

# INF5490 RF MEMS

## **LN03: Modeling, design and analysis**

Spring 2011, Oddvar Søråsen  
Department of Informatics, UoO

# Today's lecture

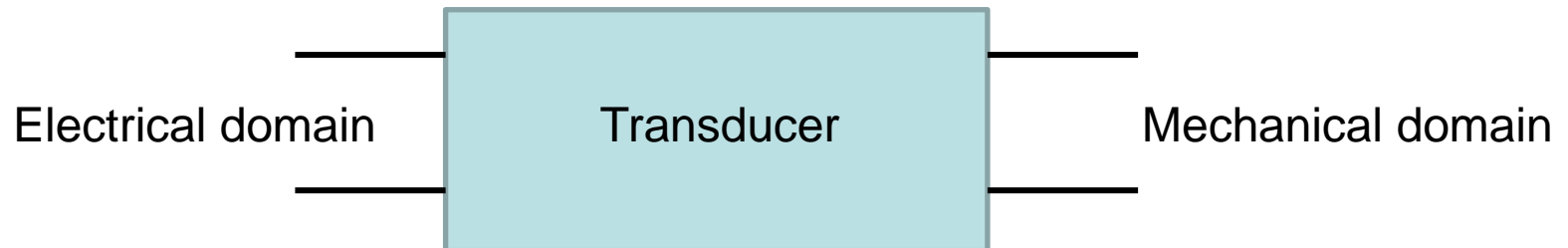
- Operating principles for MEMS components
  - Transducer principles
  - Sensor principles
- Methods for RF MEMS modeling
  - **1. Simple mathematical modeling**
  - **2. Converting to electrical equivalents**
  - **(3. Analyzing using Finite Element Methods)**
    - → LN04

# References

- In addition to the course book (Varadan et al):
- For additional reading ("cursory"):
- Senturia, *Microsystem design*, Kluwer 2002
- Course EE 5349/4328 (University of Texas at Arlington)
  - *"Fundamental MEMS, an Introduction to Microelectromechanical Systems (MEMS)"*
  - <http://arri.uta.edu/acs/jmireles/MEMSclass/lecture12.pdf>

# (RF) MEMS transducers

- **Electromechanical transducers**
  - Transforming  
**electrical energy  $\leftrightarrow$  mechanical energy**



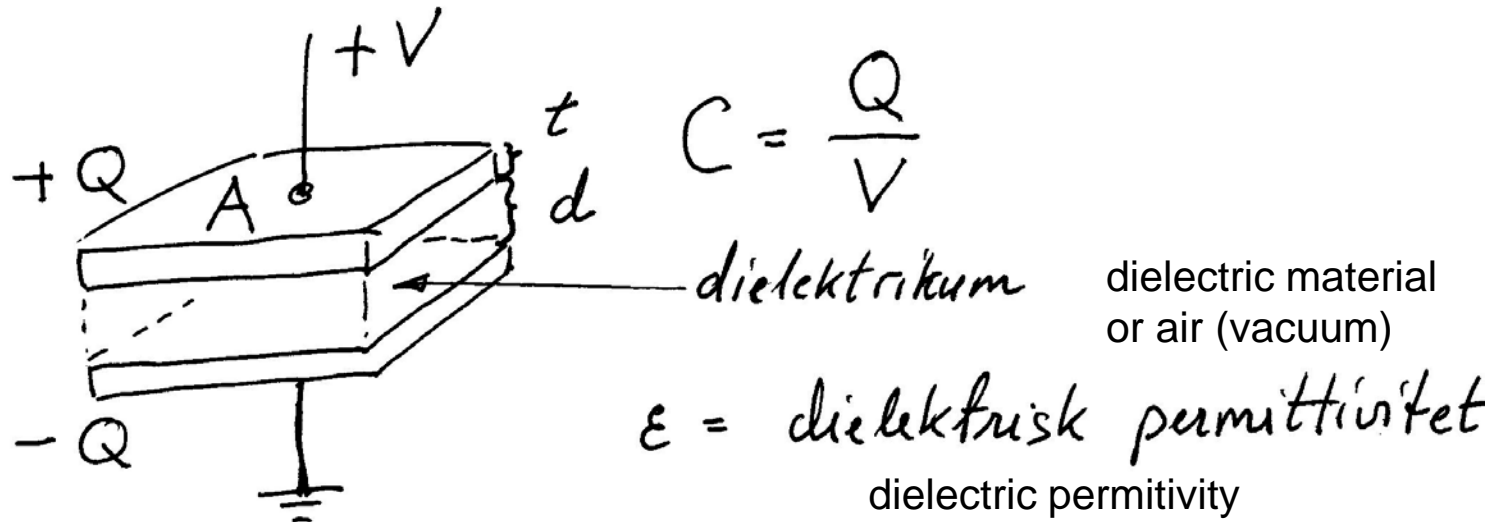
# (RF) MEMS transducers

- **Electromechanical transducers**
  - Transforming  
**electrical energy  $\leftrightarrow$  mechanical energy**
- Basic transducer principles
  - Electrostatic
  - Electromagnetic
  - Electro thermal
  - Piezoelectric

# Transducer principles

- **Electrostatic** transducers
  - Principle: **force** exists between **electric charges**
    - "Coulombs law"
  - Implemented by using a **capacitor with movable "plates"**
    - **Vertical** movement: parallel plates
    - **Horizontal** movement: comb structures

# Parallel plate capacitor



- Stored energy when mechanical work is performed on the unit can be **converted** to electrical energy
- Stored energy when electrical work is performed on the unit can be **converted** to mechanical energy

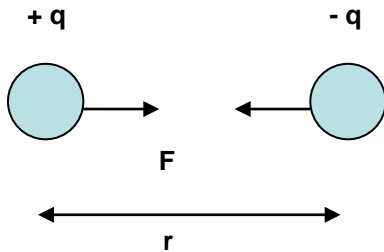
# Simple mathematical modeling

- Important equations for many RF MEMS components:
  - → **Parallel plate capacitor!**
  - Study **electrostatic** actuation of the capacitor with one plate suspended by a spring
  - Calculate **”pull-in”**
    - Formulas and figures →



# Electrostatics

Electric force between charges: **Coulombs law**



$$F = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_1 q_2}{r^2}$$

**Electric field** = force pr. unit charge

$$\vec{E} = \frac{\vec{F}}{q_0}$$

**Work** done by a force = change in potential energy

$$W_{a \rightarrow b} = \int_a^b \vec{F} \cdot d\vec{l} = U_a - U_b$$

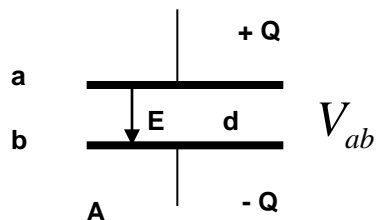
**Potential, V** = potential energy pr. unit charge

$$V = \frac{U}{q_0}$$

**Voltage** = potential difference

$$V_a - V_b = \int_a^b \vec{E} \cdot d\vec{l}$$

# Capacitance



Definition of capacitance

$$C = \frac{Q}{V_{ab}}$$

Surface charge density =  $\sigma$

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{A} \cdot \frac{1}{\epsilon_0}$$

