Models for the circulatory system

Outline

- Overview of the circulatory system
- Important quantities
- Resistance and compliance vessels
- Models for the circulatory system
- Examples and extensions

Important quantities

- Heart rate, measured in beats per minute.
- Cardiac output: The rate of blood flow through the circulatory system, measured in liters/minute.
- Stroke volume: the difference between the end-diastolic volume and the end-systolic volume, i.e. the volume of blood ejected from the heart during a heart beat, measured in liters.

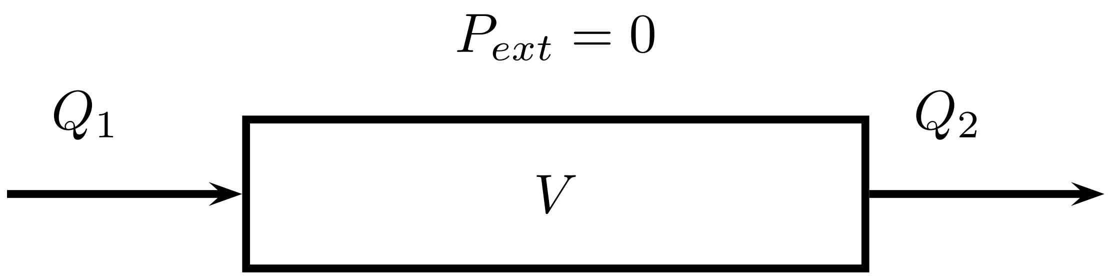
The cardiac ouput Q is given by

$$Q = FV_{stroke}$$

Typical values:

- $F = 80$ beats/minute.
- $V_{stroke} = 70\text{cm}^3/\text{beat}$ = 0.070 liters/beat.
- $Q = 5.6$ liters/minute.

Resistance and compliance vessels



- V = vessel volume,
- P_{ext} = external pressure,
- P_1 = upstream pressure,
- P_2 = downstream pressure,
- Q_1 = inflow,
- Q_2 = outflow.

Resistance vessels

Assume that the vessel is rigid, so that V is constant. Then we have

$$Q_1 = Q_2 = Q_*.$$

The flow through the vessel will depend on the pressure drop through the vessel. The simplest assumption is that Q_* is a linear function of the pressure difference $P_1 - P_2$:

$$Q_* = \frac{P_1 - P_2}{R},$$

where R is the resistance of the vessel.

Compliance vessels

Assume that the resistance over the vessel is negligible. This gives

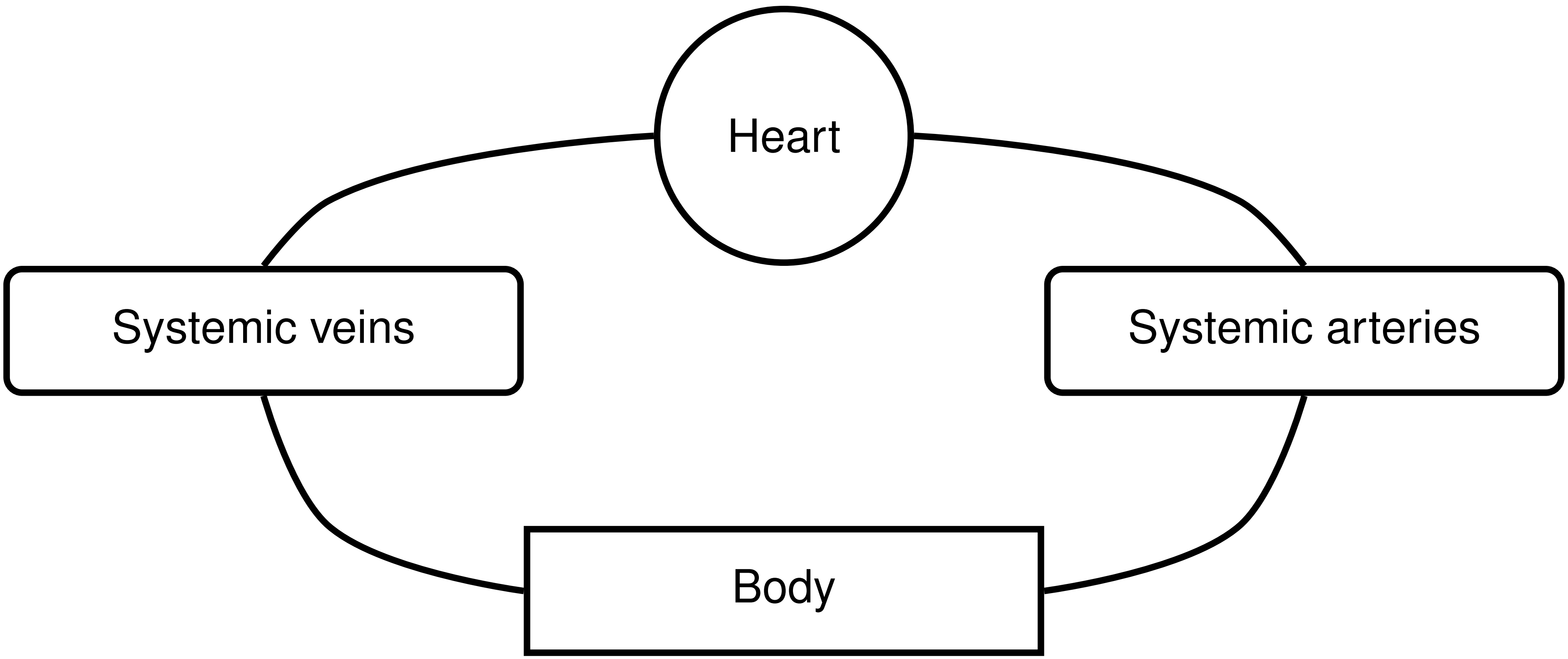
$$P_1 = P_2 = P_*$$

Assume further that the volume depends on the pressure P_*. We assume the simple linear relation

$$V = V_d + CP_*,$$

where C is the compliance of the vessel and V_d is the “dead volume”, the volume at $P_* = 0$.

