{E_{0x}}{\beta}\sin(\omega t - \beta z)$$

Phase velocity

$$v_p = \omega/\beta = 1/\sqrt{\mu\varepsilon} = c/\sqrt{\varepsilon_r}$$

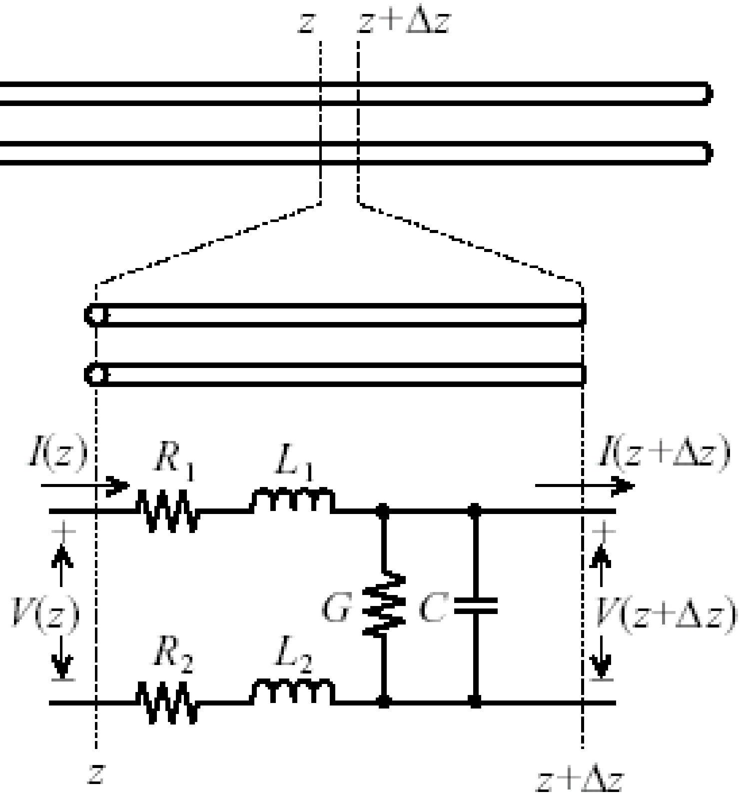
- Voltage has a time and space variation
- Space is neglected for low frequency applications
- For RF there can be a large spatial variation

Consequences of spatial voltage variations



- For low frequency (1MHz) Kirchhoff's laws apply
- For high frequency (1GHz) Kirchhoff's laws do **not apply** anymore
- **Solution:** Consider elements of infinitesimal length

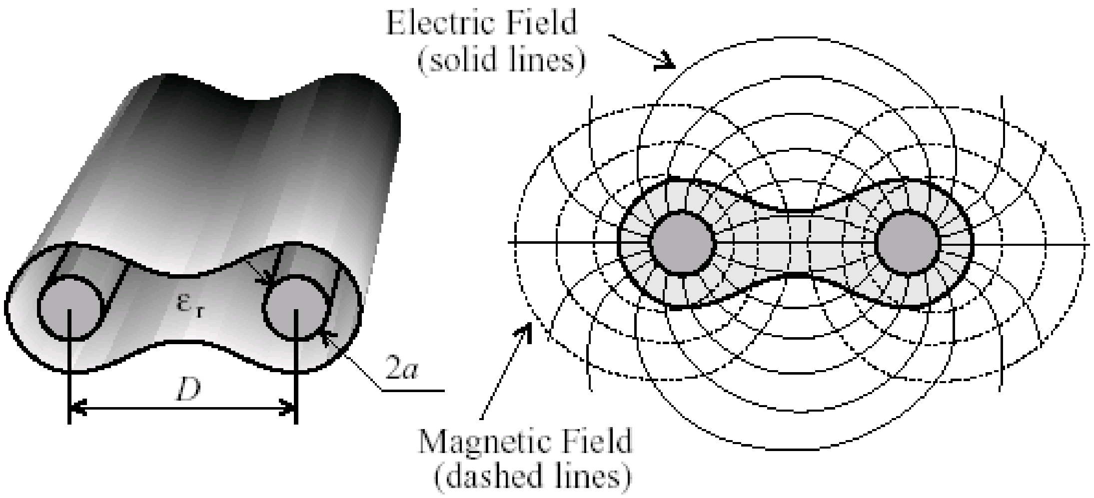
Kirchhoff's laws on a microscopic level



Distributed parameters R, G, L and C

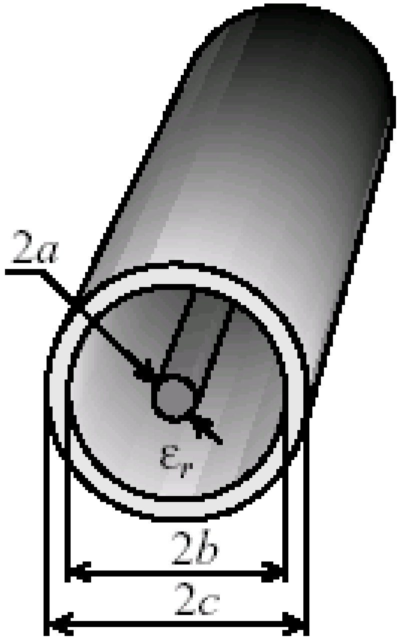
- Over a differential section we can again use basic circuit theory
- Model takes into account line losses and dielectric losses
- Ideal line (lossless) involves only L and C

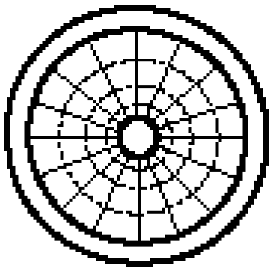
Two-wire transmission line



- Alternating electric field between conductors
- Alternating magnetic field surrounding conductors
- dielectric medium tends to confine field inside material

Coaxial line

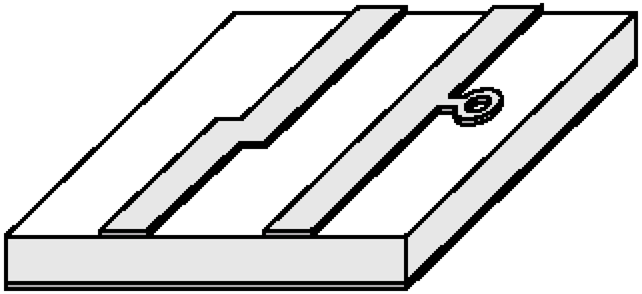




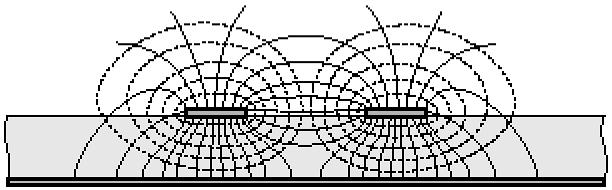
- Electric field contained between conductors
- Perfect shielding of magnetic field
- TEM mode up to a certain cutoff frequency

Always used for externally connected RF systems or measuring equipment. Also LAN.

