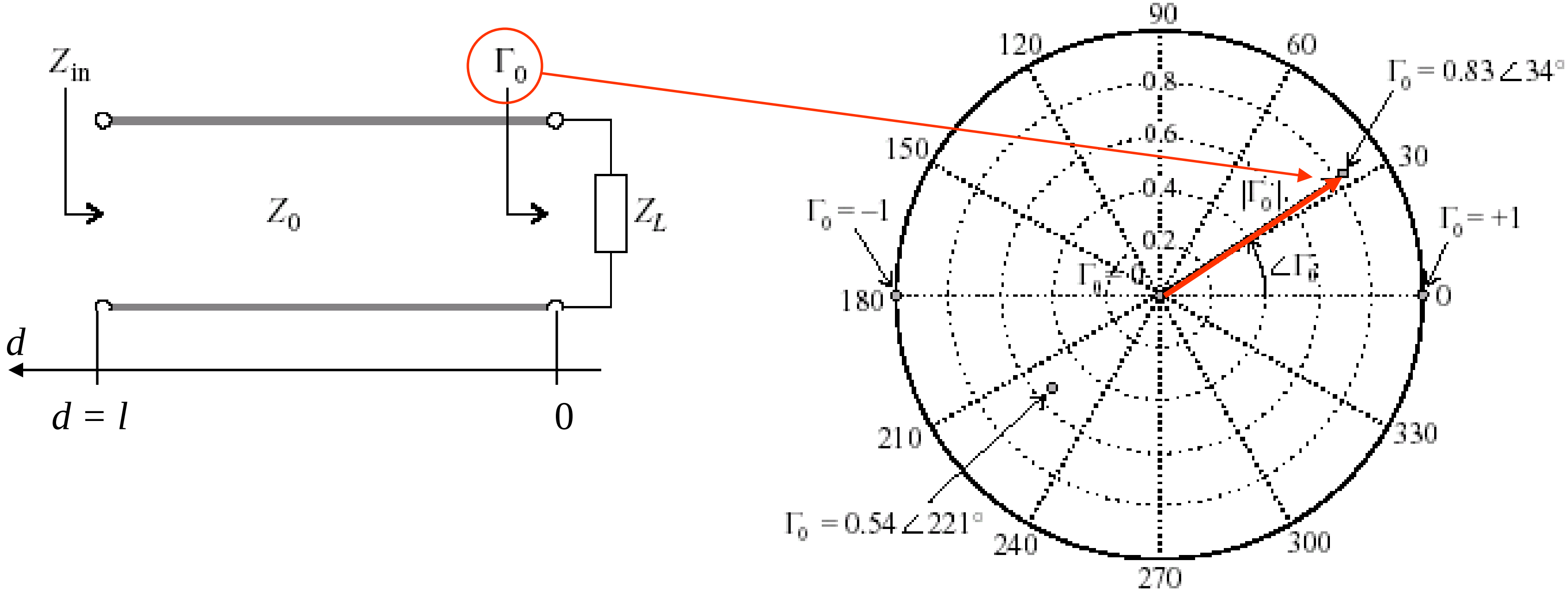
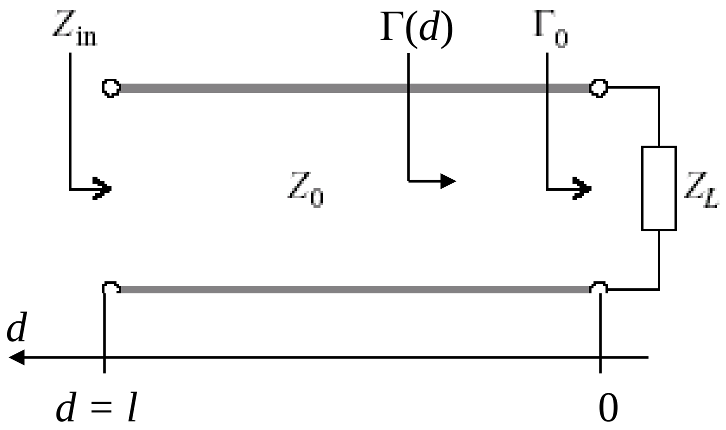
Ch. 3. The Smith Chart



$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_{0r} + j\Gamma_{0i} = |\Gamma_0| e^{j\theta_l}$$

Mapping of the reflection coefficient in the complex domain

Normalized impedance



Generalized reflection coefficient:

$$\Gamma(d) = \Gamma_0 e^{-2j\beta d} = \Gamma_r + j\Gamma_i$$

Normalized impedance:

$$\frac{Z_{in}(d)}{Z_0} = z_{in} = r + jx = \frac{1+\Gamma(d)}{1-\Gamma(d)} = \frac{1+\Gamma_r + j\Gamma_i}{1-\Gamma_r - j\Gamma_i}$$

Real part:

$$r = \frac{1-\Gamma_r^2 - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

$$\Rightarrow \quad \left(\Gamma_r - \frac{r}{r+1}\right)^2 + \Gamma_i^2 = \left(\frac{1}{r+1}\right)^2$$

Imaginary part:

$$x = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

$$\Rightarrow \quad (\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2$$

Representation of normalized resistance

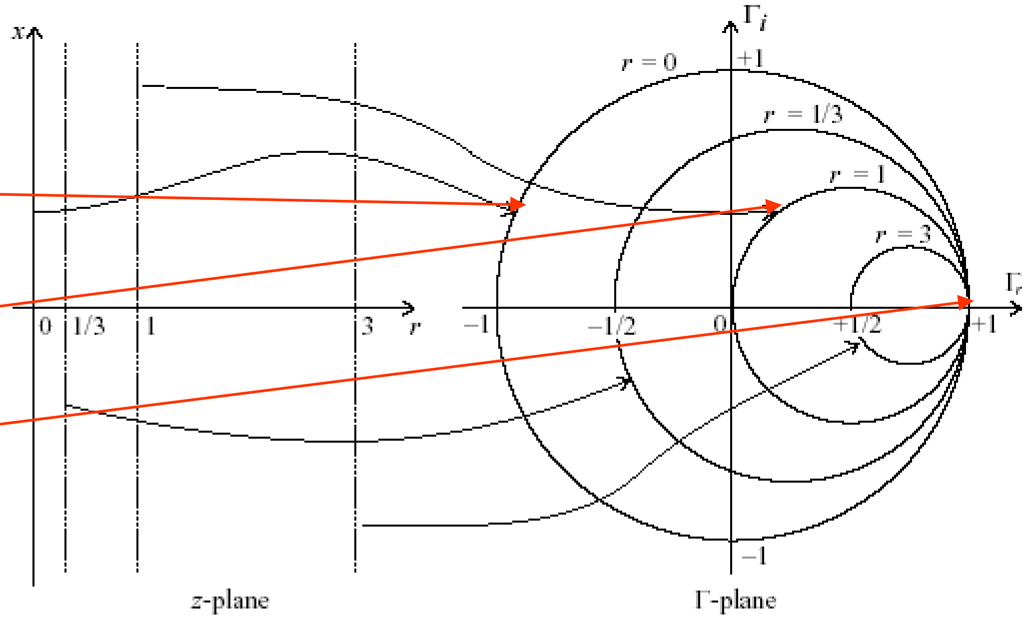
$$\left(\Gamma_r - \frac{r}{r+1}\right)^2 + \Gamma_i^2 = \left(\frac{1}{r+1}\right)^2$$

Examples:

$$r = 0 \;\Rightarrow \Gamma_r^2 + \Gamma_i^2 = 1$$

$$r = 1 \;\Rightarrow \left(\Gamma_r - \frac{1}{2}\right)^2 + \Gamma_i^2 = \left(\frac{1}{2}\right)^2$$

$$r = \infty \Rightarrow (\Gamma_r - 1)^2 + \Gamma_i^2 = 0$$



Representation of normalized reactance

$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2$$

Examples:

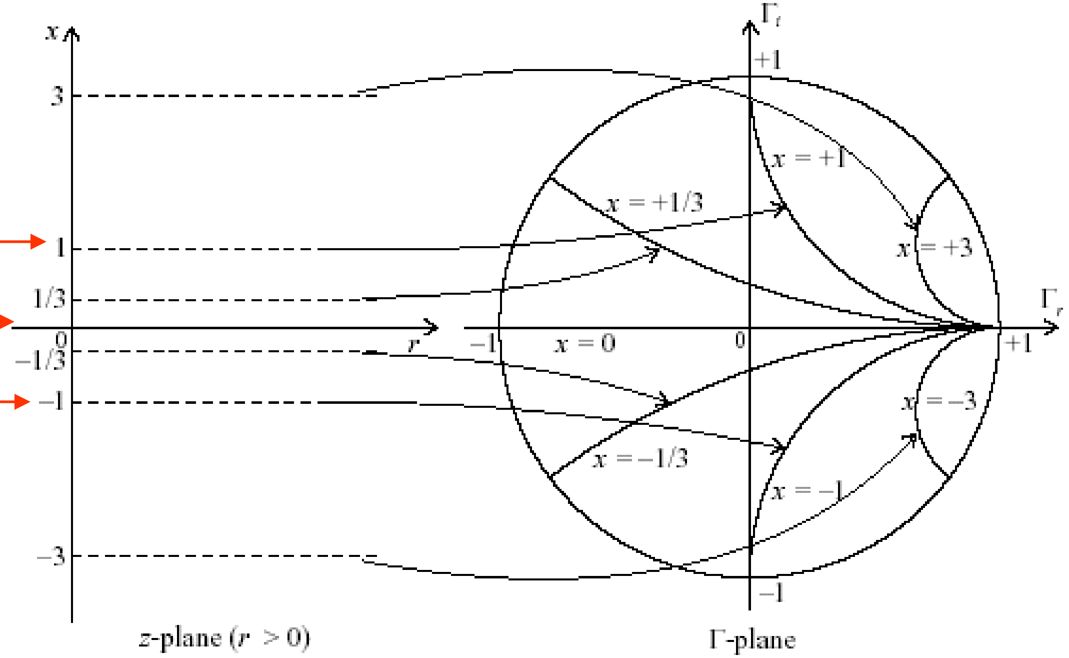
$$x = \infty \Rightarrow (\Gamma_r - 1)^2 + \Gamma_i^{\,2} = 0$$

$$x = 1 \Rightarrow (\Gamma_r - 1)^2 + (\Gamma_i - 1)^2 = 1$$

$$x = 0 \Rightarrow \Gamma_i = 0$$

$$x = -1 \Rightarrow (\Gamma_r - 1)^2 + (\Gamma_i + 1)^2 = 1$$

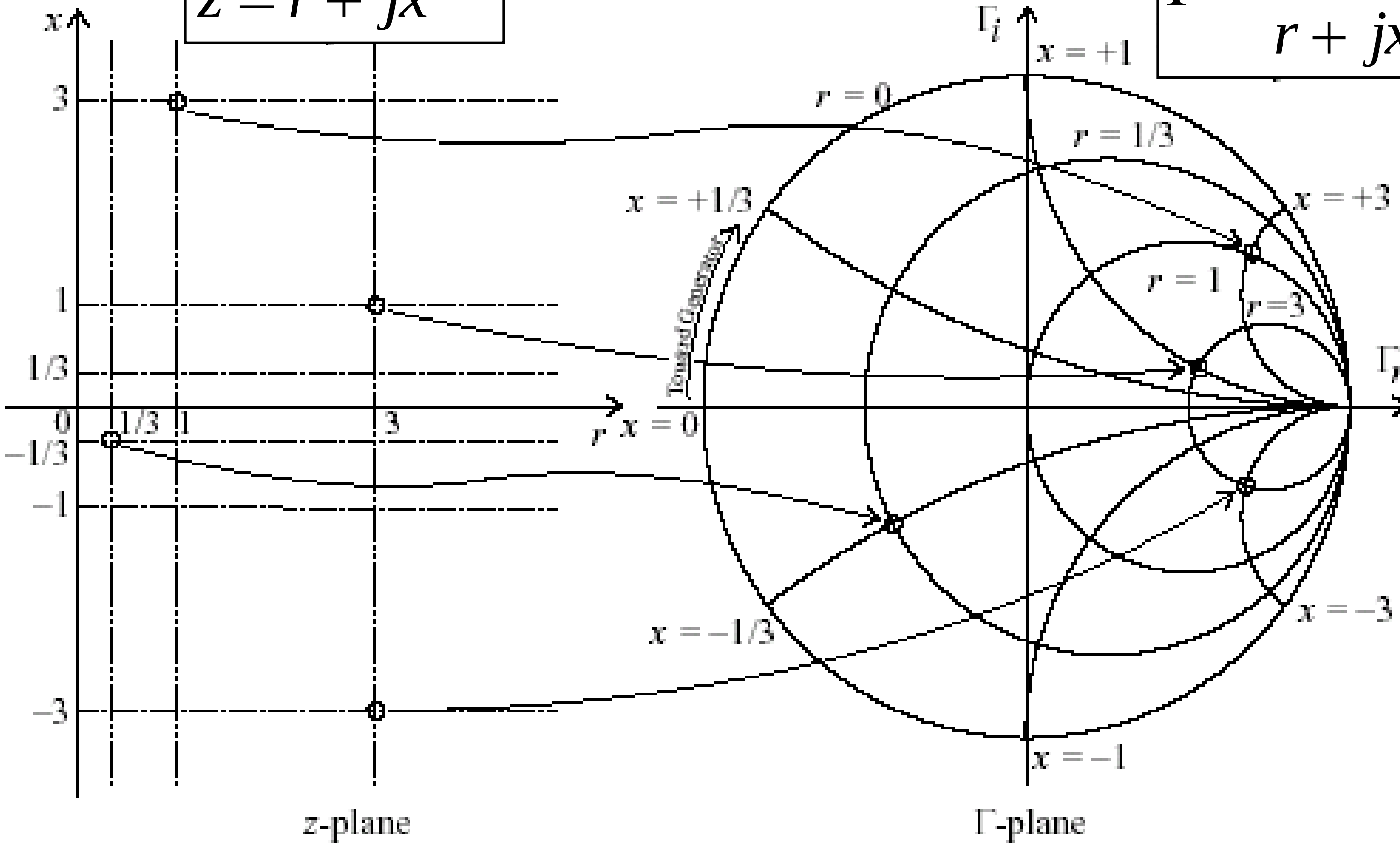
$$x = -\infty \Rightarrow (\Gamma_r - 1)^2 + \Gamma_i^{\,2} = 0$$



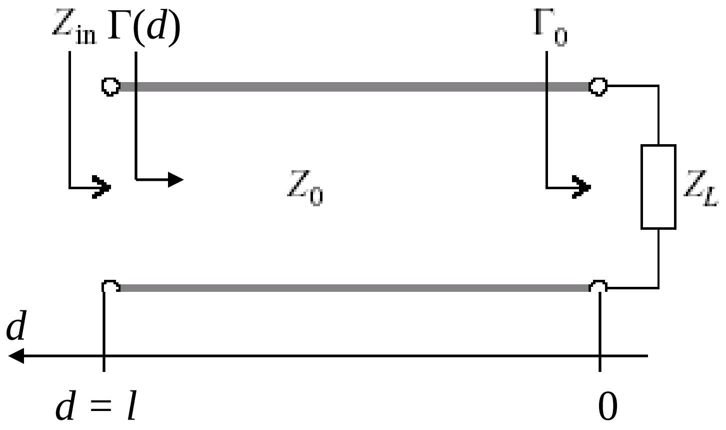
Combined diagram: Smith Chart

$$\Gamma = \frac{r + jx - 1}{r + jx + 1}$$

$$z = r + jx$$



Repetition



Reflection coefficient:

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_{0r} + j\Gamma_{0i} = |\Gamma_0| e^{j\theta_l}$$

Generalized reflection coefficient:

$$\Gamma(d) = \Gamma_0 e^{-2j\beta d} = \Gamma_r + j\Gamma_i$$

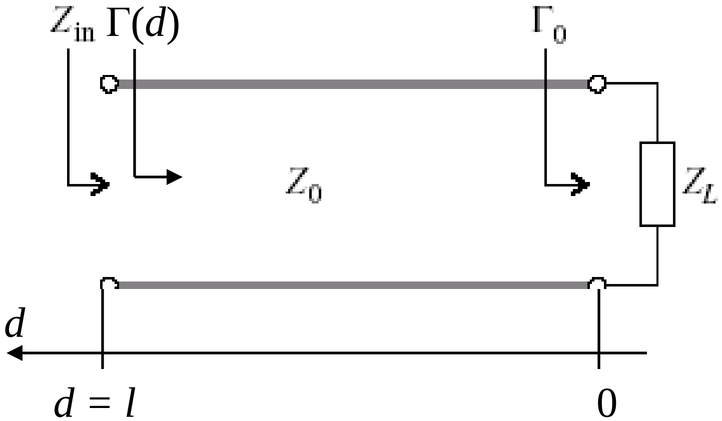
Normalized impedance: $$\frac{Z_{in}(d)}{Z_0} = z_{in} = r + jx = \frac{1+\Gamma(d)}{1-\Gamma(d)} = \frac{1+\Gamma_r + j\Gamma_i}{1-\Gamma_r - j\Gamma_i}$$

Normalized resistance: $$r = \frac{1-\Gamma_r^2 - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

Normalized reactance: $$x = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

How to use the Smith Chart

Example: determine input impedance $Z_{in}(d)$



- Normalize load impedance: $Z_L \rightarrow z_L$
- find reflection coefficient: $z_L \rightarrow \Gamma_0$
- rotate reflection coefficient: $\Gamma_0 \rightarrow \Gamma(d)$
- find normalized input impedance: $z_{in}(d)$
- de-normalize input impedance: $z_{in}(d) \rightarrow Z_{in}(d)$

Impedance transformation using Smith chart

$Z_L=30+j60\ \Omega$
$Z_0=50\ \Omega$
$d=l=2cm$
$f=2GHz \quad v_p=c/2$

1) Normalize load impedance:

$$Z_L \rightarrow z_L = Z_L/Z_0$$

2) Find reflection coefficient:

$$z_L \rightarrow \Gamma_0 = (z_L-1)/(z_L+1)$$

3) Rotate reflection coefficient:

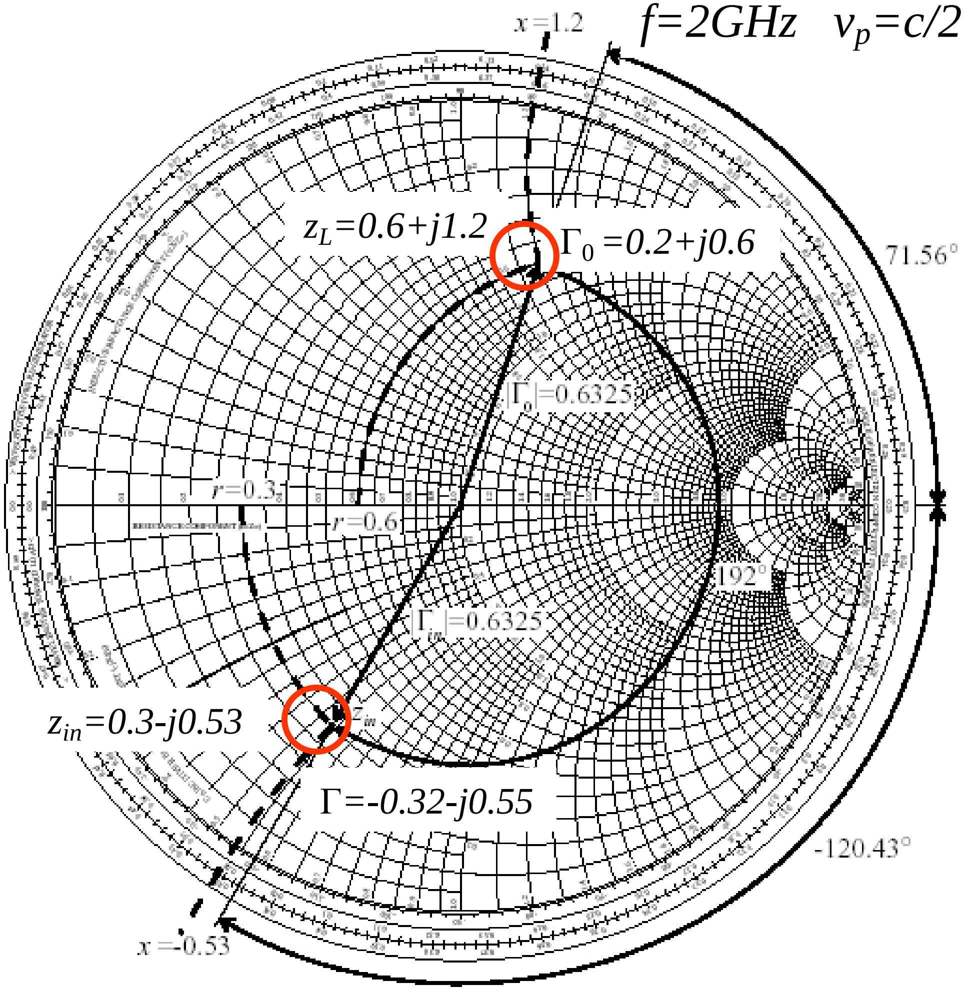
$$\Gamma_0 \rightarrow \Gamma(d) = \Gamma_0\, exp(-j2\beta d)$$

4) Find normalized input impedance:

$$\Gamma(d) \rightarrow z_{in}(d) = (1+\Gamma(d))/(1-\Gamma(d))$$

5) De-normalize input impedance:

$$z_{in}(d) \rightarrow Z_{in}(d) = Z_0\, z_{in}(d)$$



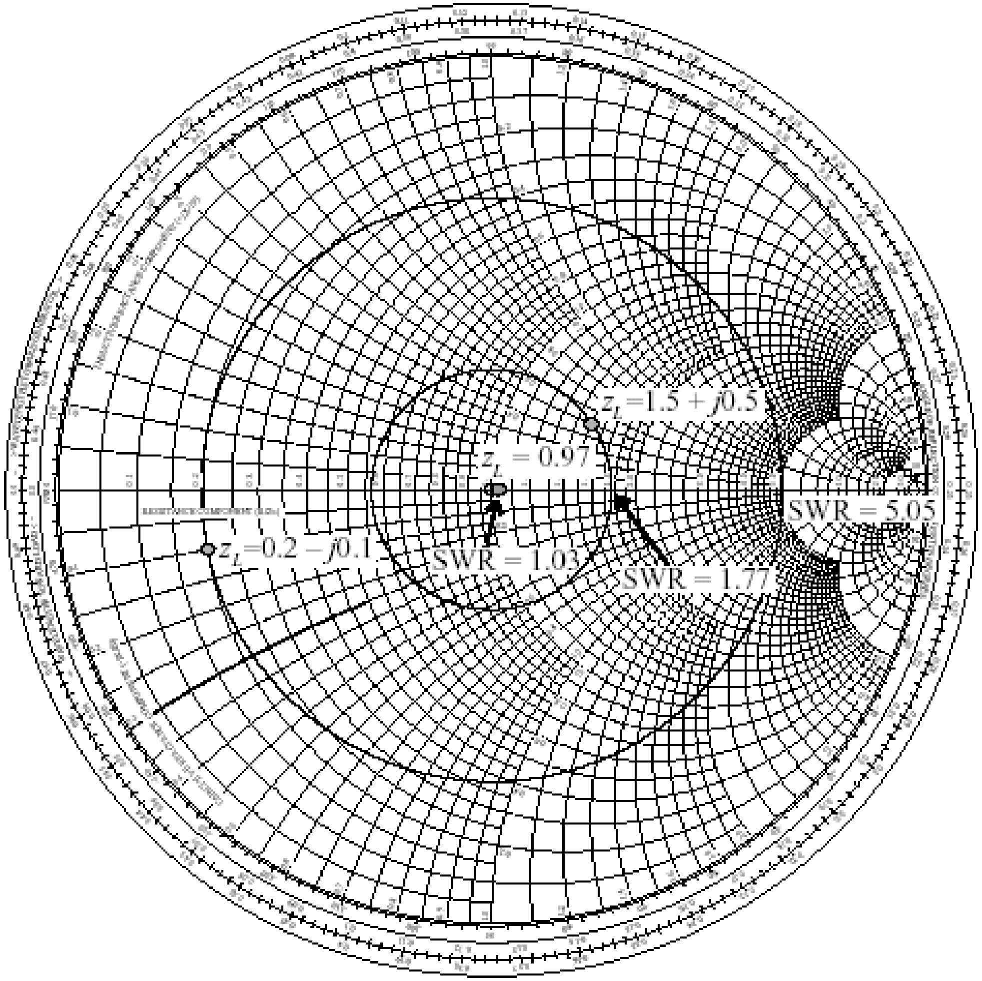
Generalized standing wave ratio

$$\Gamma(d) = \Gamma_0 e^{-j2\beta d}$$

$$SWR(d) = \frac{1+|\Gamma(d)|}{1-|\Gamma(d)|}$$

$$\Rightarrow |\Gamma(d)| = \frac{SWR-1}{SWR+1}$$

Can determine SRW for a given $\Gamma(d)$ by drawing circle with center at $\Gamma = 0$ through $\Gamma(d)$ in the Smith chart.



Open circuit TL as a reactive element

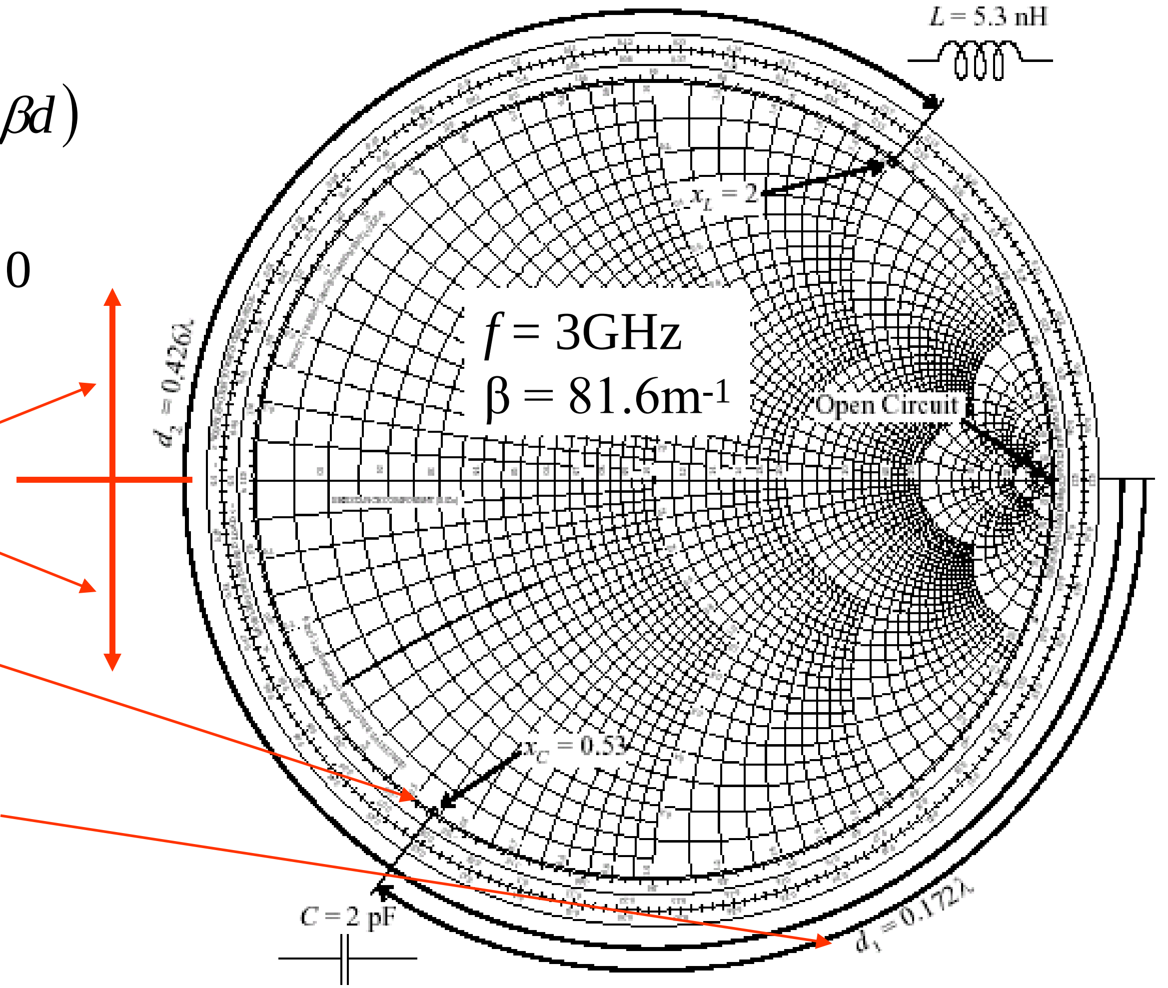
$$z_{in} = \frac{Z_L + jZ_0 \tan(\beta d)}{Z_0 + jZ_L \tan(\beta d)} \rightarrow -j\cot(\beta d)$$

Note: $Z_L = \infty$ corresponds to $r = 0$ (outer circle of Smith chart)

Upper half-circle: inductive
Lower half-circle: capacitive

Example: $z_{in} = jx = 1/j\omega C Z_0$

$$\Rightarrow d = \frac{1}{\beta}\left[\cot^{-1}\left(\frac{1}{\omega C Z_0}\right) + n\pi\right]$$