- All blood vessels can be viewed as either resistance vessels or compliance vessels. (This is a reasonable assumption, although all vessels have both compliance and resistance.)
- Large arteries and veins; negligible resistance, significant compliance.
- Arterioles and capillaries; negligible compliance, significant resistance.


Heart
Systemic veins
Systemic arteries
Body

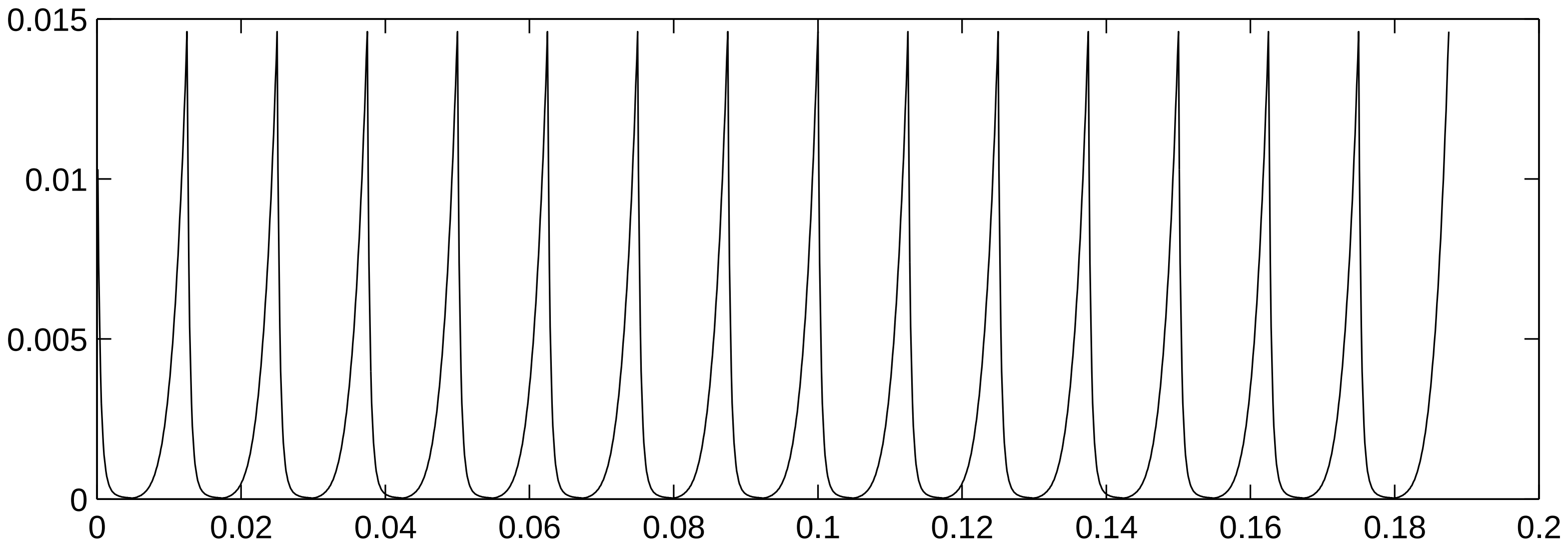
The heart as a compliance vessel

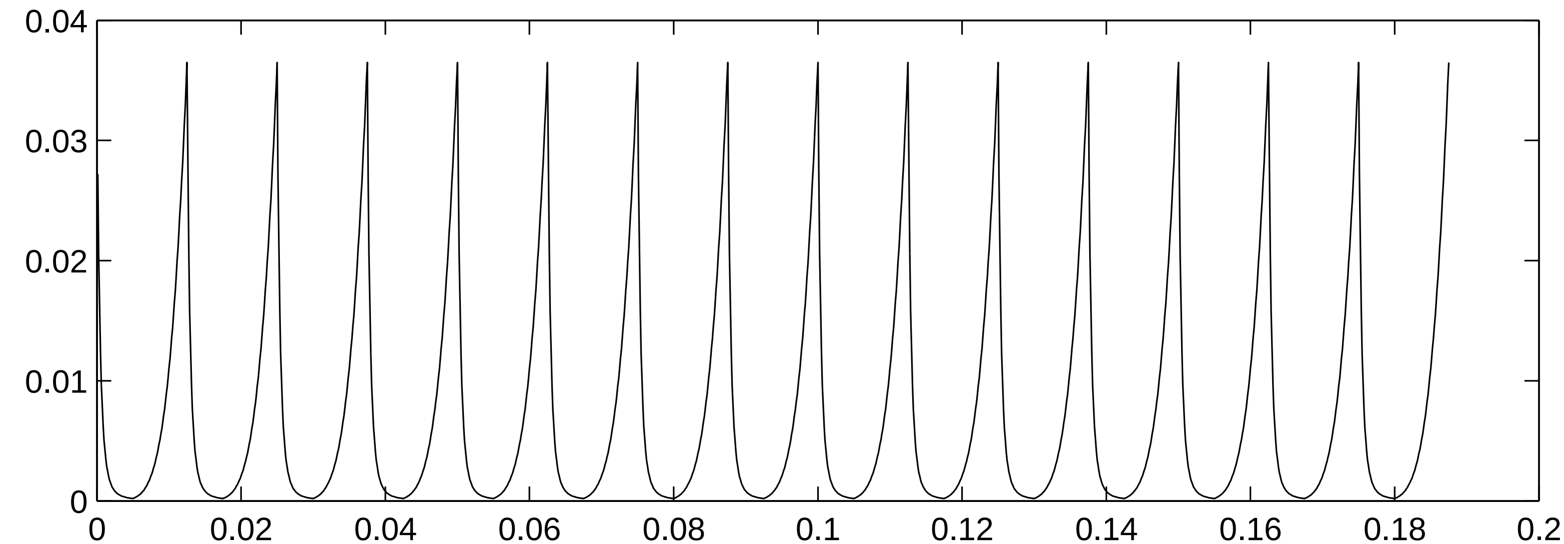
The heart may be viewed as a pair of compliance vessels, where the compliance changes with time,