Voltage

$$V_{ab} = E \cdot d = \frac{Q}{A\epsilon_0} \cdot d$$

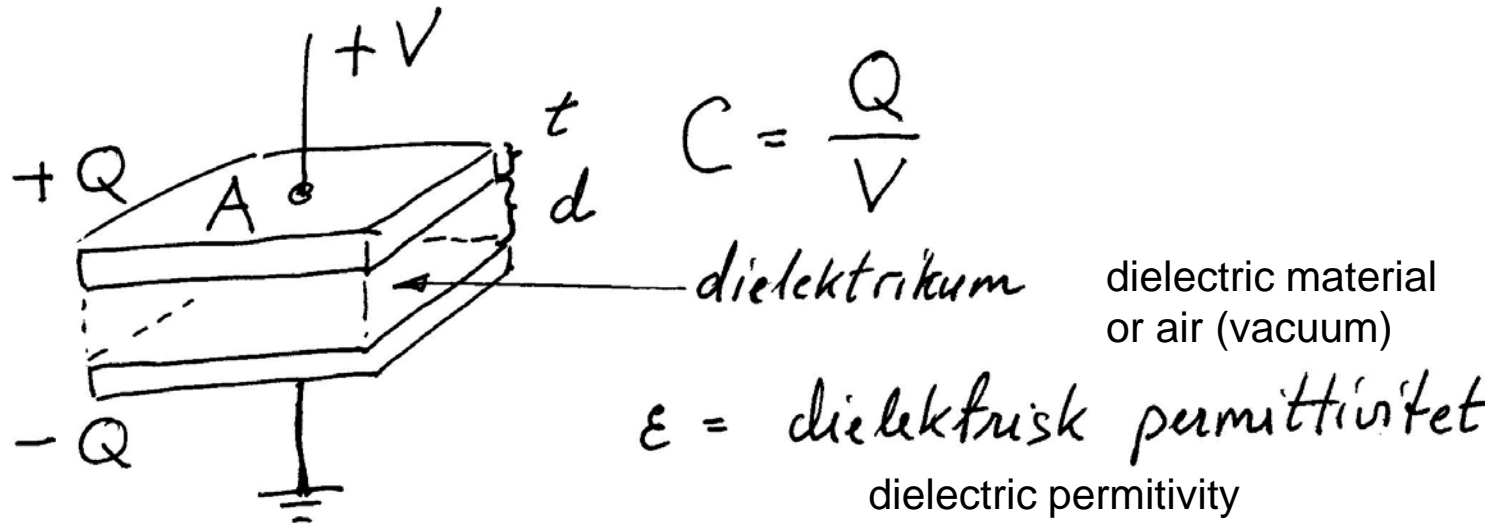
$$C = \frac{Q}{V_{ab}} = \epsilon_0 \frac{A}{d}$$

**Energy** stored in a capacitor,  $C$ ,  
that is charged to a voltage  $V_0$  at a current

$$i = \dot{Q} = C \frac{dV}{dt}$$

$$U = \int v \cdot i \cdot dt = \int v \cdot C \frac{dv}{dt} \cdot dt = C \int_0^{V_0} v \cdot dv = \frac{1}{2} C V_0^2 = \frac{\epsilon_0 A}{2d} V_0^2$$

# Parallel plate capacitor



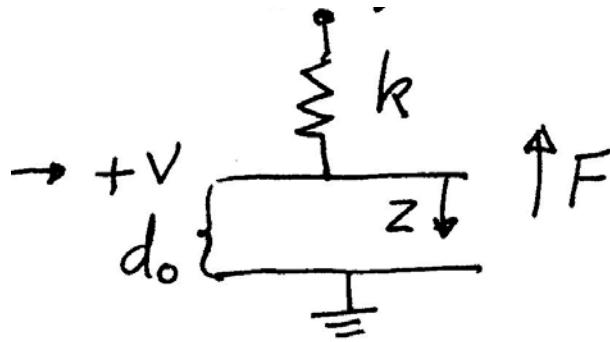
Attractive force between plates

$$F = -\frac{\partial U}{\partial d} = -\frac{\partial}{\partial d} \left( \frac{\epsilon A}{2d} V^2 \right) = \frac{\epsilon A V^2}{2d^2}$$

# Movable capacitor plate

- Assumptions for calculations:
  - Suppose air between plates
  - Spring attached to upper plate
    - Spring constant:  $k$
    - Spring force!
  - Voltage is turned on
    - Electrostatic attraction
    - Electrostatic force!
  - At equilibrium
    - Forces up and forces down are in balance →

# Force balance



$k$  = spring constant

$$F_{\text{spring}} = k \cdot x$$

deflection from start position

$d_0$  = gap at 0V and zero spring strain

$$d = d_0 - z$$

$$z = d_0 - d$$

Force on upper plate with voltage  $V$  and distance  $d$ :

$$F_{\text{net}} = - \frac{\epsilon A V^2}{2 d^2} + k (d_0 - d) = 0 \text{ at equilibrium}$$

# Two equilibrium positions

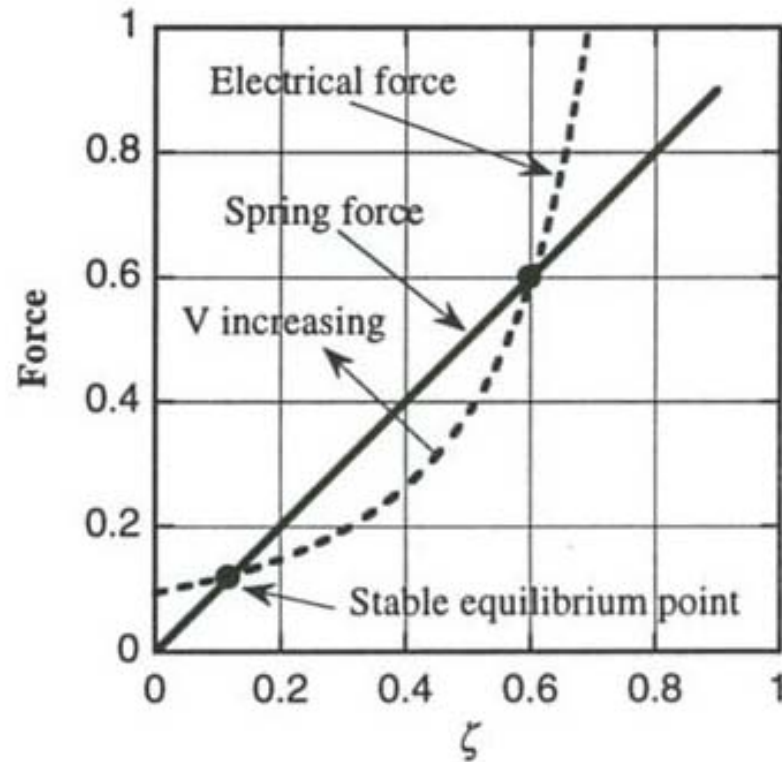


Figure 6.7. Electrical and spring forces for the voltage-controlled parallel-plate electrostatic actuator, plotted for  $V/V_{PI} = 0.8$ .

$$\zeta = 1 - d/d_0$$

Senturia

# Stability

- How the forces develop when  $d$  decreases
  - Suppose a small perturbation in the gap at constant voltage

$$\delta F_{\text{net}} = \left. \frac{\partial F_{\text{net}}}{\partial d} \right|_V \cdot \delta d$$

$$\delta F_{\text{net}} = \left( \frac{\epsilon A V^2}{d^3} - k \right) \delta d$$