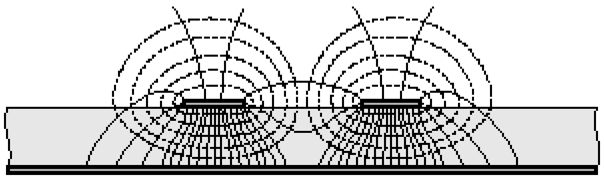
Microstrip lines



Printed circuit board (PCB) section with ground plane to prevent excessive field leakage, interference, and radiation loss



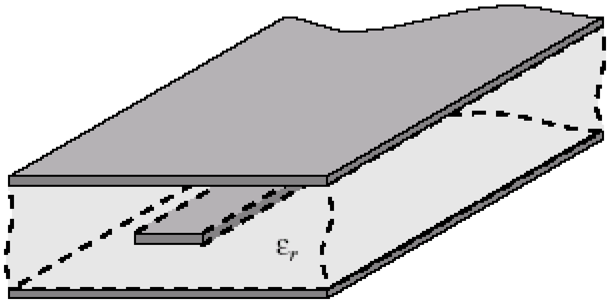
(a) Teflon epoxy ($\varepsilon_r = 2.55$)

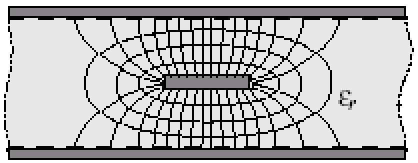


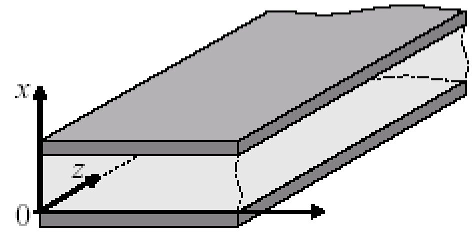
(b) Alumina ($\varepsilon_r = 10.0$)

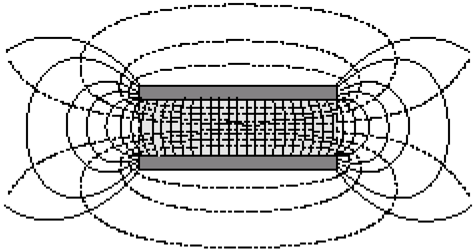
Low dielectric medium High dielectric medium

Other TEM configurations









Triple-layer line

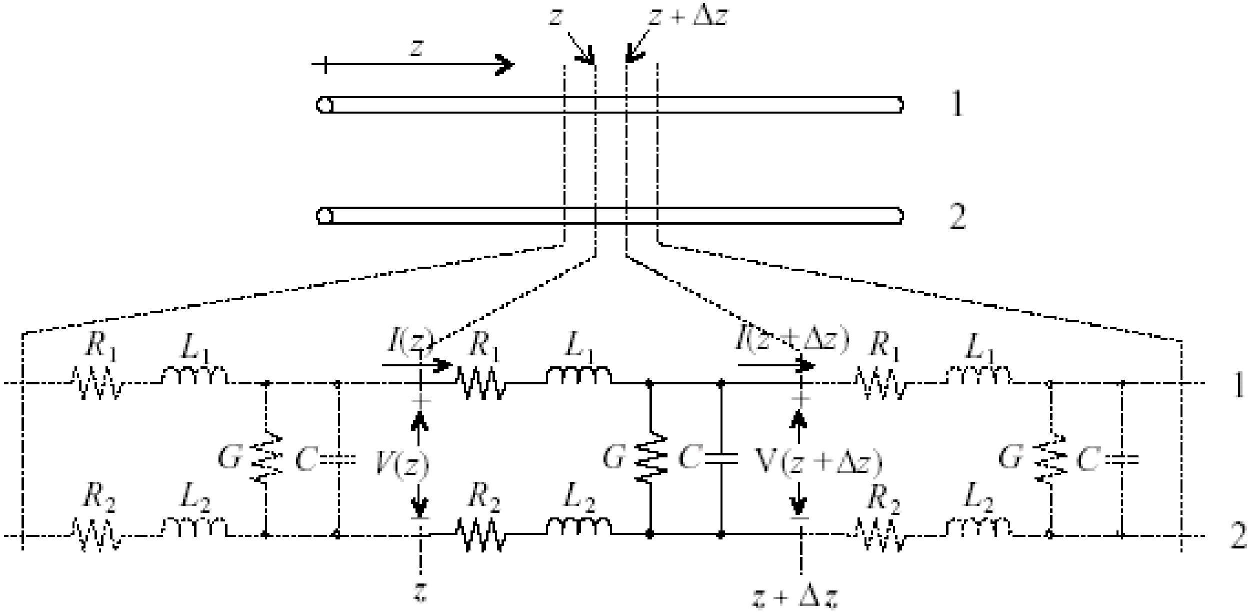
Reduced radiation losses

Parallel plate line

Low impedance, high power

Transmission line representation

- Detailed analysis is based on differential section



- Analysis applies to many types of transmission lines such as coax cables, two-wire, microstrip, etc.