$$V(t) = V_d + C(t)P.$$

The function $V(t)$ should be specified so that it takes on a large value $C_{diastole}$ when the heart is relaxed, and a small value $C_{systole}$ when the heart contracts.

Periodic functions for LV and RV compliance





Modeling the heart valves

Characteristic properties of a heart valve:

- Low resistance for flow in the “forward” direction.
- High resistance for flow in the “backward” direction.

The operation of the valve can be seen as a switching function that depends on the pressure difference across the valve. The switching function can be expressed as

$$S = \begin{cases} 1 & \text{if} P_1 > P_2 \\ 0 & \text{if} P_1 < P_2 \end{cases}$$

- The flow through the valve can be modeled as flow through a resistance vessel multiplied by the switching function. We have

$$Q_* = \frac{(P_1 - P_2)S}{R},$$

where R will typically be very low for a healthy valve.

- Inserting for S, we have

$$Q_* = \max((P_1 - P_2)/R, 0).$$

Dynamics of the arterial pulse

For a compliance vessel that is not in steady state, we have

$$\frac{dV}{dt} = Q_1 - Q_2.$$

From the pressure-volume relation for a compliance vessel we get

$$\frac{d(CP)}{dt} = Q_1 - Q_2.$$

When C is constant (which it is for every vessel except for the heart muscle itself) we have

$$C\frac{dP}{dt} = Q_1 - Q_2.$$

The circulatory system can be viewed as a set of compliance vessels connected by valves and resistance vessels. For each compliance vessel we have

$$\frac{d(C_i P_i)}{dt} = Q_i^{in} - Q_i^{out},$$

while the flows in the resistance vessels follow the relation

$$Q_j = \frac{P^{in} - P^{out}}{R_j}.$$

A simple model for the circulatory system

Consider first a simple model consisting of three compliance vessels; the left ventricle, the systemic arteries, and the systemic veins. These are connected by two valves, and a resistance vessel describing the flow through the systemic tissues. For the left ventricle we have