Suppose the gap decreases  $\delta d < 0$

If the upward force also decreases,  
the system is **UNSTABLE!**

$$\delta F_{\text{net}} < 0,$$

# Stability, contd.

Stability condition:

$$\left. \frac{\partial F_{net}}{\partial d} \right|_V < 0$$

$$k > \frac{\epsilon A V^2}{d^3}$$

**Pull-in** when:

$$k = \frac{\epsilon A V_{PI}^2}{d_{PI}^3}$$



# Pull-in

$$F_{net} = 0$$

$$\frac{\epsilon A V_{PI}^2}{2 d_{PI}^2} = k (d_0 - d_{PI})$$

$\uparrow = \frac{\epsilon A V_{PI}^2}{d_{PI}^3}$

Pull-in when:

$$d_{PI} = \frac{2}{3} d_0$$

$$V_{PI} = \sqrt{\frac{8 k d_0^3}{27 \epsilon A}}$$

# Pull-in

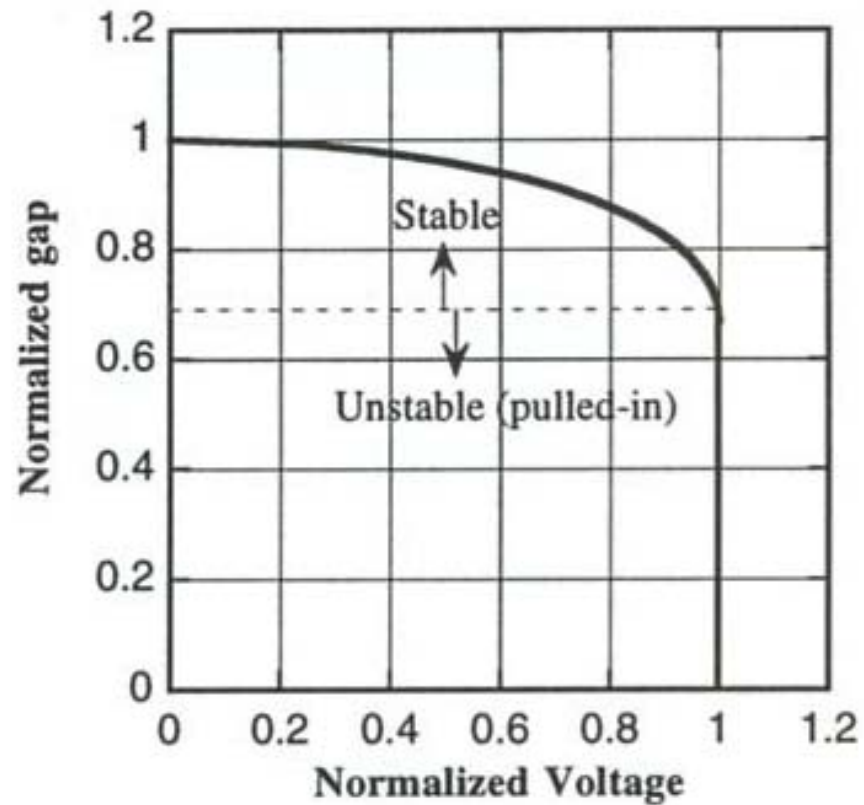


Figure 6.8. Normalized gap as a function of normalized voltage for the electrostatic actuator.

Senturia

# Electrostatic transducers: summary

- + Simple principle and fabrication
- + Actuation (movement) controlled by voltage
  - voltage  $\rightarrow$  charges  $\rightarrow$  attractive force  $\rightarrow$  movement
- + Movement gives current
  - movement  $\rightarrow$  variable capacitor  $\rightarrow$  current when voltage is constant:  
 $Q = V C$  and  $i = dQ/dt = V dC/dt$
- $\div$  Need environmental protection (dust)
  - Packaging required (vacuum)
- $\div$  Transduction mechanism is non-linear
  - ... for variation of distance between plates ...
  - Force is not proportional to voltage
  - Solution: small signal variations around a DC voltage
- The most used form of electromechanical energy conversion

# (RF) MEMS transducers

- **Electromechanical transducers**
  - Transforming  
**electrical energy  $\leftrightarrow$  mechanical energy**
- Basic transducer principles
  - Electrostatic
  - **Electromagnetic**
  - **Electro thermal**
  - **Piezoelectric**

# Transducer principles, contd.

- **Electromagnetic** transducers
  - Magnetic windings pull the element
  - ÷ More complicated processes
- **Electro thermal** actuators
  - Different thermal expansion due to temperature gradients
    - Different materials
      - Each with its: TCE – Thermal Coefficient of Expansion
    - Different locations
  - **Large deflections** can be obtained
  - **Slow!**

# Transducer principles, contd.

- **Piezoelectric** transducers
  - In some **anisotropic** crystalline materials the charges will be displaced when **stressed** → electric field
    - stress = "mechanical stress" (Norw: "*mekanisk spenning*")
  - Similarly, **strain** results when an electric field is applied (relative shrinking or prolongation of unit)
    - strain = "mechanical strain" (Norw: "*mekanisk tøyning*")
  - Ex. PZT (lead zirconate titanates) – ceramic material
- (Electrostrictive transducers
  - Mechanical deformation by electric field
- (Magnetostrictive transducers
  - Deformation by magnetic field)

# Comparing different transducer principles

**Table 1.4** Comparison of electromechanical transducers

Actuator	Fractional stroke (%)	Maximum energy density ( $\text{J cm}^{-3}$ )	Efficiency	Speed
Electrostatic	32	0.004	High	Fast
Electromagnetic	50	0.025	Low	Fast
Piezoelectric	0.2	0.035	High	Fast
Magnetostrictive	0.2	0.07	Low	Fast
Electrostrictive	4	0.032	High	Fast
Thermal	50	25.5	Low	Slow

Source: Wood, Burdess and Hariss, 1996.

# Sensor principles

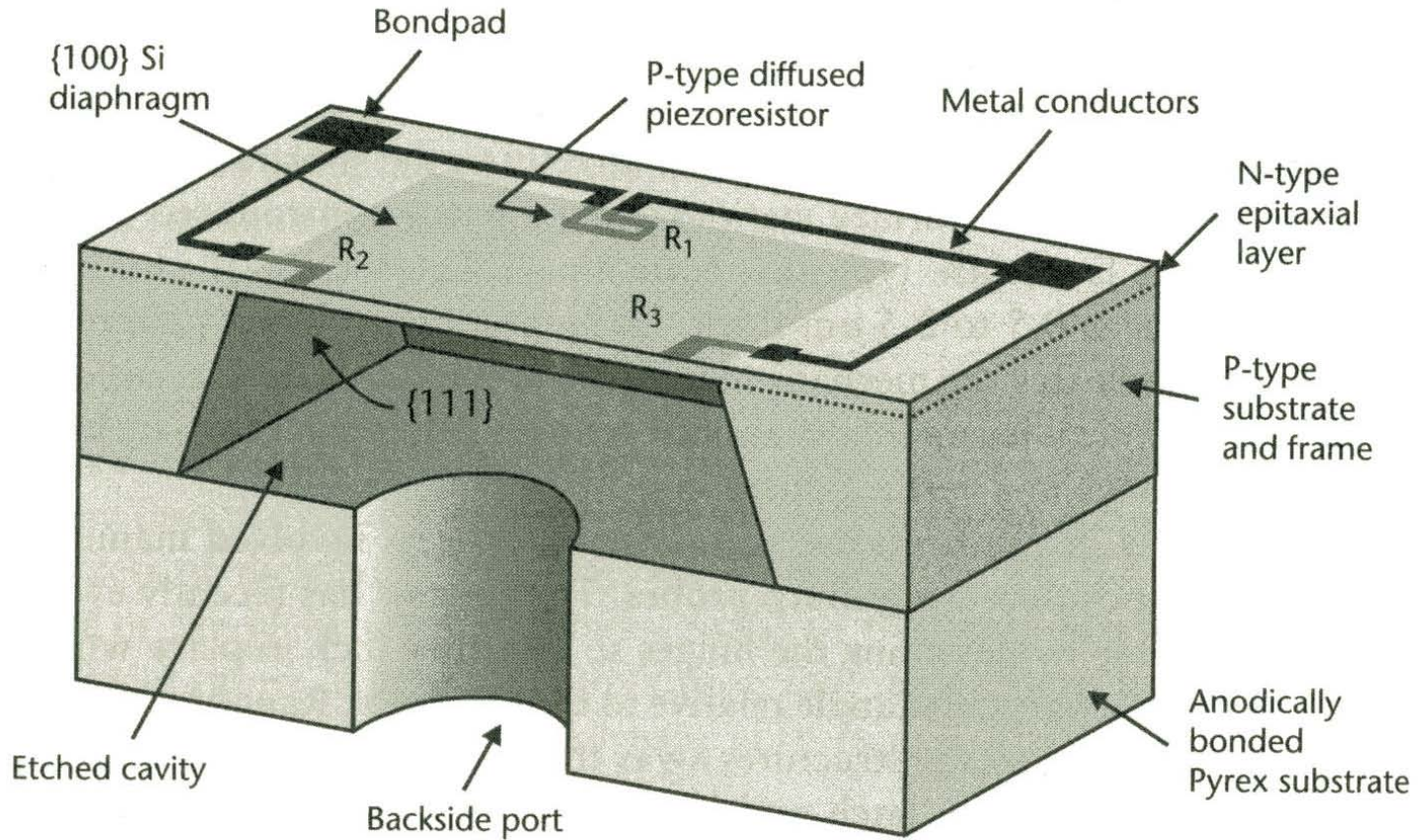
- Transducers can be used as sensors
  - Piezoresistive detection
  - Capacitive detection
  - Piezoelectric detection
  - Resonance detection