Pros and cons of electric circuit representation

- Clear intuitive physical picture
- Yields a standardized two-port network representation
- Serves as building blocks to go from microscopic to macroscopic forms

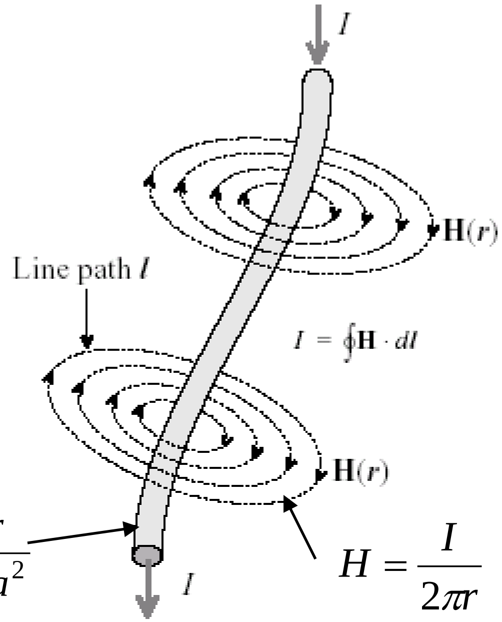
- Basically a one-dimensional representation (cannot take into account interferences)
- Material nonlinearities, hysteresis, and temperature effects are not accounted for

Basic electromagnetism

- Ampère's law

$$\oint \mathbf{H} \cdot d\mathbf{l} = \iint \mathbf{J} \cdot d\mathbf{S}$$

$$\Rightarrow \quad \nabla \times \mathbf{H} = \mathbf{J}$$

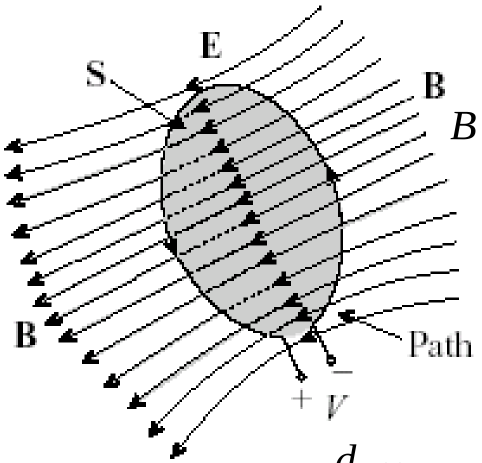


$$H = \frac{Ir}{2\pi a^2} \qquad H = \frac{I}{2\pi r}$$

- Faraday's law

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \iint \mathbf{B} \cdot d\mathbf{S}$$

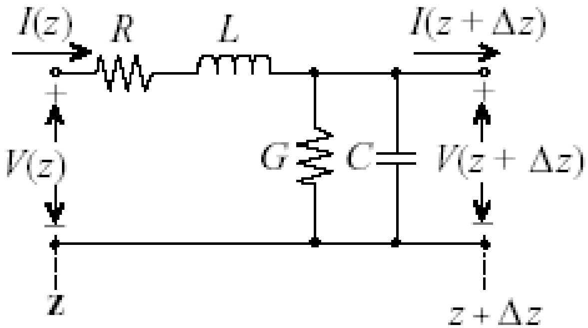
$$\Rightarrow \quad \nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt}$$



$$B = \mu_0 H_0 \cos(\omega t)$$

$$V = \frac{d}{dt} \iint \mathbf{B} d\mathbf{S} = -\pi a^2 \omega \mu_0 H_0 \sin(\omega t)$$

Line parameters for specific cases



Generic electric equivalent circuit representation

Parameter	Two-Wire Line	Coaxial Line	Parallel-Plate Line
R Ω/m	$\frac{1}{\pi a \sigma_{\text{cond}} \delta}$	$\frac{1}{2\pi \sigma_{\text{cond}} \delta}\left(\frac{1}{a}+\frac{1}{b}\right)$	$\frac{2}{w \sigma_{\text{cond}} \delta}$
L $\mathrm{H/m}$	$\frac{\mu}{\pi} \cosh^{-1}\left(\frac{D}{2a}\right)$	$\frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right)$	$\mu \frac{d}{w}$
G $\mathrm{S/m}$	$\frac{\pi \sigma_{\text{diel}}}{\cosh^{-1}(D/(2a))}$	$\frac{2\pi \sigma_{\text{diel}}}{\ln(b/a)}$	$\sigma_{\text{diel}} \frac{w}{d}$
C $\mathrm{F/m}$	$\frac{\pi \varepsilon}{\cosh^{-1}(D/(2a))}$	$\frac{2\pi \varepsilon}{\ln(b/a)}$	$\varepsilon \frac{w}{d}$

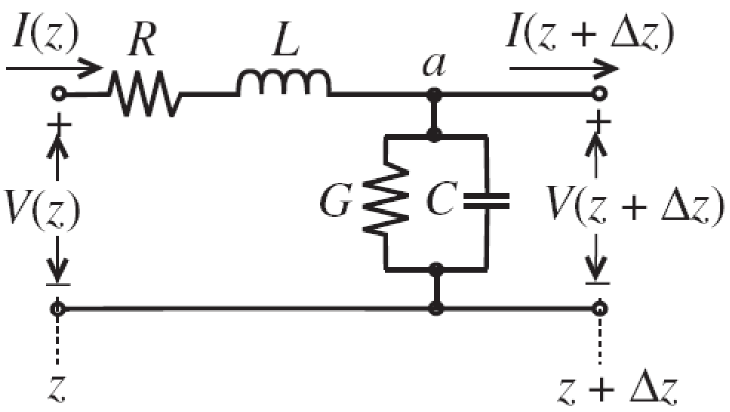
Check out example 2.3 in text book to get feeling of numbers!

General transmission line equation

KVL:

$$V(z) = (R + j\omega L)I(z)\Delta z + V(z+\Delta z)$$

$$\xrightarrow[\Delta z \to 0]{} \quad \frac{dV(z)}{dz} = -(R + j\omega L)I(z)$$



KCL:

$$I(z) = V(z+\Delta z)(G + j\omega C)\Delta z + I(z+\Delta z)$$

$$\xrightarrow[\Delta z \to 0]{} \quad \frac{dI(z)}{dz} = -(G + j\omega C)V(z)$$

$\Rightarrow$ Coupled first-order differntial equations

Traveling voltage and current waves

$$\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0 \qquad \frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0$$

Complex propagation constant:

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

Solutions:

$$V(z) = V^+ e^{-\gamma z} + V^- e^{+\gamma z} \qquad I(z) = I^+ e^{-\gamma z} + I^- e^{+\gamma z}$$

General line impedance definition

$$\frac{dV(z)}{dz} = -(R + j\omega L)I(z) \quad \Rightarrow \quad I(z) = \frac{\gamma}{R + j\omega L}\left(V^+ e^{-\gamma z} - V^- e^{+\gamma z}\right)$$