$$\frac{d(C(t)P_{lv})}{dt} = Q^{in} - Q^{out},$$

with Q^{in} and Q^{out} given by

$$Q_{in} = \max((P_{sv} - P_{lv})/R_{mi}, 0), \quad (1)$$

$$Q_{out} = \max((P_{lv} - P_{sa})/R_{ao}, 0). \quad (2)$$

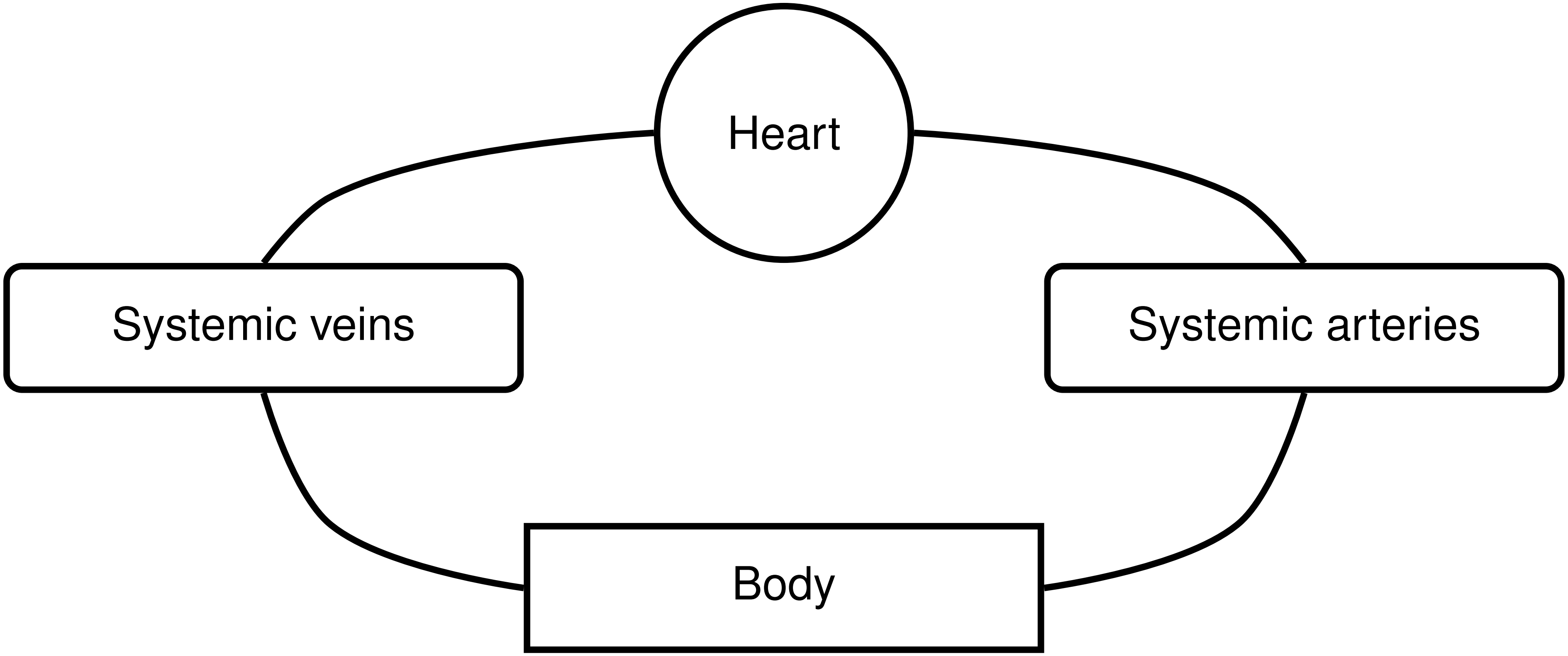
- We get

$$\frac{d(C(t)P_{lv})}{dt} = \max((P_{sv} - P_{lv})/R_{mi}, 0) - \max((P_{lv} - P_{sa})/R_{ao}, 0)$$

- Note: this is an ODE with discontinuous right hand side (if-tests), which is not ideal from a numerical viewpoint. Implicit solvers must be used, and the order of convergence is limited to 1.

Similar calculations for the two other compliance vessels gives the system

$$\begin{aligned}
\frac{d(C(t)P_{lv})}{dt} =& \max((P_{sv} - P_{lv})/R_{mi}, 0) \\
& - \max((P_{lv} - P_{sa})/R_{ao}, 0), \\
C_{sa}\frac{dP_{sa}}{dt} =& \max((P_{lv} - P_{sa})/R_{ao}, 0) - \frac{P_{sa} - P_{sv}}{R_{sys}}, \\
C_{sv}\frac{dP_{sv}}{dt} =& \frac{P_{sa} - P_{sv}}{R_{sys}} - \max((P_{sv} - P_{lv})/R_{mi}, 0).
\end{aligned}$$


Heart
Systemic veins
Systemic arteries
Body

With a specification of the parameters $R_{mi}, R_{ao}, R_{sys}, C_{sa}, C_{sv}$ and the function $C_{lv}(t)$, this is a system of ordinary differential equations that can be solved for the unknown pressures P_{lv}, P_{sa}, and P_{sv}. When the pressures are determined they can be used to compute volumes and flows in the system.

A more realistic model

The model can easily be improved to a more realistic model describing six compliance vessels:

- The left ventricle, $P_{lv}, C_{lv}(t)$,
- the right ventricle, $P_{rv}, C_{rv}(t)$,
- the systemic arteries, P_{sa}, C_{sa},
- the systemic veins, P_{sv}, C_{sv},
- the pulmonary arteries, and P_{pv}, C_{pv},
- the pulmonary veins, P_{pv}, C_{pv}.

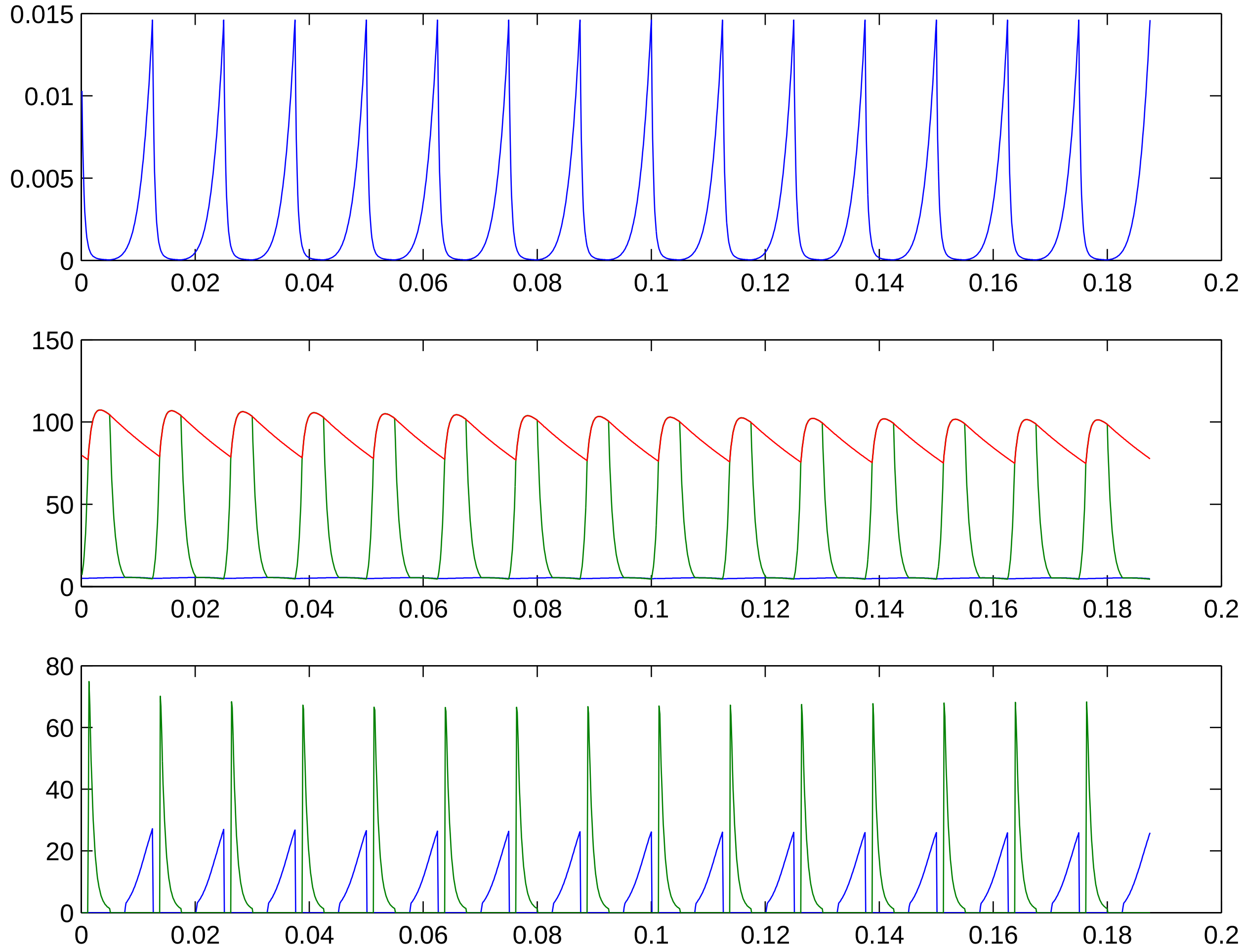
The flows are governed by two resistance vessels and four valves:

- Systemic circulation; R_{sys},
- pulmonary circulation; R_{pu},
- aortic valve (left ventricle to systemic arteries); R_{ao}
- tricuspid valve (systemic veins to right ventricle); R_{tri},
- pulmonary valve (right ventricle to pulmonary arteries); R_{puv},
- mitral valve (pulmonary veins to left ventricle); R_{mi},

This gives the ODE system

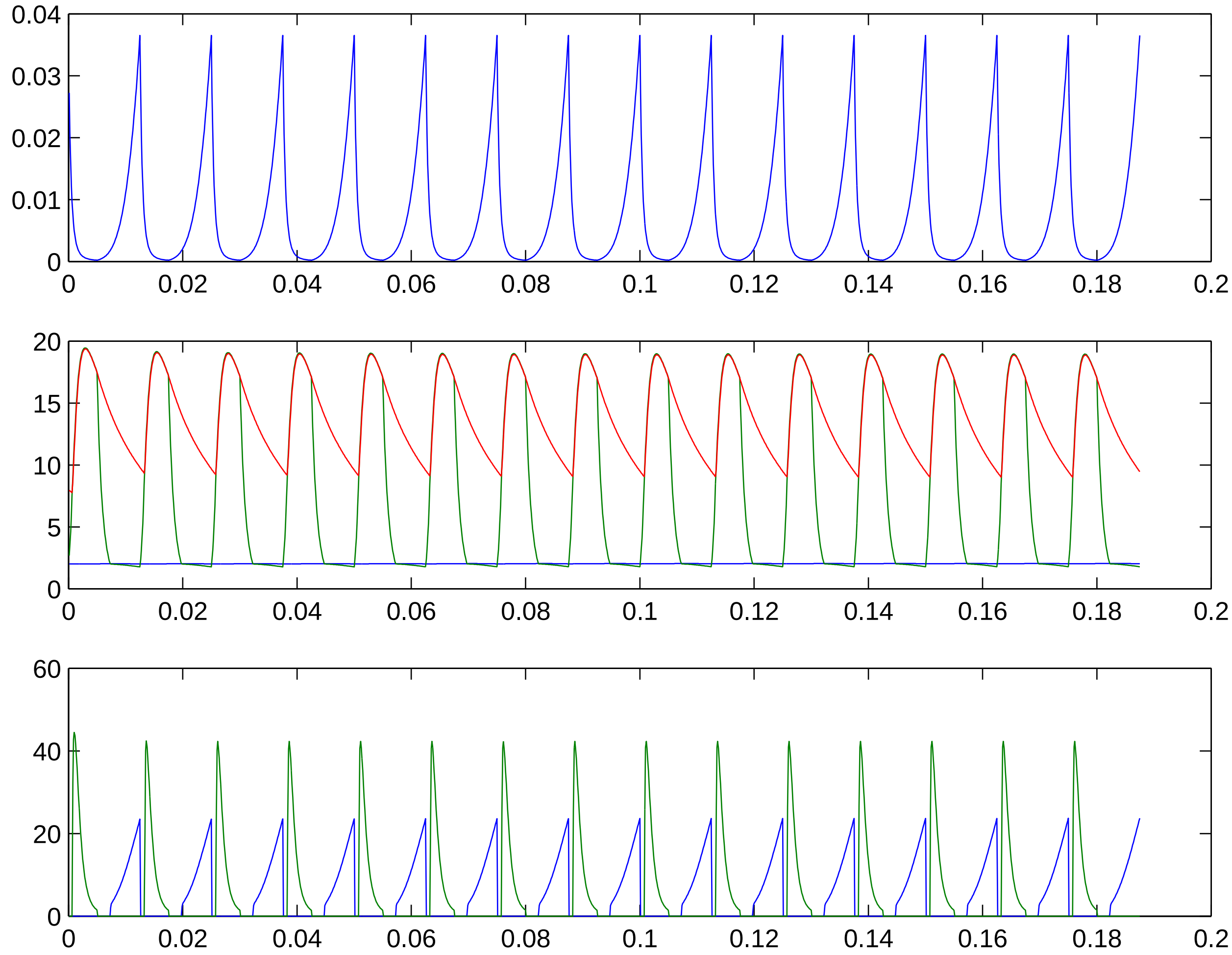
$$
\begin{aligned}
\frac{d(C_{lv}(t)P_{lv})}{dt} &= \max((P_{pv} - P_{lv})/R_{mi}, 0) - \max((P_{lv} - P_{sa})/R_{ao}, 0) \\
\frac{dC_{sa}P_{sa}}{dt} &= \max((P_{lv} - P_{sa})/R_{ao}, 0) - \frac{P_{sa} - P_{sv}}{R_{sys}}, \\
\frac{dC_{sv}P_{sv}}{dt} &= \frac{P_{sa} - P_{sv}}{R_{sys}} - \max((P_{sv} - P_{rv})/R_{tri}, 0), \\
\frac{d(C_{rv}(t)P_{rv})}{dt} &= \max((P_{sv} - P_{rv})/R_{tri}, 0) - \max((P_{rv} - P_{pa})/R_{puv}, 0), \\
\frac{d(C_{pa}P_{pa})}{dt} &= \max((P_{rv} - P_{pa})/R_{puv}, 0) - \frac{P_{pa} - P_{pv}}{R_{pu}}, \\
\frac{dC_{pv}P_{pv}}{dt} &= \frac{P_{pa} - P_{pv}}{R_{pu}} - \max((P_{pv} - P_{lv})/R_{mi}, 0).
\end{aligned}
$$