# Sensor principles

- **Piezoresistive** detection
  - Resistance varies due to external pressure/stress
  - Resistor value is proportional to strain (“tøyning”)
  - Piezoresistors placed on membrane where **strain is maximum**
    - At the peripheries
  - Used in pressure sensors
    - Deflection of membrane influences R values
  - + **Simple** principle
  - ÷ Performance of piezoresistive micro sensors is **temperature dependent**

# Pressure sensor



# Sensor principles, contd.

- **Capacitive** detection
  - Exploiting capacitance variations
  - Pressure  $\rightarrow$  change in  $C \rightarrow$  electric signal
    - Can influence oscillation frequency, charge or voltage
  - Potentially higher performance than piezoresistive detection
    - + Better sensitivity
    - + Can detect small pressure variations
    - + **High stability**

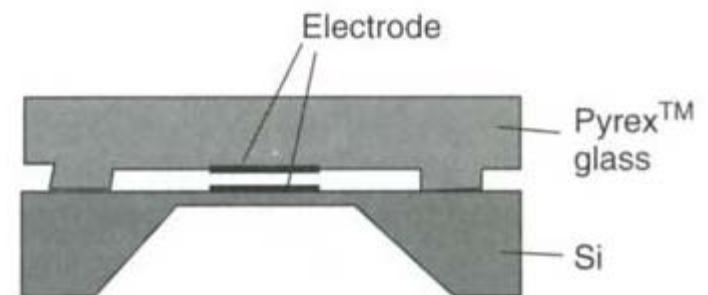


Figure 1.19 Capacitive sensing structure

# Sensor principles, contd.

- **Piezoelectric** detection
  - Electric charge distribution changed due to external force → electric field → current
- **Resonance** detection
  - Using resonating structures
  - Analogy: stress variation on a string gives strain and is changing the “**natural**” **resonance frequency**

# Methods for modeling RF MEMS

- **1. Simple mathematical modeling**
  - Ex. As explained for the parallel plate capacitor
- Use equations, formulas describing the physical phenomena
  - Simplification, approximations necessary
  - a) Explicit solutions for simple problems
    - Linearization around a bias point
  - b) Numerical solution of a set of equations
    - Typical: differential equations
- **+ Gives the designer insight/ understanding**
  - How the performance changes by parameter variations
  - May be used for initial estimates

# Methods for modeling RF MEMS

- 1. Simple mathematical modeling
- **2. Converting to electrical equivalents**
- **3. Analysis using Finite Element Methods**

# A mechanical system can be modeled as an electrical circuit

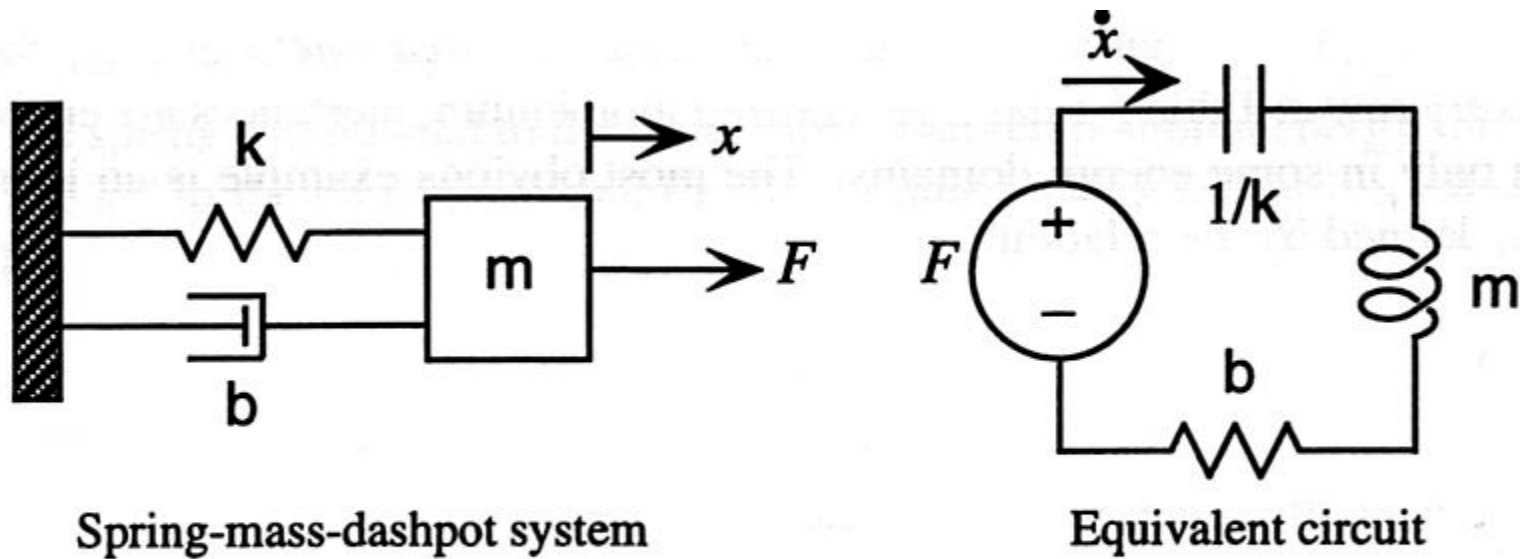


Figure 5.9. Translating mechanical to electrical representations.

$$C=1/k \quad L=m \quad R=b$$

## 2. Converting to electrical equivalents

- Mechanical behavior can be modeled using **electrical circuit elements**
  - Mechanical structure → simplifications → equivalent electrical circuit
    - ex. spring/mass/damper system → R, C, L -equivalent
  - Then: Possible to “interconnect” electrical and mechanical **energy domains**
    - Simplified modeling and co-simulation of electronic and mechanical parts of the system
  - Proper **analysis-tools** can be used
    - Ex. SPICE



# Converting to electrical equivalents, contd.

- We will discuss:
  - Relevant circuit theory
  - Conversion principles
    - **effort - flow**
  - Example of conversion
    - Mechanical resonator
  - In a future lecture:
    - Co-existence and coupling between various energy domains

# Circuit theory

- Basic circuit elements: R, C, L
- Current and voltage equations for basic elements (low frequency)
  - Ohms law, C and L-equations
    - $V = RI$ ,  $I = C \, dV/dt$ ,  $V = L \, dI/dt$
  - Laplace transformation
    - From differential equations to algebraic (s-polynomial)
    - → Complex impedances:  $R$ ,  $1/sC$ ,  $sL$
- Kirchhoffs equations
  - $\Sigma$  currents into nodes = 0,  $\Sigma$  voltages in loops = 0