Short circuit TL as a reactive element

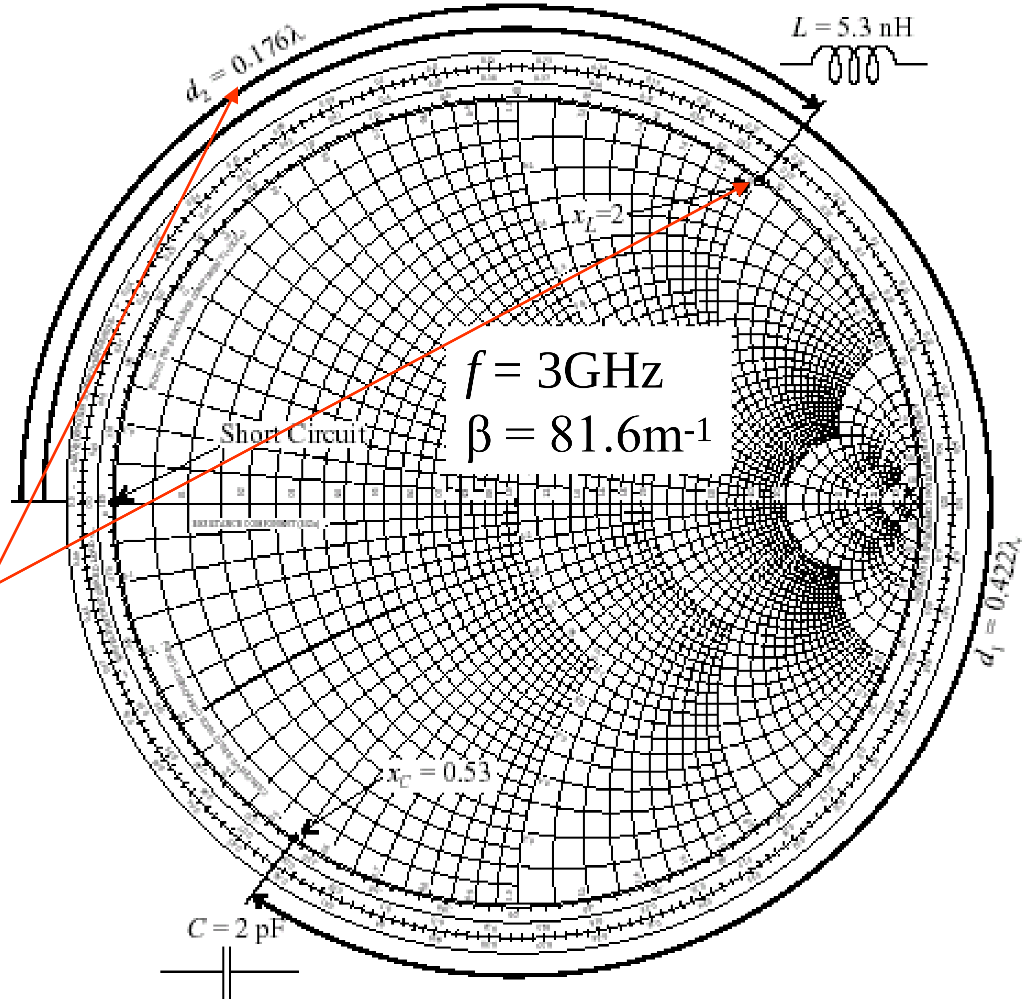
$$z_{in} = \frac{Z_L + jZ_0 \tan(\beta d)}{Z_0 + jZ_L \tan(\beta d)} \rightarrow j\tan(\beta d)$$

Note: $Z_L = 0$ also corresponds to $r = 0$ (outer circle of Smith chart)

Upper half-circle: inductive
Lower half-circle: capacitive

Example: $z_{in} = jx = j\omega L/Z_0$

$$\Rightarrow \quad d = \frac{1}{\beta}\left[\tan^{-1}\left(\frac{\omega L}{Z_0}\right) + n\pi\right]$$