Characteristic line impedance:

$$Z_0 \equiv \frac{R + j\omega L}{\gamma} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \frac{V^+}{I^+} = -\frac{V^-}{I^-}$$

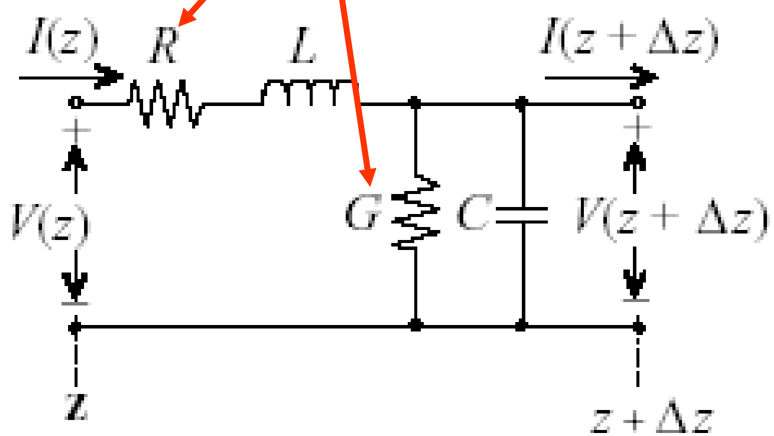
$$\Rightarrow \quad I(z) = \frac{1}{Z_0}\left(V^+ e^{-\gamma z} - V^- e^{+\gamma z}\right)$$

Note: Z_0 is not a conventional impedance, but is a characteristic of the positive and negative traveling waves

Lossless transmission line

Lossless implies:

R = 0   and G = 0



Consider lossless parallell plate transmission line with:

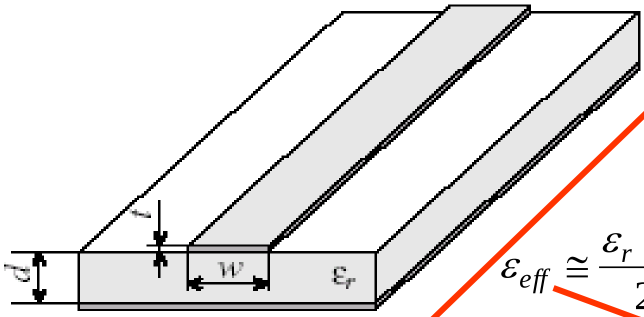
$$L = \mu \frac{d}{w} \quad \text{and} \quad C = \varepsilon \frac{w}{d}$$

Characteristic impedance:

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{\varepsilon}} \frac{d}{w} \cong \frac{377\Omega}{\sqrt{\varepsilon_r}} \frac{d}{w}$$

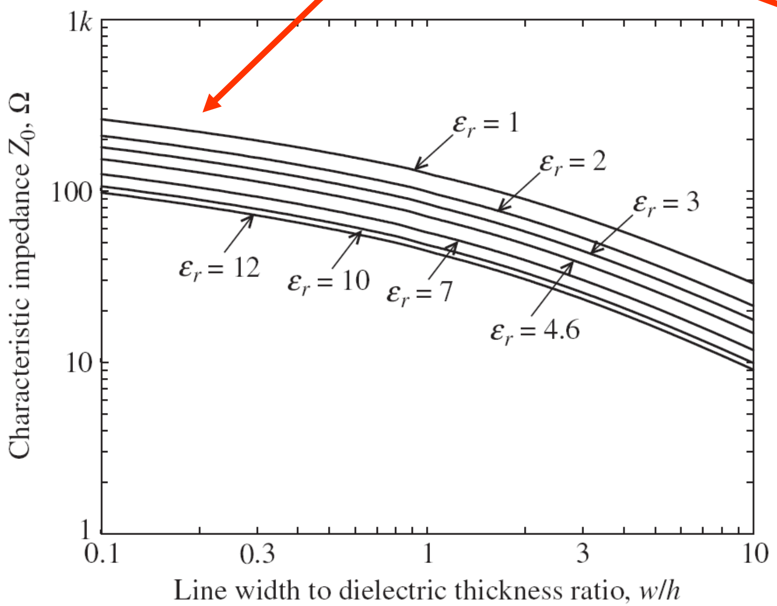
$Z_f = \sqrt{\mu_0 / \varepsilon_0} \cong 377\Omega$ is the wave impedance of free space

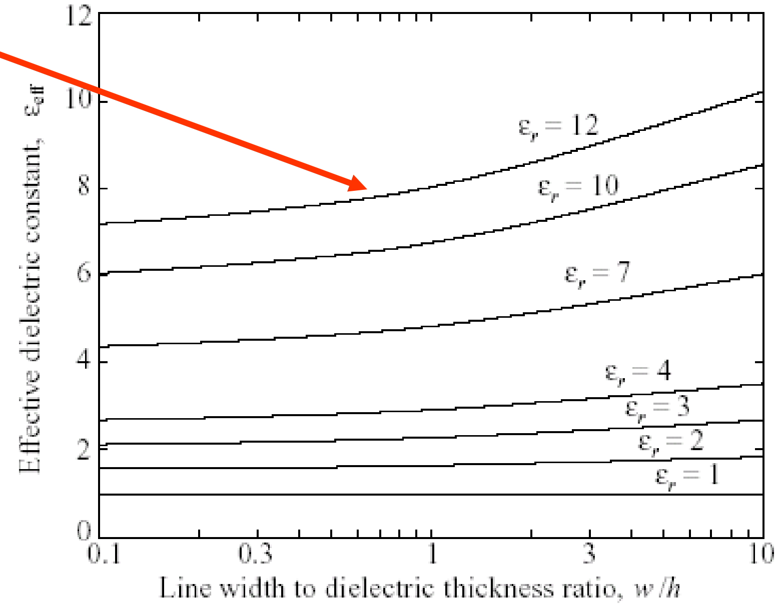
Microstrip transmission line



$$Z_0 \cong \frac{Z_f}{2\pi\sqrt{\varepsilon_{ff}}}\ln\left(\frac{8h}{w}+\frac{w}{4h}\right)\Bigg|_{w<h} \quad or \quad \frac{Z_f}{\sqrt{\varepsilon_{ff}}\left(1.393+\frac{w}{h}+\frac{2}{3}\ln\left(1.444+\frac{w}{h}\right)\right)}\Bigg|_{w>h}$$

$$\varepsilon_{eff} \cong \frac{\varepsilon_r+1}{2}+\frac{\varepsilon_r-1}{2}\left[\left(1+\frac{12h}{w}\right)^{-\frac{1}{2}}+0.04\left(1-\frac{w}{h}\right)^2\right]\Bigg|_{w<h} \quad or \quad \frac{\varepsilon_r+1}{2}+\frac{\varepsilon_r-1}{2}\left(1+\frac{12h}{w}\right)^{-\frac{1}{2}}\Bigg|_{w>h}$$

