Numerical techniques for the bidomain model

A review

Explain some of the challenges in doing realistic simulations using the bidomain model.

- Highly complex system of PDEs and ODEs
- Difficult to construct efficient solution algorithms
- Simulation software tends to be complex and difficult to maintain

What is the advantage of using operator-splitting techniques to solve the bidomain equations?

Operator splitting splits the complex bidomain equation system into smaller parts that are easier to solve. The nonlinear system of PDEs and ODEs is reduced to a system of linear PDEs and nonlinear ODEs, which are solved sequentially.

Explain each step in the algorithms for Godunov and Strang splitting. You may use a simplified model problem instead of the bidomain equations.

We will use the initial value problem

$$\begin{aligned} \frac{du}{dt} &= \nabla^2 u + f(u), \\ u(0) &= u_0. \end{aligned}$$

With Godunov splitting, we first solve

$$\begin{aligned} \frac{dv}{dt} &= \nabla^2 v, \\ v(0) &= u_0 \end{aligned}$$

for $t \in [0, \Delta t]$. This gives us $v(\Delta t)$.

Next, we solve

$$\begin{aligned} \frac{dw}{dt} &= f(w), \\ w(0) &= v(\Delta t) \end{aligned}$$

for $t \in [0, \Delta t]$ to get $w(\Delta t)$ which we set equal to

$$\tilde{u}(\Delta t) = w(\Delta t)\,.$$

This is an approximation of the solution $u(\Delta t)$ to the original (unsplit) problem because both splitting and the numerical solution of the subproblems introduce error.

For Strang splitting, we have a 3-step algorithm. First we solve

$$\begin{aligned} \frac{dv}{dt} &= \nabla^2 v, \\ v(0) &= u_0 \end{aligned}$$

for $t \in [0, \Delta t/2]$ to get $v(\Delta t/2)$. Next, we solve

$$\begin{aligned} \frac{dw}{dt} &= f(w), \\ w(0) &= v(\Delta t/2) \end{aligned}$$

for $t \in [0, \Delta t]$ and get $w(\Delta t)$. Finally, we solve

$$\begin{aligned} \frac{dv}{dt} &= \nabla^2 v, \\ v(\Delta t/2) &= w(\Delta t) \end{aligned}$$

for $t \in [\Delta t/2, \Delta t]$ and set $\tilde{u}(\Delta t) = v(\Delta t)$.

The convergence results of Godunov and Strang splitting were based on exact solutions of each step. These steps are in reality solved numerically. What demands must the numerical solutions fulfill in order for the convergence results to hold?

To obtain the overall second-order accuracy of Strang splitting, the ODE systems in steps 1 and 3 and the PDE system is step 2 must be solved with at least second-order accuracy. The overall accuracy is limited by the splitting error, so solving the subproblems with a higher order of accuracy will be of no benefit.

Mention some of the challenges for solving the cell model equations effectively.

- The spatial discretization of the bidomain equations results in a cell model ODE system for each node in the finite element grid--several million nodes for realistic simulations.

- The complexity of the cell model can vary. The FitzHugh-Nagumo model has 2 variables, more realistic ones have more; for example, the Winslow model has 31.

- Cell model equations tend to be “stiff”, which makes them challenging to solve numerically.

- Explicit methods, which are easy to implement, are not suitable for stiff problems.

- Implicit methods are more suitable, but more difficult to implement.

Give advantages and disadvantages of the implicit ODE solvers (compared to the explicit solvers).

- Disadvantages
  - More difficult to implement
  - Require the solution of a system of non-linear equations
  - More expensive per step
- Advantages
  - Better stability properties
  - Allow for larger time steps
  - More efficient for stiff problems