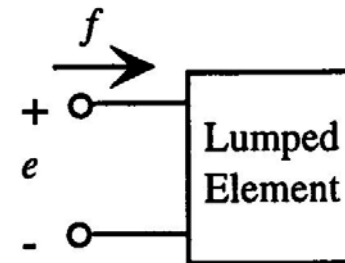
# Effort - flow

- Electrical circuits are described by a **set of variables: *conjugate power variables***
  - Voltage  $V$ : **across** or **effort** variable
  - Current  $I$ : **through** or **flow** variable
  - *An effort variable drives a flow variable through an impedance,  $Z$*

- Circuit element is modeled as a **1-port** with terminals

- Same current ( $f = \text{flow}$ ) in and out and through the element

- ***Positive flow*** into a terminal defining a ***positive effort***



# Energy-domains, analogies

- Various energy domains exist
  - Electric, mechanical/elastic, thermal, for liquids etc.
- ***For every energy domain it is possible to define a set of **conjugate power variables** that may be used as basis for lumped component modeling using equivalent circuits elements***
- Table 5.1 Senturia ->

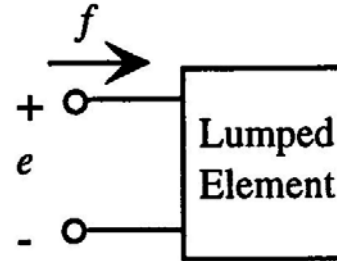
# Ex. of conjugate power variables

Energy Domain	Effort	Flow	Momentum	Displacement
Mechanical translation	Force $F$	Velocity $\dot{x}, v$	Momentum $p$	Position $x$
Fixed-axis rotation	Torque $\tau$	Angular velocity $\omega$	Angular momentum $J$	Angle $\theta$
Electric circuits	Voltage $V, v$	Current $I, i$	...	Charge $Q$
Magnetic circuits	Magnetomotive force MMF	Flux rate $\dot{\phi}$	...	Flux $\phi$
Incompressible fluid flow	Pressure $P$	Volumetric flow $Q$	Pressure momentum $\Gamma$	Volume $V$
Thermal	Temperature $T$	Entropy flow rate $\dot{S}$	...	Entropy $S$

# Conjugate power variables: e, f

- Properties

- $e * f = \text{power}$
- $e / f = \text{impedance}$



- Generalized **displacement** represents the state, f. ex. position or charge

$$f(t) = \dot{q}(t)$$

$$q(t) = \int_{t_0}^t f(t) dt + q(t_0)$$

- $e * q = \text{energy}$

# Generalized momentum

$$p(t) = \int_{t_0}^t e(t) dt + p(t_0)$$

– Mechanics: “impulse”

- $F \cdot dt = mv - mv_0$

– General:  **$p \cdot f = \text{energy}$**

# Ex.: Mechanical energy domain

$$e = F \quad (\text{kraft})$$

force

$$f = v, \dot{x} \quad (\text{hastighet})$$

velocity

$$g = x \quad (\text{posisjon}) = \int \dot{x} dt$$

position

$$p = p \quad (\text{momentum}) = \int F dt$$

momentum

$$(\text{kraft} \times \text{tid})$$

force x time

$$e \cdot f \rightarrow F \cdot \dot{x} = \frac{F \Delta x}{\Delta t} = \frac{\text{arbeid}}{\text{tid}} = \text{effekt}$$

work/time = power

$$e \cdot g \rightarrow F \cdot x = \text{kraft} \times \text{vei} = \text{arbeid} = \text{energi}$$

force\*distance = work = energy

$$p \cdot f \rightarrow p \cdot \dot{x} = m v \cdot v = m v^2 = \text{energi}$$

energy



# Ex.: Electrical energy domain

$$e = V \quad (\text{spenning}) \quad \text{voltage}$$

$$f = I \quad (\text{strøm}) \quad \text{current}$$

$$q = \int I dt = Q \quad (\text{ladning}) \quad \text{charge}$$

$$p = \text{n.a.}$$

$$e \cdot f \rightarrow V \cdot I = \text{effekt} \quad \text{power}$$

$$e \cdot q \rightarrow V \cdot Q = V \int I dt = \text{energi} \quad \text{energy}$$

# $e \rightarrow V$ - convention

- **Senturia and Tilmans** use the  **$e \rightarrow V$  -convention**
- Ex. electrical and mechanical circuits
  - $e \rightarrow V$  (voltage)      equivalent to  $F$  (force)
  - $f \rightarrow I$  (current)      equivalent to  $v$  (velocity)
  - $q \rightarrow Q$  (charge)      equivalent to  $x$  (position)
  - $e * f =$  "power" injected into the element

H. Tilmans, Equivalent circuit representation of electromagnetical transducers:  
I. Lumped-parameter systems, J. Micromech. Microeng., Vol. 6, pp 157-176, 1996

# Other conventions

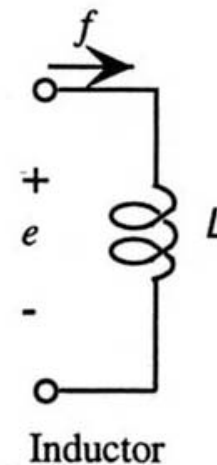
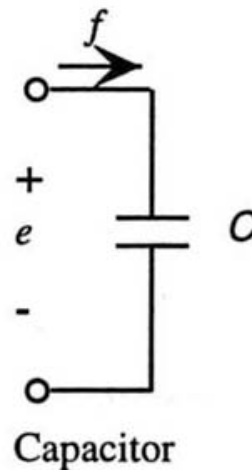
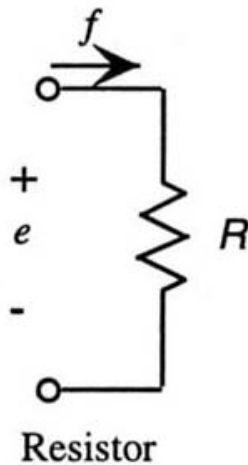
- Different conventions exist for defining **through-** or **across-variables**

*Table 5.2.* Different conventions for assigning circuit variables.

Convention	Across Variable	Through Variable	Product	Principal Use
$e \rightarrow V$ *	$e$	$f$	power	electric circuit elements
$f \rightarrow V$ alternative	$f$	$e$	power	mechanical circuit elements
Thermal	T	$\dot{Q}$	Watt-Kelvin	thermal circuits
HDL	q	e	energy	HDL circuit representation of mechanical elements

# Generalized circuit elements

- **One-port** circuit elements
  - R, dissipating element
  - C, L, energy-storing elements
  - Elements can have a **general function!**
    - Can be used in **various energy domains**



# Generalized capacitance

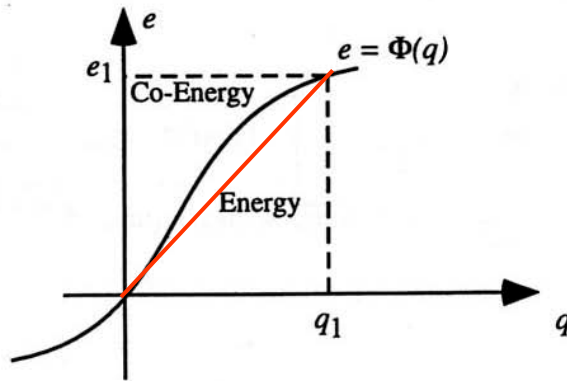


Figure 5.5. Illustrating energy and co-energy for a generalized capacitor.