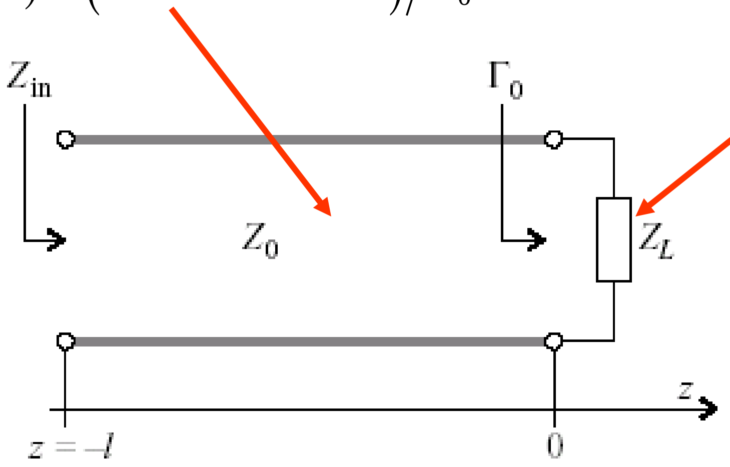




Teminated lines - Voltage reflection coefficient

$$V(z) = V^{+}e^{-\gamma z} + V^{-}e^{+\gamma z}$$

$$I(z) = \left(V^{+}e^{-\gamma z} - V^{-}e^{+\gamma z}\right)/Z_0$$



Load impedance:

$$Z(0) = Z_L = \frac{V(0)}{I(0)} = Z_0\frac{V^{+} + V^{-}}{V^{+} - V^{-}} = Z_0\frac{1+\Gamma_0}{1-\Gamma_0}$$

Reflection coefficient:

$$\Gamma_0 \equiv \frac{V^{-}}{V^{+}} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Open line: $Z_L \rightarrow \infty$, $\Gamma_0 = 1$ Wave fully reflected with same polarity as incident wave

Short circuit: $Z_L = 0$, $\Gamma_0 = -1$ Wave fully reflected with opposite polarity of incident wave

Load match: $Z_L = Z_0$, $\Gamma_0 = 0$ No reflection when load matches line impedance

Lossless transmission line

For lossless line ($R = G = 0$):

$$\gamma = \alpha + j\beta = j\omega\sqrt{LC}$$

$$\alpha = 0 \qquad \beta = \omega\sqrt{LC}$$

Voltage and current waves:

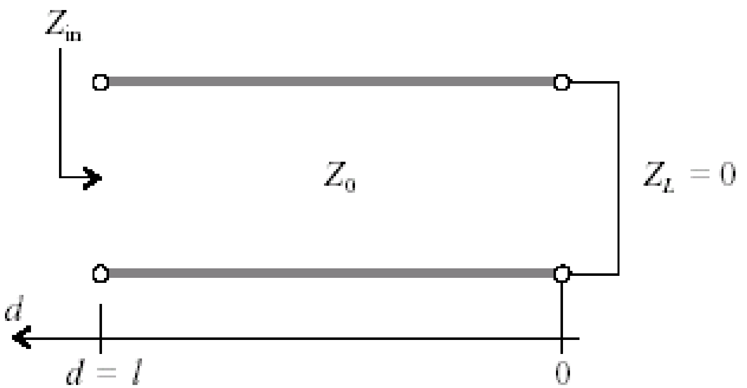
$$V(z) = V^{+}\left(e^{-j\beta z} + \Gamma_0 e^{+j\beta z}\right) \qquad I(z) = \frac{V^{+}}{Z_0}\left(e^{-j\beta z} - \Gamma_0 e^{+j\beta z}\right)$$

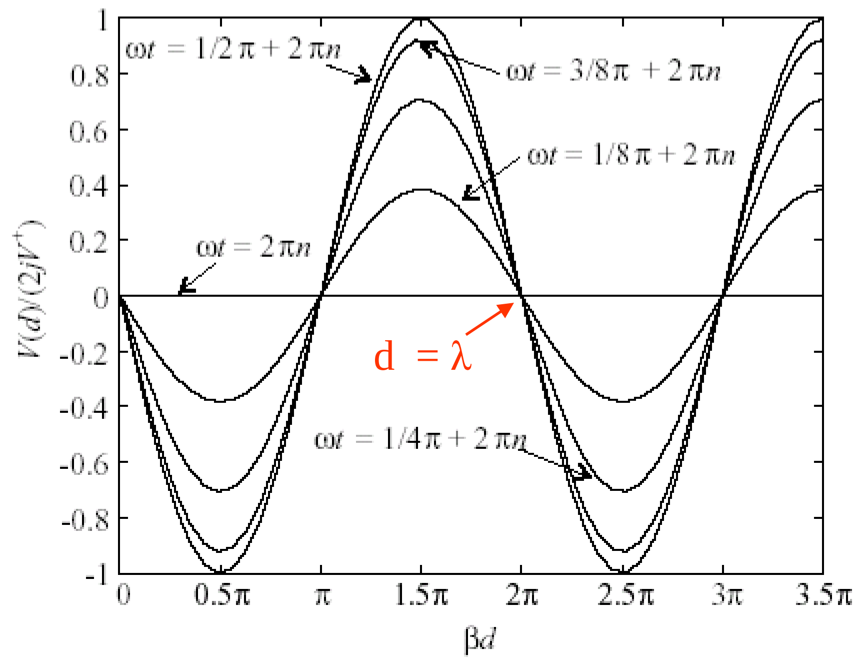
Phase velocity:

$$v_p = \lambda f = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$

Standing waves

Short circuit: $Z_L = 0$, $\Gamma_0 = -1$
Wave fully reflected with opposite polarity of incident wave





Note: d = -z

⇒ Standing wave pattern:

$$V(d) = V^+\left(e^{+j\beta d} - e^{-j\beta d}\right) = 2jV^+ \sin(\beta d)$$

$$v(d,t) = \mathrm{Re}\left\{Ve^{j\omega t}\right\} = 2V^+ \sin(\beta d)\cos(\omega t + \pi/2)$$