LV compliance, pressures and flows



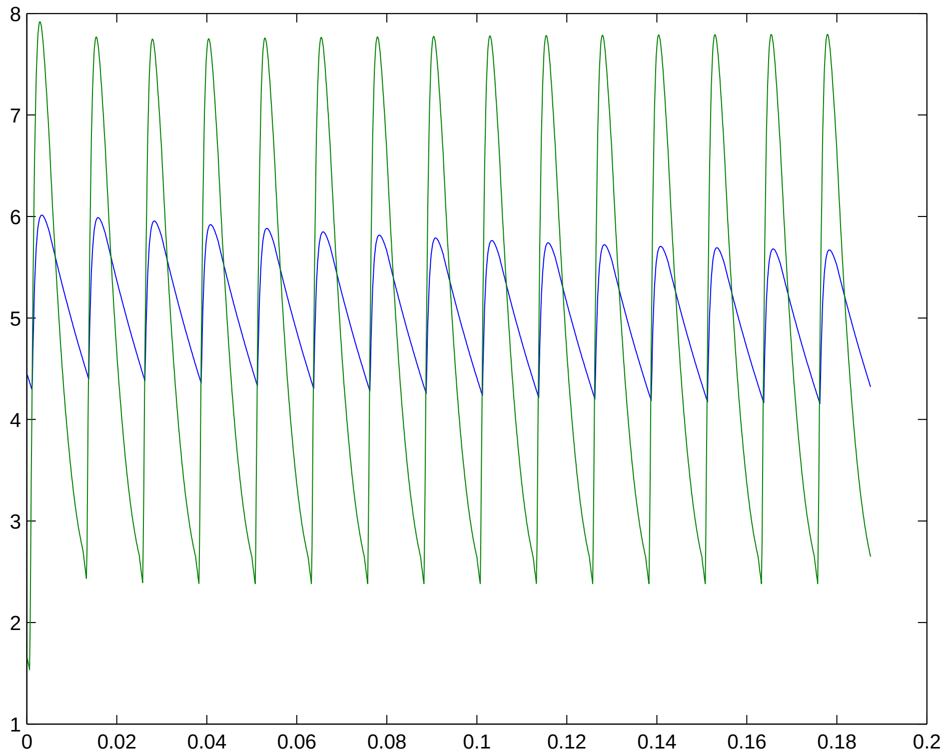
Top: C_lv, Middle: P_{pv} (blue), P_{lv} (green), P_{sa} (red), Bottom: Q_{mi}, Q_{ao}.