Compare with a **simplified case**:

- a **linear** capacitor

$$Q = V \cdot C$$

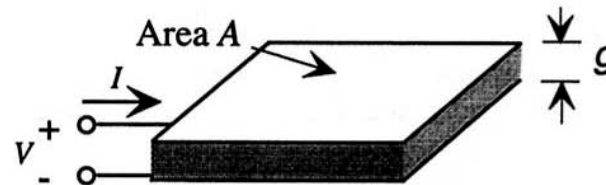
$$V = \frac{1}{C} \cdot Q$$

$\Downarrow$

$$e = \frac{1}{C} \cdot q$$

$$C = \frac{\epsilon A}{g}$$

definition of C



# Generalized capacitance, contd.

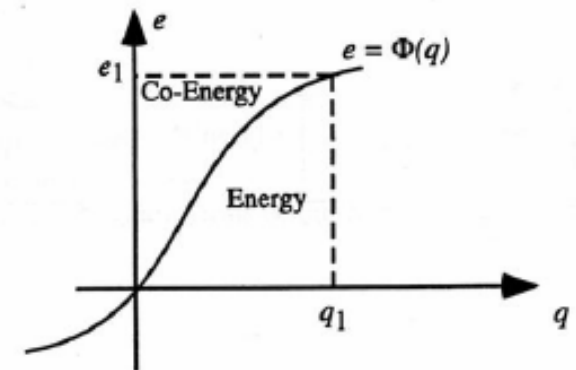
**Capacitance** is associated with stored **potential energy**

$$\mathcal{W}(q_1) = \int_0^{q_1} e \, dq = \int_0^{q_1} \Phi(q) \, dq \quad (5.10)$$

**Co-energy:**

$$\mathcal{W}^*(e) = eq - \mathcal{W}(q) \quad (5.11)$$

$$\mathcal{W}^*(e_1) = \int_0^{e_1} q \, de = \int_0^{e_1} \Phi^{-1}(e) \, de \quad (5.12)$$



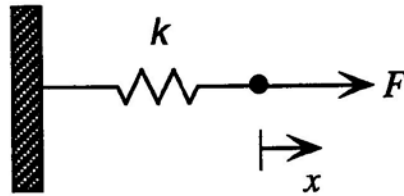
# Energy stored in parallel plate capacitor

Energy: 
$$W(Q) = \int_0^Q e \cdot dq = \int_0^Q \frac{q}{C} \cdot dq = \frac{Q^2}{2C}$$

Co-energy: 
$$W^*(V) = \int_0^V q \cdot de = \int_0^V C \cdot v \cdot dv = \frac{CV^2}{2}$$

$$W^*(V) = W(Q) \quad \text{for linear capacitance}$$

# Mechanical spring



Hook's law:  $F = k \cdot x$

Stored energy  $W(x_1) = \int_0^{x_1} F(x)dx = \frac{1}{2}kx_1^2$  (5.18)

Compare with capacitor  $W(Q) = \frac{1}{2} \cdot \frac{1}{C} \cdot Q^2$

$Q$  displacement  
 $x_1$  displacement

→ 1/C equivalent to k



# "Compliance"

- "Compliance" = "inverse stiffness"

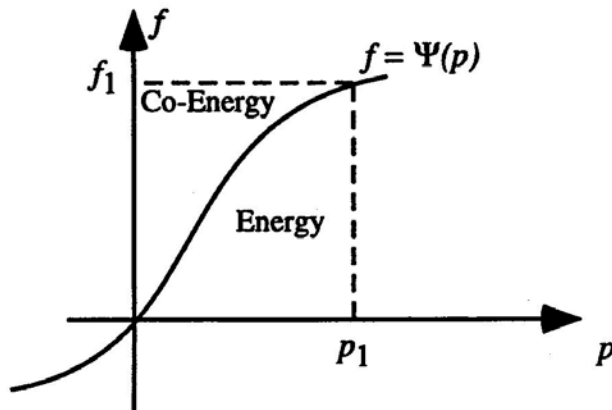
$$C_{spring} = \frac{1}{k}$$

- Stiff spring → small capacitor
- Soft spring → large capacitor

# Generalized inductance

Energy also defined as:

$$\int e dt = \int v dt = \int m \cdot v dt = \text{flow} \times \text{momentum}$$




**Energy = stored kinetic energy**

$$W(p_1) = \int_0^{p_1} f(p) dp$$

## Ex.: Electrical inductor

Co-energy:  $W^*(f) = \int_0^{f_1} p(f) df$

  $V = L \frac{dI}{dt}$

$$p = \int e dt = \int V dt = \int L \frac{dI}{dt} dt = \int L dI$$

$$p(f) = p(I) = LI$$

$$W^*(f_1) = W^*(I_1) = \int_0^{I_1} L \cdot I \cdot dI = \frac{1}{2} L I_1^2$$

# Analogy between mass (mechanical inertance) and inductance L

A mechanical system has **linear momentum**:  $p = mv$

Flow:  $\phi = v = \frac{p}{m}$

$$W(p_1) = \int_0^{p_1} f(p) dp = \int_0^{p_1} \frac{p}{m} dp = \frac{p_1^2}{2m}$$

**Co-energy:**

$$W^*(v_1) = \int_0^{v_1} p(v) dv = \int_0^{v_1} (mv) dv = \frac{1}{2} m v_1^2$$

# Analogy between m and L

$$W^*(f_1) = W^*(I_1) = \int_0^{I_1} L \cdot I \cdot dI = \frac{1}{2} L I_1^2$$

Compare with:  $W^*(v_1) = \frac{1}{2} m v_1^2$

$$I_1 = \text{flow}$$

$$v_1 = \text{---}$$

L is equivalent to m

**m = L** inertance

**Mechanical inertance = mass m**  
**is analog to inductance L**

# Interconnecting elements

- $e \rightarrow V$  follows two basic principles
  - Elements that share a **common flow**, and hence a common variation of displacement, are connected in **series**
  - Elements that share a **common effort** are connected in **parallel**