Standing wave ratio (SWR)

Generally: $V(d) = V^{+} e^{+j\beta d}\left(1 + \Gamma_0 e^{-2j\beta d}\right) = A(d)[1 + \Gamma(d)]$

$$I(d) = \frac{A(d)}{Z_0}[1 - \Gamma(d)]$$

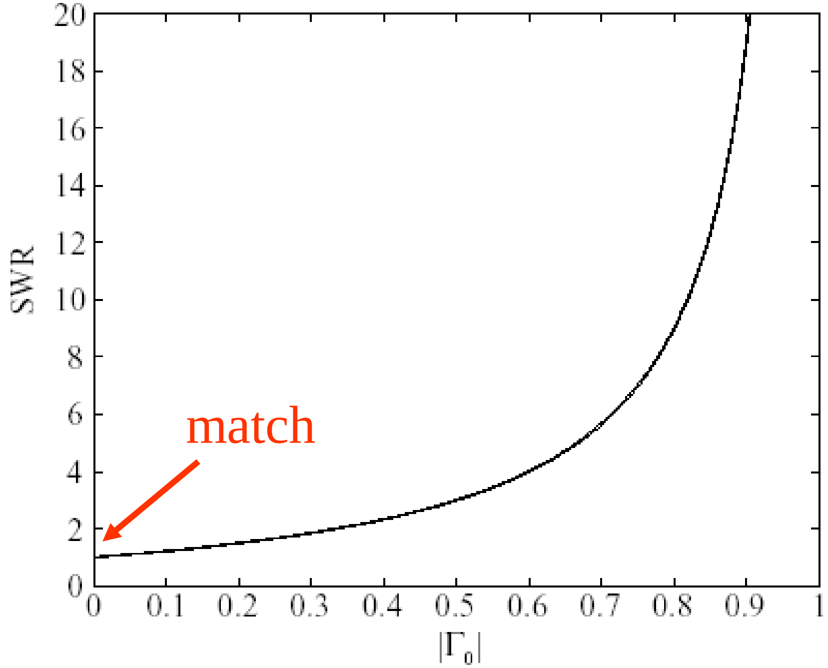
Reflection coefficient: $\Gamma(d) = \Gamma_0 e^{-2j\beta d}$

$$SWR = \frac{|V_{\max}|}{|V_{\min}|} = \frac{|I_{\max}|}{|I_{\min}|} = \frac{1 + |\Gamma_0|}{1 - |\Gamma_0|}$$

SWR is a measure of mismatch of the load to the line

SWR=1 (matched)

SWR $\rightarrow \infty$ (total mismatch)



Note: SRW applies to lossless lines, but also works well in low-loss cases

Transformation of load impedance

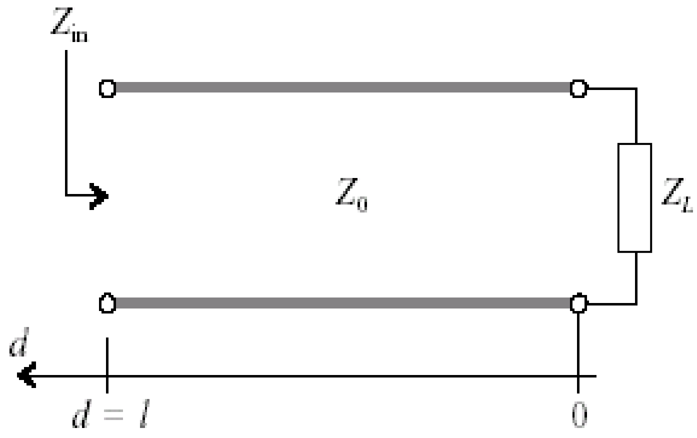
- Terminated lossless transmission line ($Z_0 = \sqrt{L/C}$)

- Input impedance:

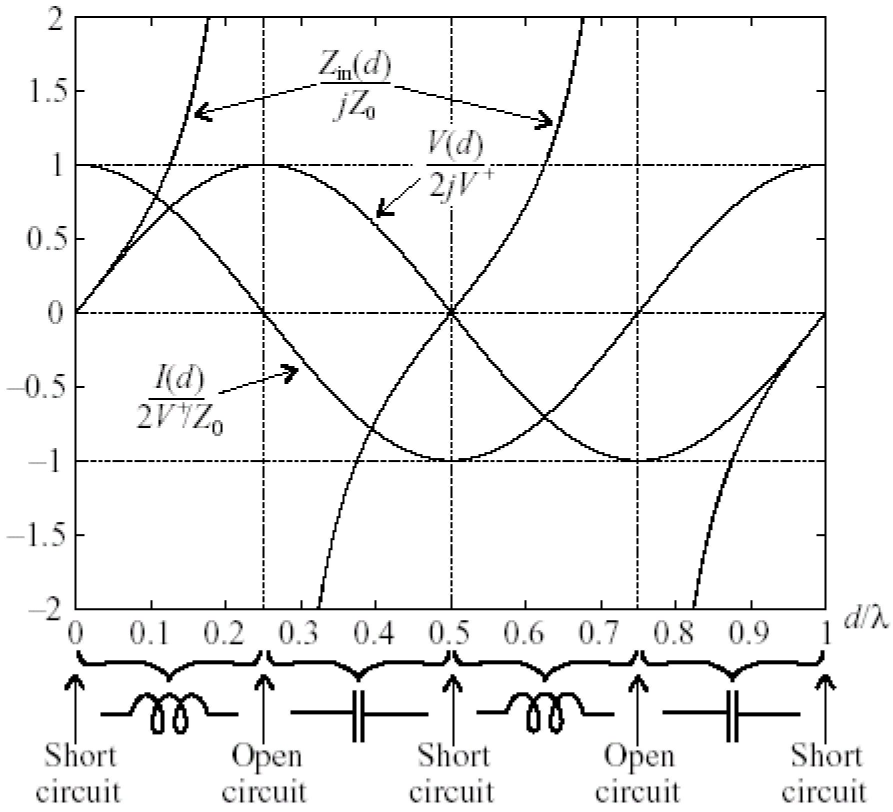
$$Z_{in}(d=l) = \frac{V(l)}{I(l)} = Z_0 \frac{e^{j\beta l} + \Gamma_0 e^{-j\beta l}}{e^{j\beta l} - \Gamma_0 e^{-j\beta l}} = Z_0 \frac{(Z_L + Z_0)e^{j\beta l} + (Z_L - Z_0)e^{-j\beta l}}{(Z_L + Z_0)e^{j\beta l} - (Z_L - Z_0)e^{-j\beta l}}$$