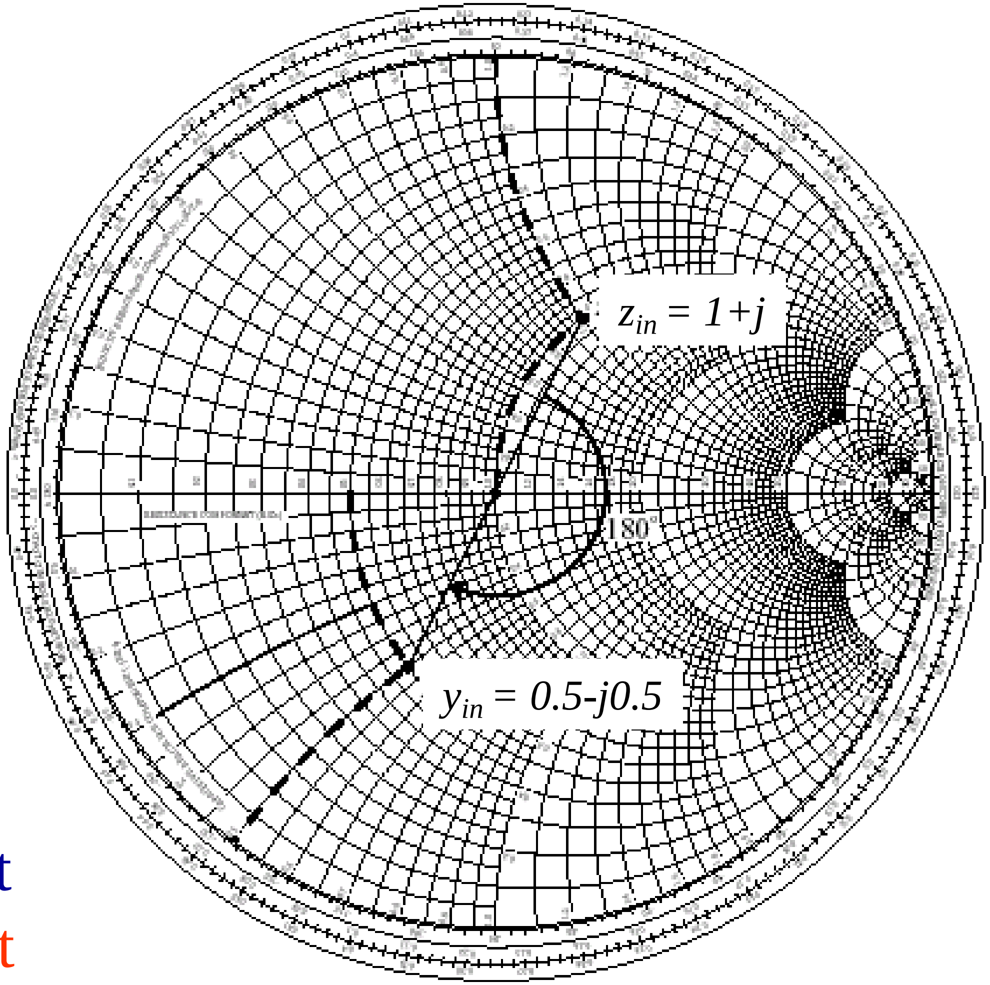
Admittance transformation

$$Y_{in} = \frac{1}{Z_{in}} = \frac{1}{Z_0}\frac{1-\Gamma(d)}{1+\Gamma(d)}$$

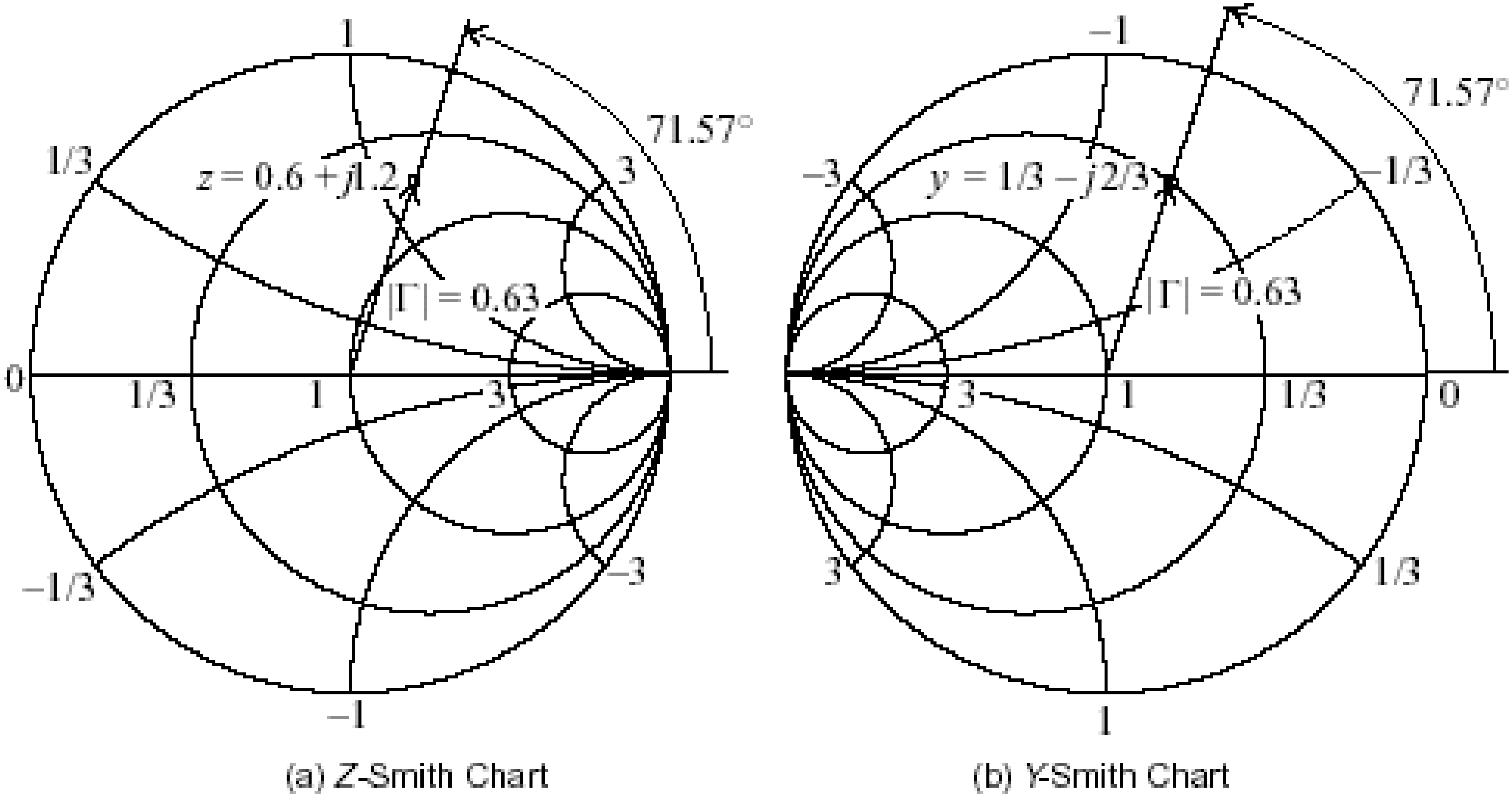
$$y_{in} = \frac{1}{z_{in}} = \frac{1-\Gamma(d)}{1+\Gamma(d)} = \frac{1+e^{-j\pi}\Gamma(d)}{1-e^{-j\pi}\Gamma(d)}$$

$e^{-j\pi}\,\Gamma(d)$ corresponds to 180º rotation of $\Gamma(d)$ in Smith chart. This converts impedance to admittance

Alternatively: Rotate Smith chart by 180º : Admittance Smith chart



Admittance Smith chart

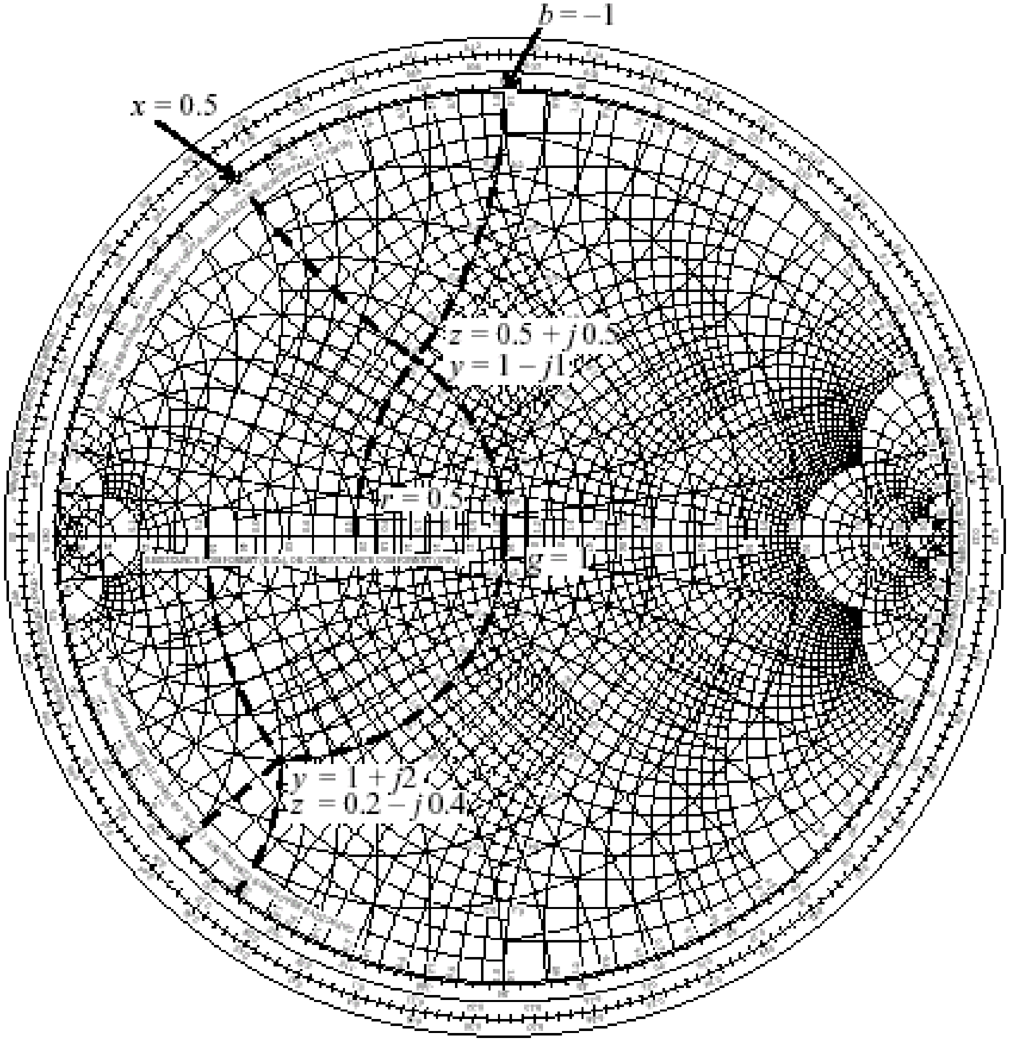


ZY Smith chart

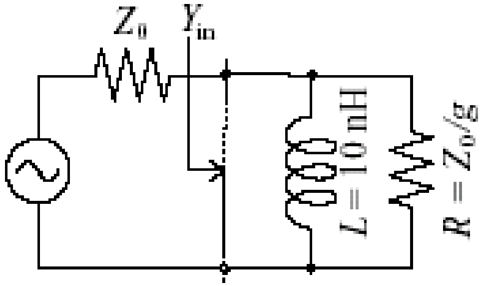
$$y_{in} = g + jb = \frac{1}{z_{in}} = \frac{1}{r + jx}$$

$$\Rightarrow g = \frac{r}{r^2 + x^2}, \quad b = \frac{-x}{r^2 + x^2}$$

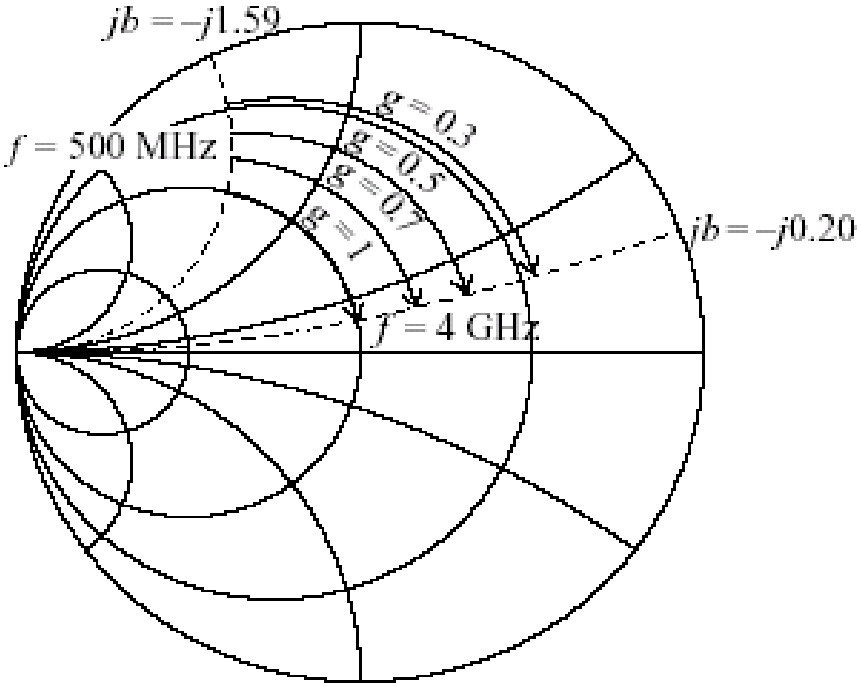
Use original Smith chart to display impedances and rotated chart to display admittances.