# Ex. of interconnection:

”Direct transformation”: **series connection**

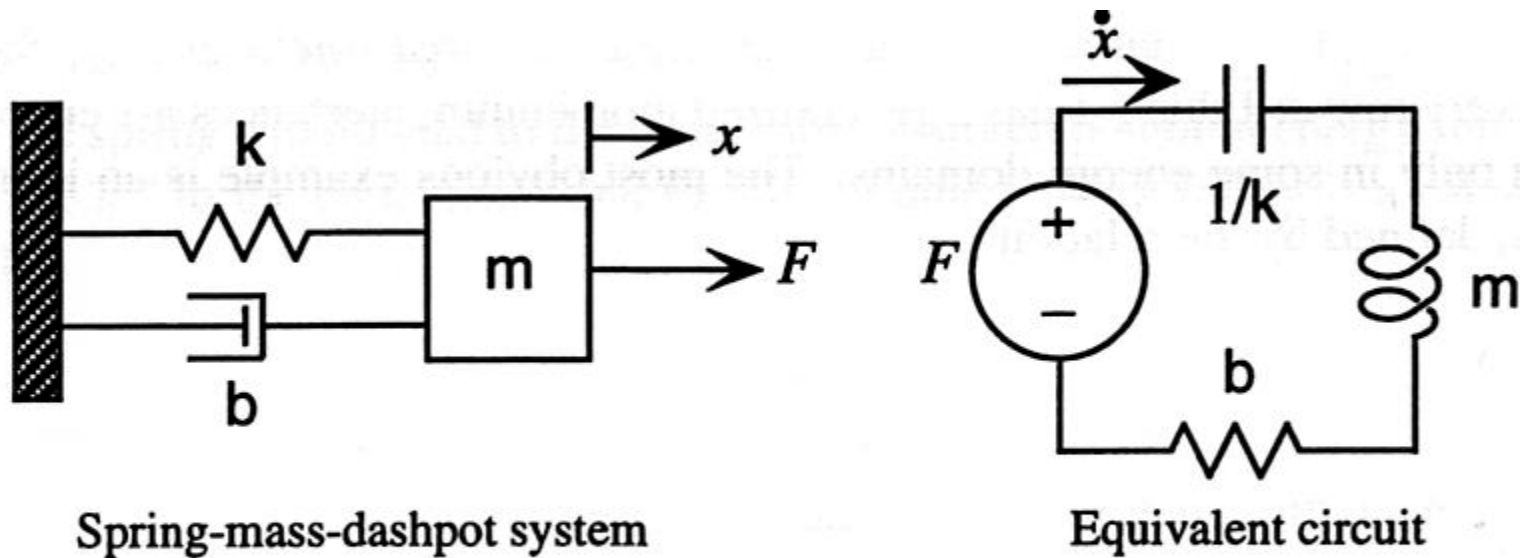
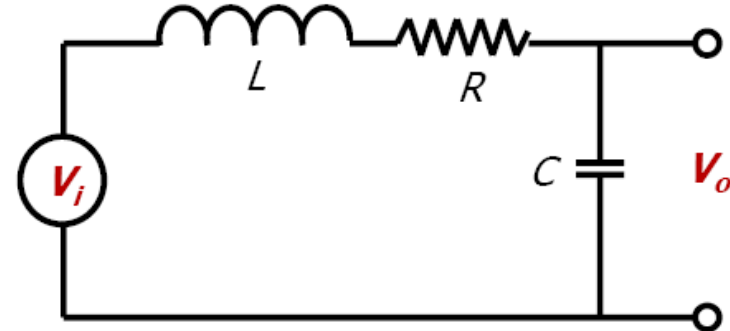
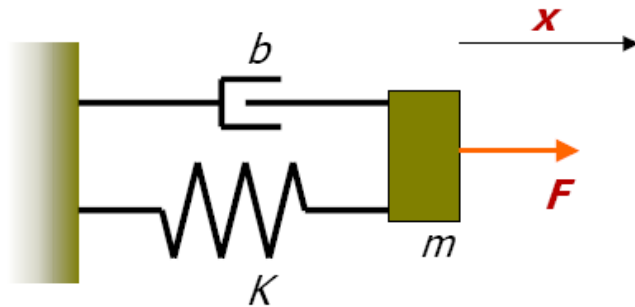


Figure 5.9. Translating mechanical to electrical representations.

# Mechanical / Electrical Systems



Input : external force  $F$

Output : displacement  $x$

$$m\ddot{x}(t) + b\dot{x}(t) + Kx(t) = F$$

$m$  mass,  $b$  damping,  $K$  stiffness

Transfer function :

$$H(s) = \frac{x}{F} = \frac{\frac{1}{m}}{s^2 + \frac{b}{m}s + \frac{K}{m}}$$

Input : voltage  $V_i$

Output : voltage  $V_o$

$$L\ddot{q}(t) + R\dot{q}(t) + \frac{1}{C}q(t) = V_i$$

$L$  induct.,  $R$  resist.,  $C$  capacit.

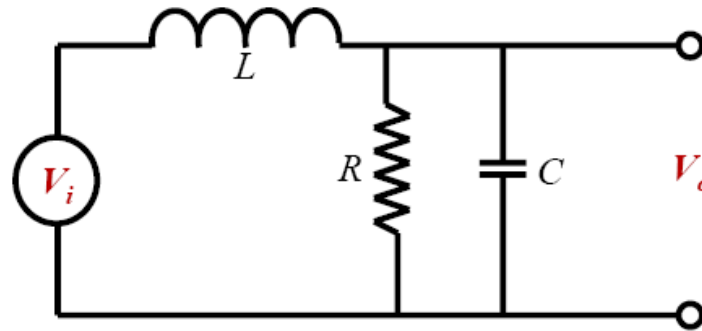
Transfer function :

$$H(s) = \frac{V_o}{V_i} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$



# Mechanical / Electrical Systems

Alternative circuit:



Input : voltage  $V_i$

Output : voltage  $V_o$

$$L\ddot{q}(t) + \frac{L}{RC}\dot{q}(t) + \frac{1}{C}q(t) = V_i$$

$L$  inductance,  $R$  resistance,  $C$  capacitance

Transfer function :

$$H(s) = \frac{V_o}{V_i} = \frac{\frac{1}{LC}}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

# Resonators

- Analogy between mechanical and electrical system:
  - Mass  $m$  - inductivity  $L$
  - Spring  $K$  - capacitance  $C$
  - Damping  $b$  - resistance  $R$  (depending where  $R$  is placed in circuit)
- Solution to 2nd order differential equation:

$$H(s) = \frac{\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

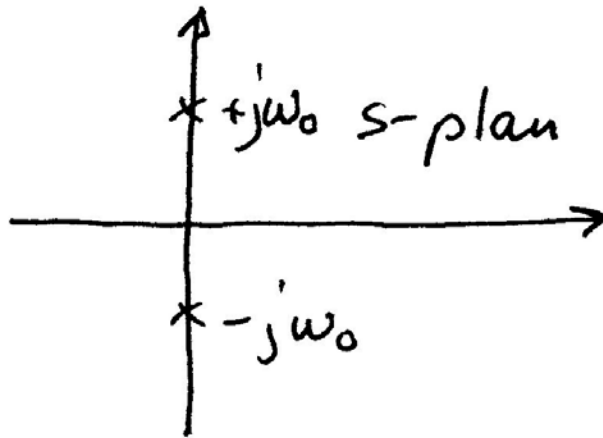
$$\omega_0 = 2\pi f_0 \text{ natural frequency}$$

$$\omega_0 = \sqrt{\frac{K}{m}} \text{ mechanical system, } \omega_0 = \sqrt{\frac{1}{LC}} \text{ electrical system}$$

$$Q \text{ quality factor}$$

# System without damping ( $b=0$ , $R=0$ )

$$H(s) = \frac{\omega_0^2}{s^2 + \omega_0^2} = \frac{\omega_0^2}{(s + j\omega_0)(s - j\omega_0)}$$



$$|H(j\omega_0)| = \infty$$

$$H(j\omega) = \frac{1}{1 - \left(\frac{\omega}{\omega_0}\right)^2}$$

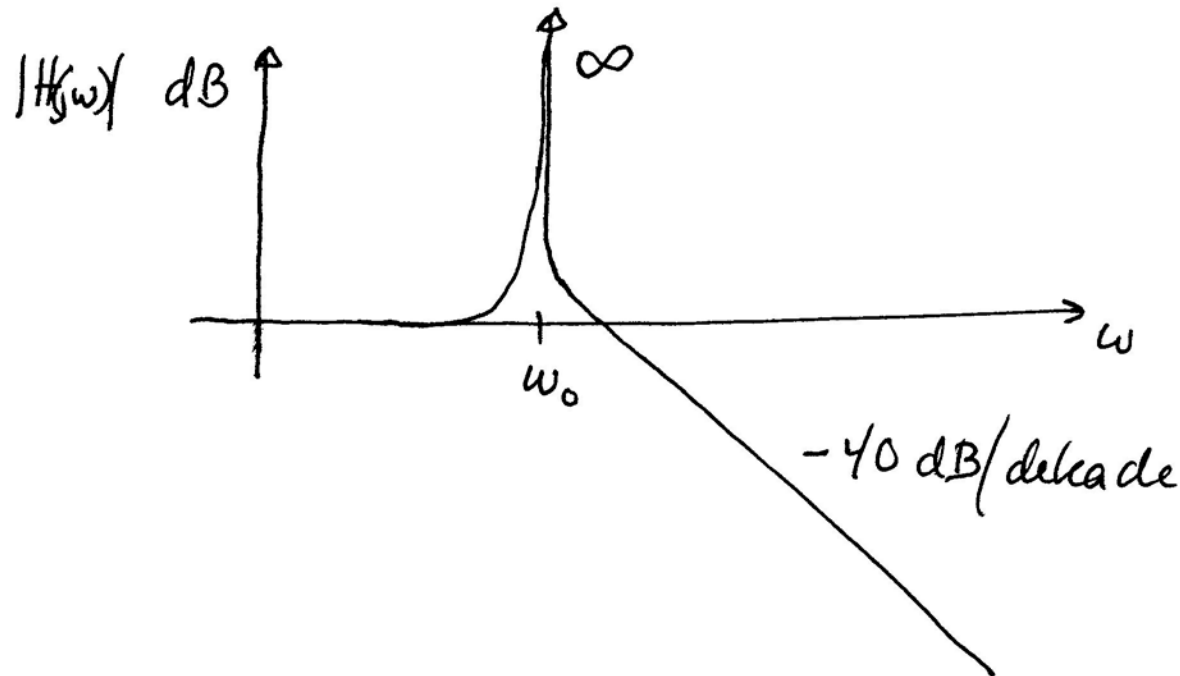
$$\omega_0 = \sqrt{\frac{1}{LC}}, \omega_0 = \sqrt{\frac{k}{m}}$$