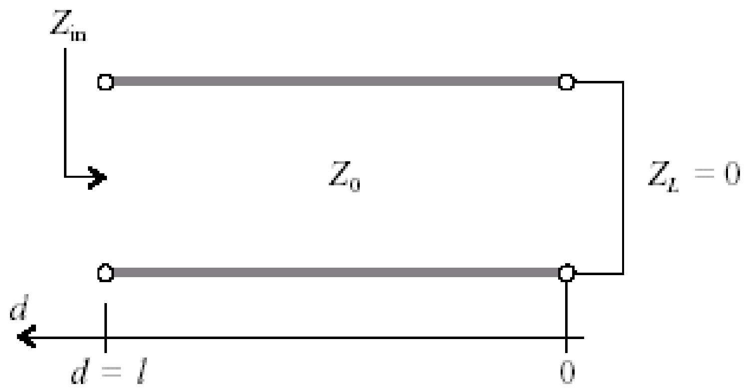
- Used: $\Gamma_0 \equiv \frac{V^-}{V^+} = \frac{Z_L - Z_0}{Z_L + Z_0}$

$$\Rightarrow \boxed{Z_{in}(d) = Z_0 \frac{Z_L + jZ_0 \tan(\beta d)}{Z_0 + jZ_L \tan(\beta d)}}$$