RV compliance, pressures and flows



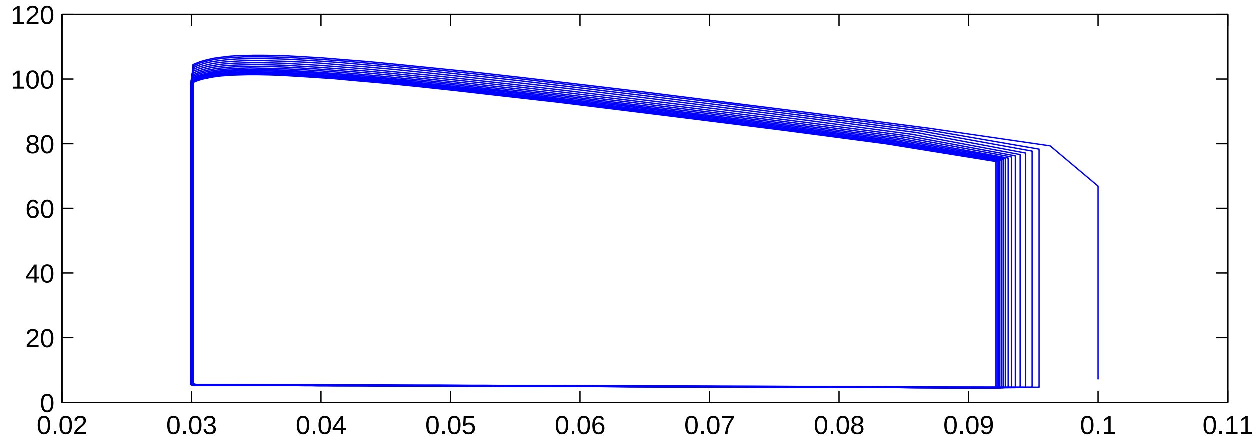
Top: C_{rv}, Middle: P_{sv} (blue), P_{rv} (green), P_{pa} (red), Bottom: Q_{tri}, Q_{puv}.

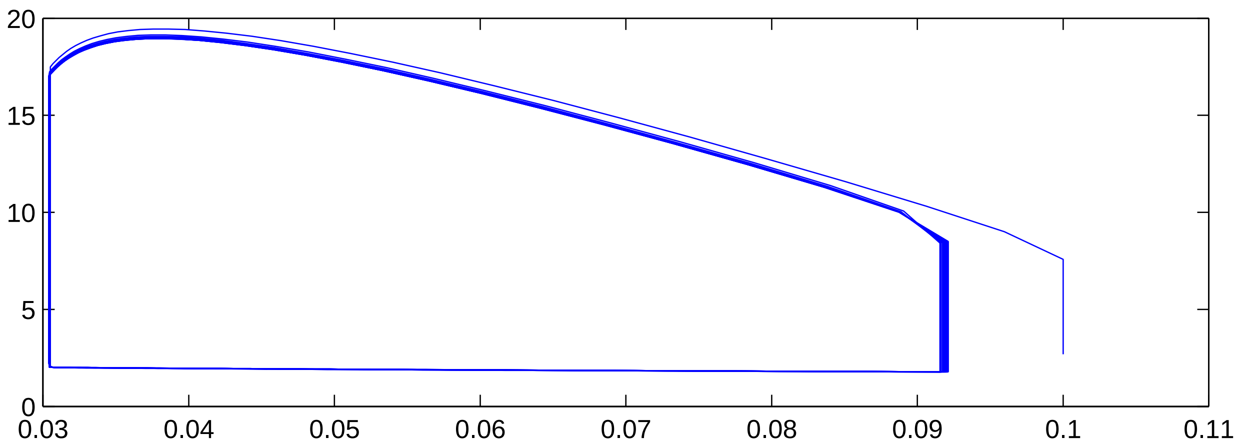
Systemic and pulmonary flows



Q_{sys} (blue) and Q_{pu} (green). Note the higher maximum flow in the pulmonary system despite the lower pressure. This is caused by the low resistance in the pulmonaries.