# System without damping, contd.

$$H(j\omega) = \frac{1}{1 - \left(\frac{\omega}{\omega_0}\right)^2}$$

$$|H(j\omega)| = 1 \quad \text{near } \omega \ll \omega_0 \quad 0 \text{ dB}$$

$$|H(j\omega)| = -\left(\frac{\omega_0}{\omega}\right)^2 \quad \text{near } \omega \gg \omega_0 \quad -40 \text{ dB/decade}$$

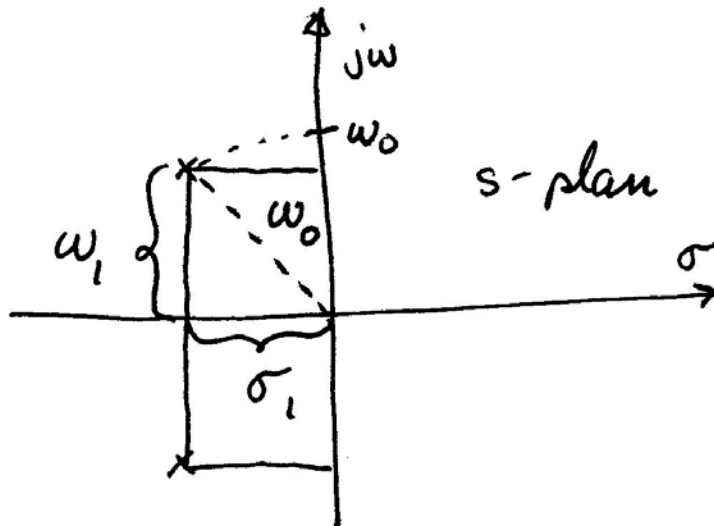


# With damping

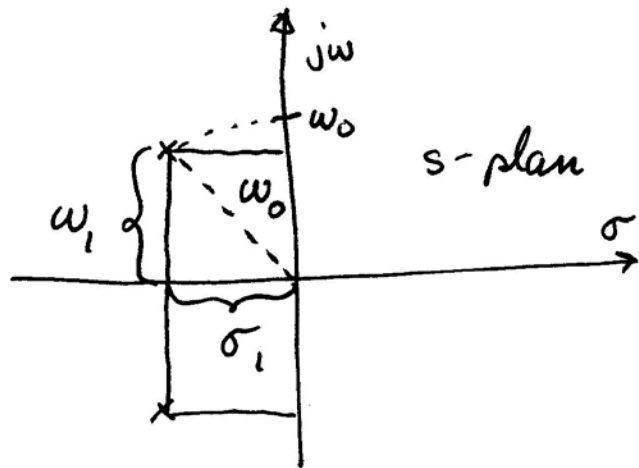
$$s^2 + \frac{\omega_0}{Q} s + \omega_0^2 = 0$$

$$s = -\frac{\omega_0}{2Q} \pm j \omega_0 \sqrt{1 - \frac{1}{4Q^2}}$$

$$= -\sigma_1 \pm j \omega_1$$



# Damped system, contd.



$$\frac{\omega_0}{Q} = \frac{b}{m} = \frac{1}{\tau}$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$\sigma_1 = \frac{1}{2\tau} = \frac{b}{2m}$$

$$\omega_1 = \omega_0 \sqrt{1 - \frac{1}{4\omega_0^2 \tau^2}} = \omega_0 \sqrt{1 - \frac{b^2}{4km}}$$

$$\omega_1^2 + \sigma_1^2 = \omega_0^2$$

# Mechanical Resonator

- Frequency and phase shift under damping:

$$x(t) = Ae^{-t/2\tau} \cos(\omega_1 t + \varphi)$$

$$\tau = m/b \text{ damping time}$$

$$\omega_1 = \omega_0 \sqrt{1 - \frac{1}{4\omega_0^2 \tau^2}} = \omega_0 \sqrt{1 - \frac{b^2}{4Km}}$$

$\varphi$  phase shift

- Energy dissipation:

$$E(t) = E_0 e^{-t/\tau}$$

# What is the meaning of "damping time"?

$\tau$  = damping time

$$e^{-t/2\tau} \Big|_{t=\tau} = e^{-1/2} = \frac{1}{\sqrt{e}}$$

Power

Effektum

$$|x(t)|^2 \Big|_{t=\tau} = \frac{1}{e}$$

$$x(t) = A e^{-t/2\tau} \cos(\omega_1 t + \varphi)$$

$$x(0) = A \cdot \cos \varphi \quad \begin{array}{l} \text{initialbedingungen} \\ \text{initial conditions} \end{array}$$



# Q-factor and damping time

Generell ligning

General equation

$$s^2 + \frac{\omega_0}{Q} s + \omega_0^2 = 0$$

$$\Rightarrow s^2 + \frac{1}{\tau} s + \omega_0^2 = 0$$

$$Q = \omega_0 \tau$$

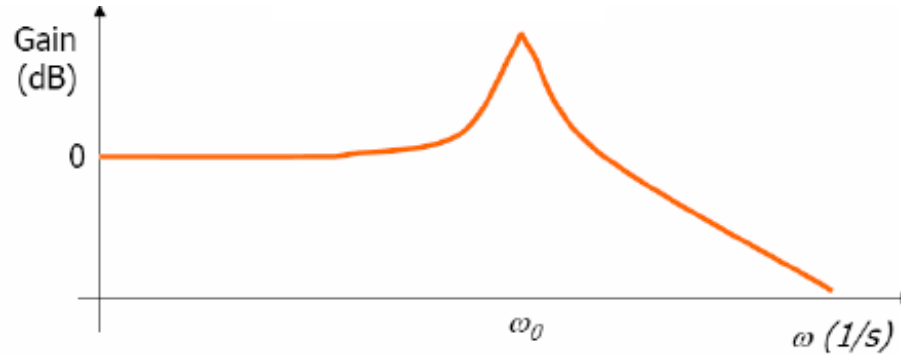
$$\tau = \frac{m}{b} \quad \begin{array}{l} \text{mekanisk} \\ \text{mechanical} \end{array}$$

$$\tau = \frac{L}{R} \quad \begin{array}{l} \text{elektrisk} \\ \text{electrical} \end{array}$$

$$Q_{\text{mek}} = \frac{\omega_0 m}{b}$$

$$Q_{\text{el}} = \frac{\omega_0 L}{R}$$

# Amplitude at resonance for forced vibrations



$$H(s) = \frac{\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

$$H(j\omega) = \frac{\omega_0^2}{(\omega_0^2 - \omega^2) + j \frac{\omega \omega_0}{Q}}$$

$$|H(j\omega_0)| = \left| \frac{\omega_0^2}{0 + j \frac{\omega_0^2}{Q}} \right| = Q$$