Short circuit transmission line



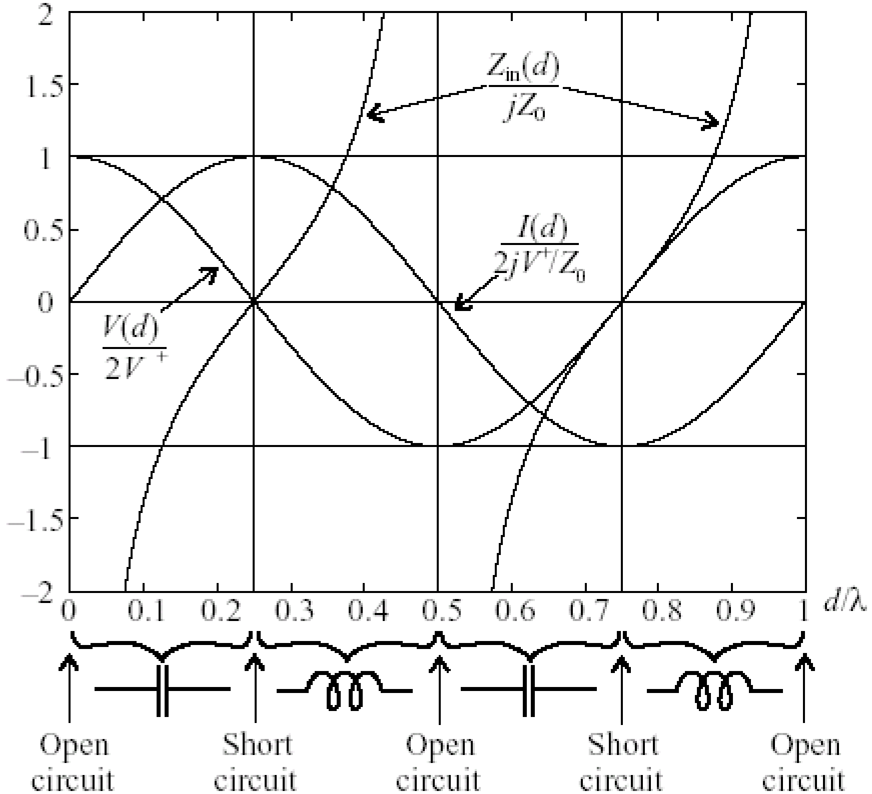


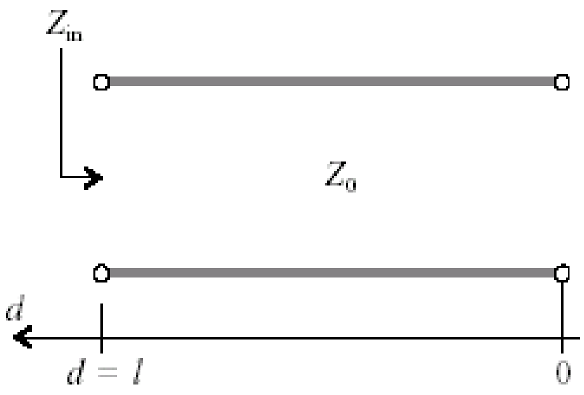
$$Z_{in}(d) = jZ_0 \tan(\beta d)$$

$$V(d) = 2jV^+ \sin(\beta d)$$

$$I(d) = \frac{2V^+}{Z_0}\cos(\beta d)$$

Open circuit transmission line



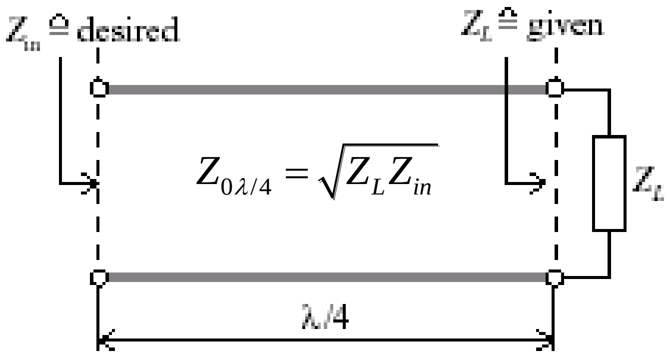


$$Z_{in}(d) = -jZ_0 \cot(\beta d)$$

$$V(d) = 2V^+ \cos(\beta d)$$

$$I(d) = \frac{2jV^+}{Z_0} \sin(\beta d)$$

Quarter-wave transmission line

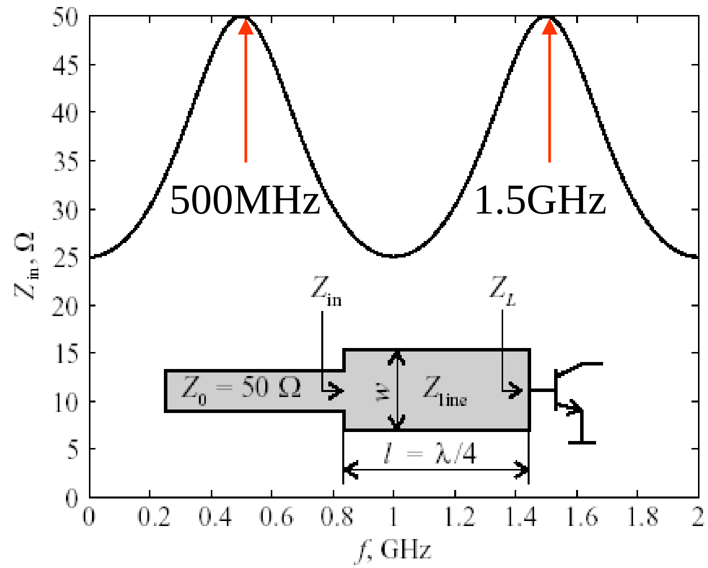


For $d = \lambda/4$ we have:

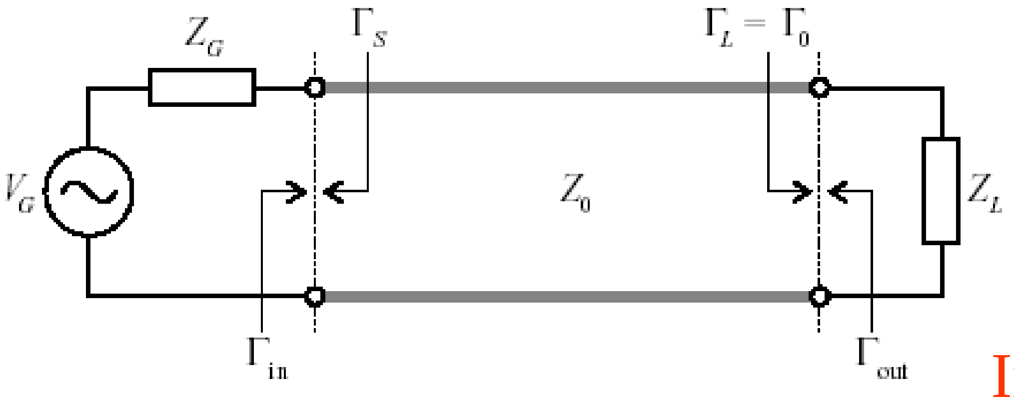
$$Z_{in}\left(d = \frac{\lambda}{4}\right) = Z_{0\lambda/4} \frac{Z_L + jZ_{0\lambda/4} \tan\left(\frac{2\pi}{\lambda}\frac{\lambda}{4}\right)}{Z_{0\lambda/4} + jZ_L \tan\left(\frac{2\pi}{\lambda}\frac{\lambda}{4}\right)} = \frac{Z_{0\lambda/4}^2}{Z_L}$$

Lamda-quarter transformer matches given input and output impedances by choosing a line with characteristic impedance (narrowband matching, $Z_{in} = Z_0$):

$$\boxed{Z_{0\lambda/4} \equiv Z_{line} = \sqrt{Z_L Z_0}}$$



Transmission line with source and load



Also have to consider the impedance matching at the source!

Input voltage ($d = l$):

$$V_{in} = V_{in}^{+} + V_{in}^{-} = V_{in}^{+}\left(1 + \Gamma_{in}\right) = V_G \frac{Z_{in}}{Z_{in} + Z_G}$$

Transmission coefficients:

$$T_{in} = 1 + \Gamma_{in} = \frac{2Z_{in}}{Z_{in} + Z_0}$$

$$T_0 = 1 + \Gamma_0 = \frac{2Z_L}{Z_L + Z_0}$$

Reflection coefficients:

$$\Gamma_{in} = \Gamma(d = l) = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} = \Gamma_0 e^{-2j\beta l}$$

$$\Gamma_S = \frac{Z_G - Z_0}{Z_G + Z_0} \qquad \Gamma_{out} = \Gamma_S e^{-2j\beta l}$$

Input Power

$$P_{in} = \frac{1}{2}\mathrm{Re}\{V_{in} I_{in}^*\} = \frac{1}{2}\frac{|V_{in}^+|^2}{Z_0}\left(1-|\Gamma_{in}|^2\right)$$

$$V_{in}^+ = \frac{V_{in}}{1+\Gamma_{in}} = \frac{V_G}{1+\Gamma_{in}}\frac{Z_{in}}{Z_{in}+Z_G} \qquad Z_{in} = Z_0\frac{1+\Gamma_{in}}{1-\Gamma_{in}} \qquad Z_G = Z_0\frac{1+\Gamma_S}{1-\Gamma_S}$$

$$\boxed{P_{in} = \frac{1}{8}\frac{|V_G|^2}{Z_0}\frac{|1-\Gamma_S|^2}{|1-\Gamma_S\Gamma_{in}|^2}\left(1-|\Gamma_{in}|^2\right)}$$

Lossless TL:

$$\boxed{P_{in} = \frac{1}{8}\frac{|V_G|^2}{Z_0}\frac{|1-\Gamma_S|^2}{|1-\Gamma_S\Gamma_0 e^{-2j\beta l}|^2}\left(1-|\Gamma_0|^2\right)}$$

Lossless TL - special cases

Load and source matched to line:

$$\Gamma_0 = \Gamma_S = 0 \quad \Rightarrow \quad P_{in} = \frac{1}{8}\frac{|V_G|^2}{Z_0}$$

Maximum available power provided by the source

Match at load and mismatch at source:

$$\Gamma_0 = 0 \quad \Rightarrow \quad P_{in} = \frac{1}{8}\frac{|V_G|^2}{Z_0}|1-\Gamma_s|^2$$

Power usually measured in dBm: $P[dBm] = 10\log\left(\frac{P[W]}{1mW}\right)$

Return and insertion losses

$$RL = -10\log\left(\frac{P_r}{P_i}\right) = -10\log|\Gamma_{in}|^2 = -20\log|\Gamma_{in}|$$

$$IL = -10\log\left(\frac{P_t}{P_i}\right) = -10\log\left(\frac{P_i - P_r}{P_i}\right) = -10\log\left(1 - |\Gamma_{in}|^2\right)$$