Frequency dependence: parallel R and L

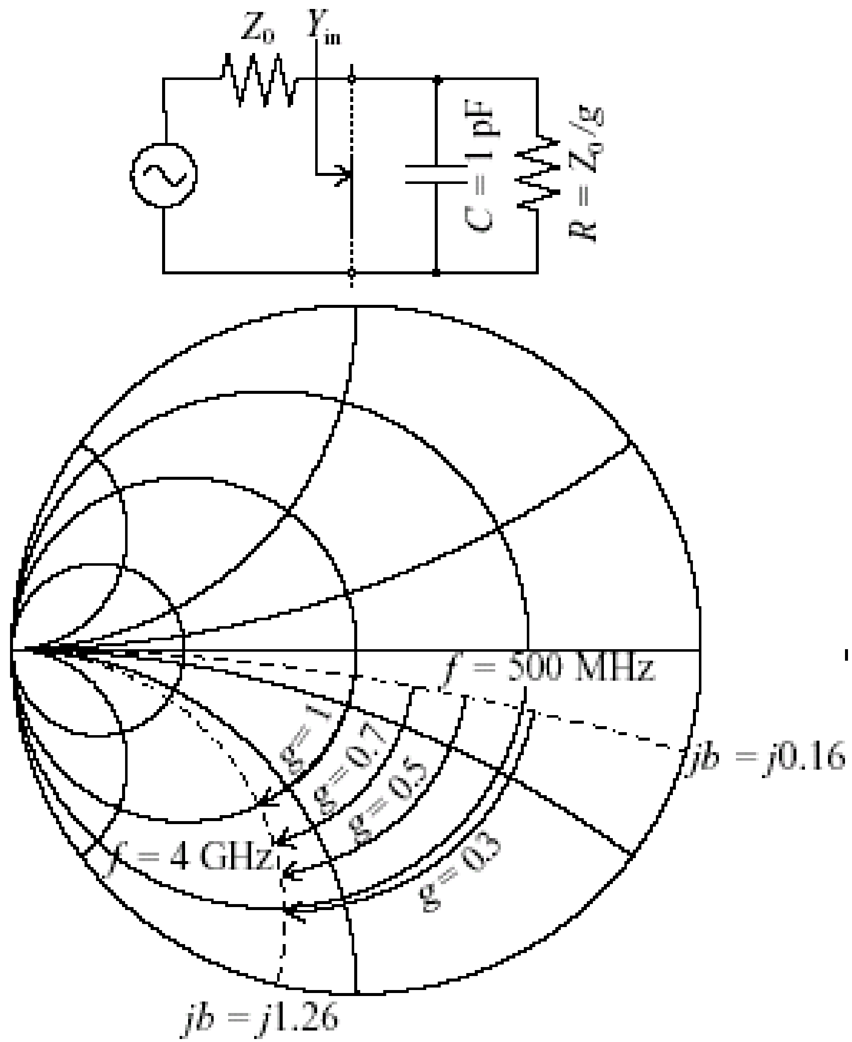


$$y_{in} = g + jb = \frac{Z_o}{R} - j\frac{Z_o}{\omega L}$$



Example:
Z_0 = 50 Ω, L = 10 nH
g = 0.3, 0.5, 0.7, 1
f = 500 MHz to 4 GHz

Frequency dependence: parallel R and C



$$y_{in} = g + jb = \frac{Z_o}{R} + jZ_o\omega C$$

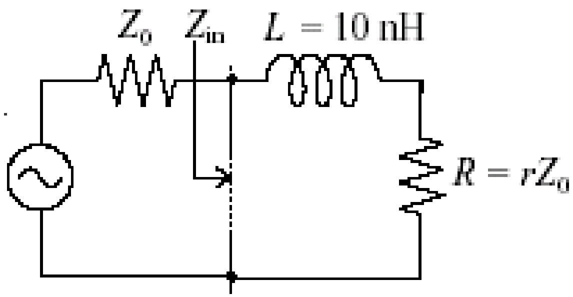
Example:

$Z_0 = 50\ \Omega$, C = 1 pF

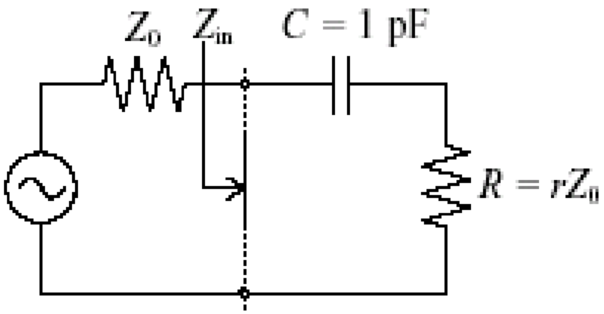
g = 0.3, 0.5, 0.7, 1

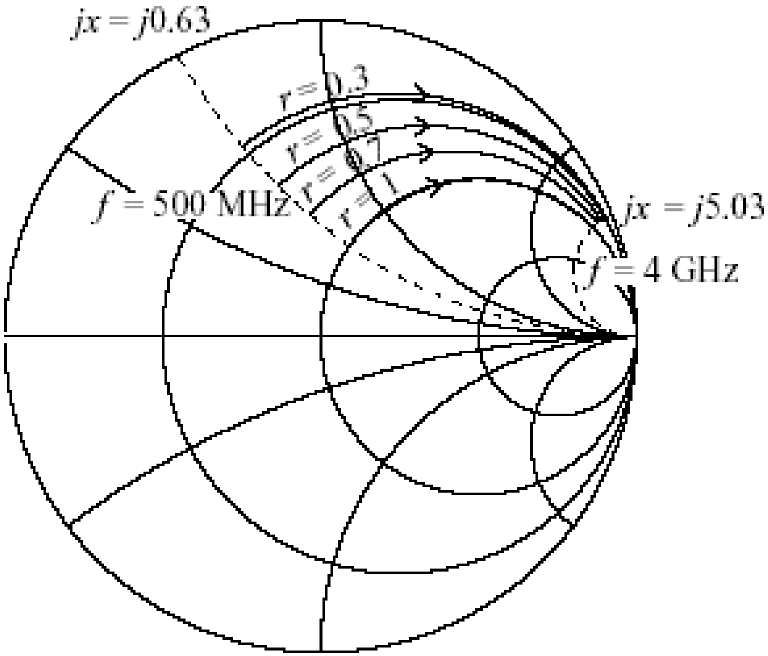
f = 500 MHz to 4 GHz

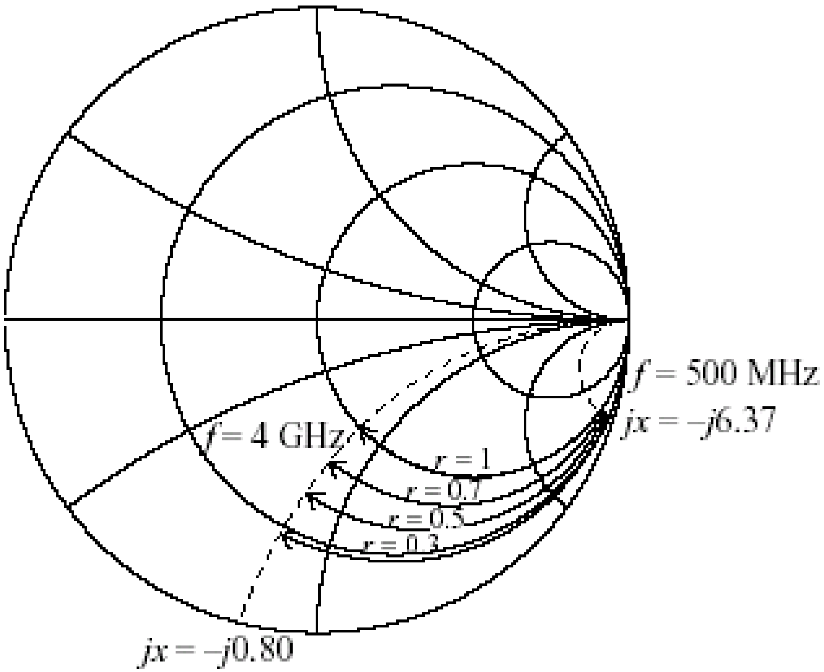
Frequency dependence: series connections



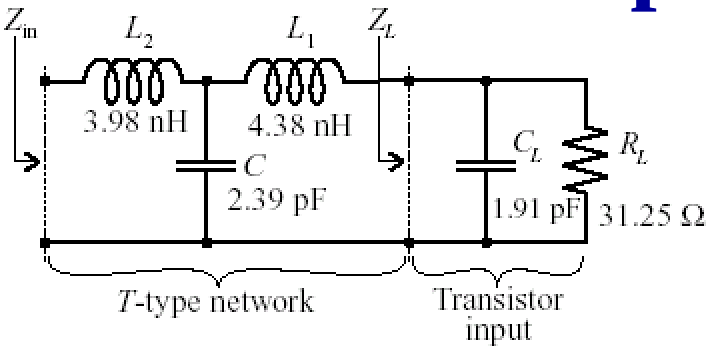
$$z_{in} = r + jx$$
$$= \frac{R}{Z_o} + j\frac{\omega L}{Z_o} - \frac{j}{Z_o \omega C}$$







Example: T-type network



$Z_0 = 50\ \Omega$ f = 2 GHz

A: $g_A = Z_0/R_L = 1.6$

B: $y_B = g_A + jZ_0\omega C_L = 1.6 + j1.2$

$\Rightarrow z_B = 0.4 - j0.3$

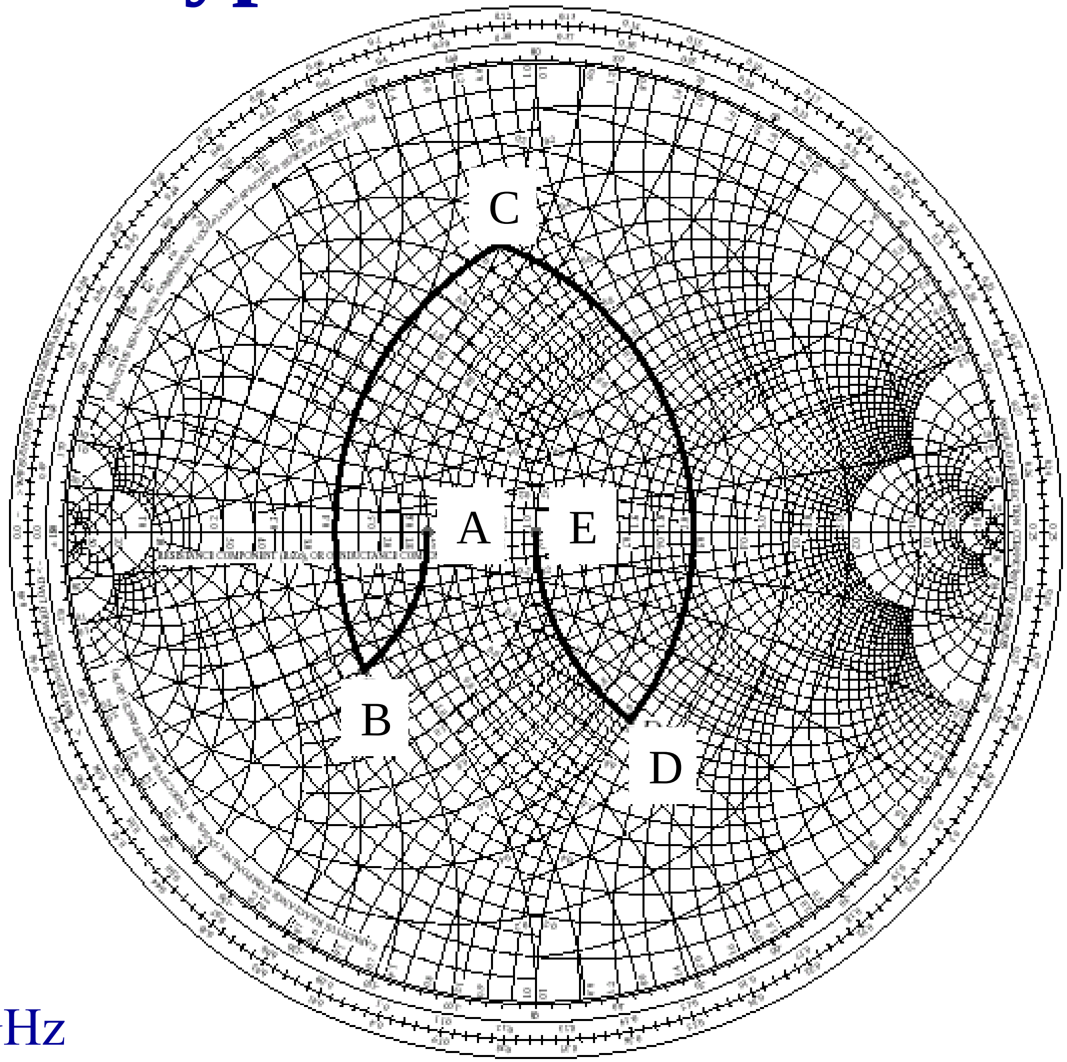
C: $z_C = z_B + j\omega L_1/Z_0 = 0.4 + j0.8$

$\Rightarrow y_C = 0.5 - j1.0$

D: $y_D = y_C + jZ_0\omega C = 0.5 + j0.5$

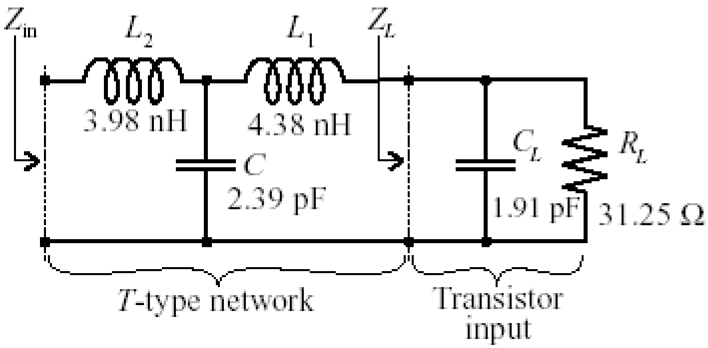
$\Rightarrow z_D = 1 - j1$

E: $z_E = z_D + j\omega L_2/Z_0 = 1$



$\Rightarrow Z_{in} = Z_0 = 50\ \Omega$: Match at 2 GHz

Simulation of Z_{in} for T-network



CAD simulation of Z_{in} for frequences 0.5 – 4 GHz

Note that C behaves as a short at the highest frequencies and the network will be dominated by L_2 (purely inductive)