Pressure volume loops



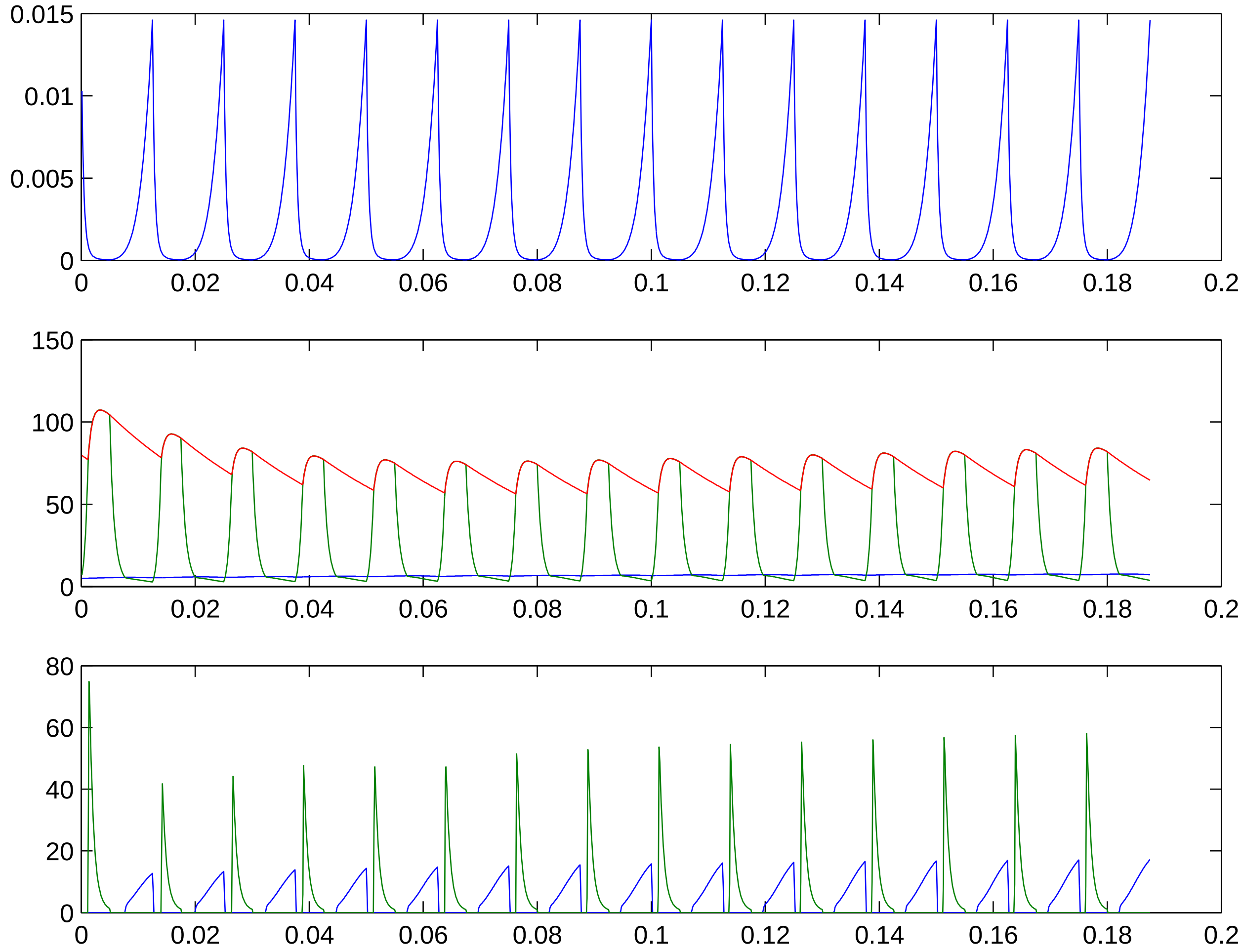


Top: left ventricle, bottom: right ventricle.

Mitral valve stenosis

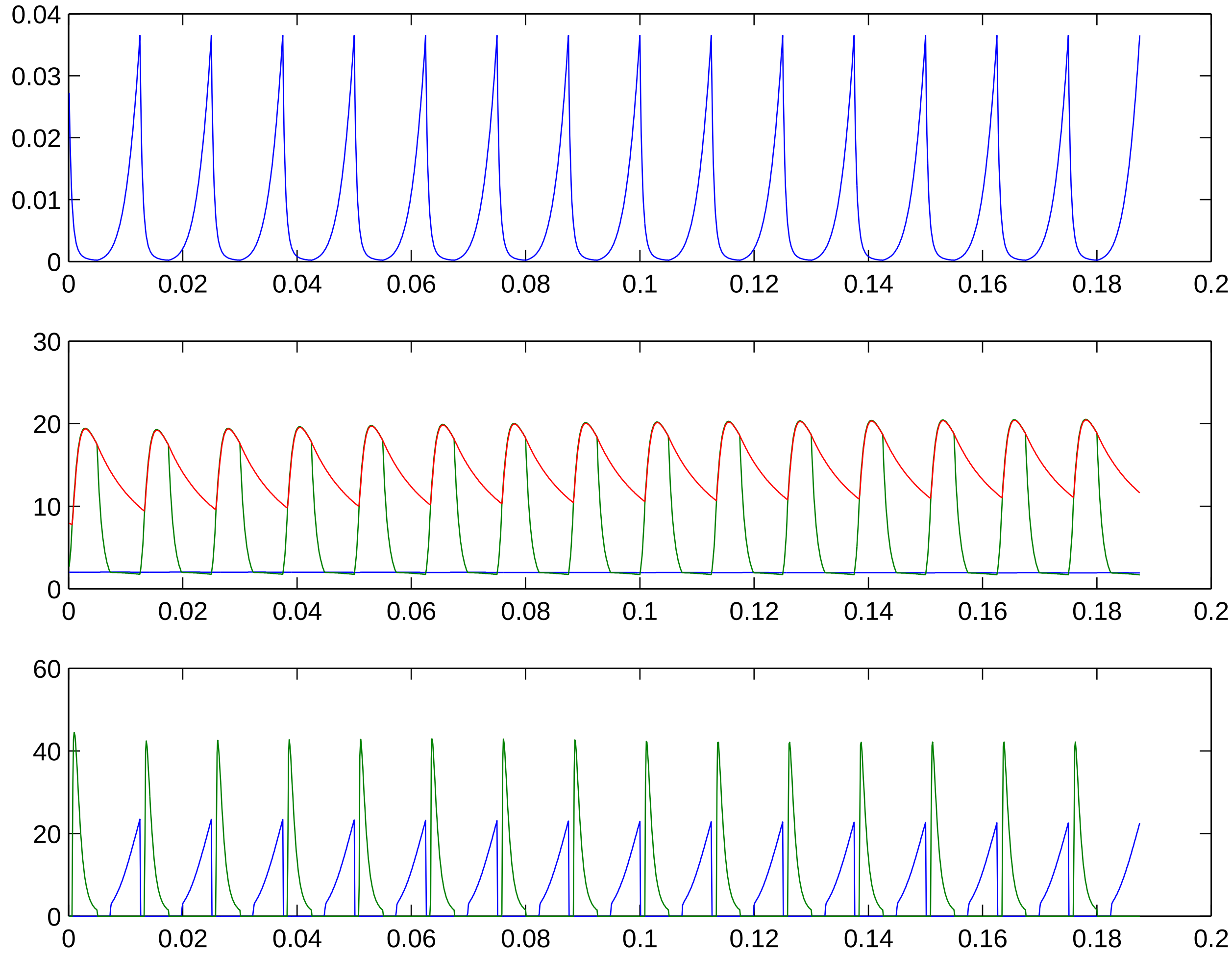
- R_{mi} changes from 0.01 to 0.2.

LV compliance, pressures and flows



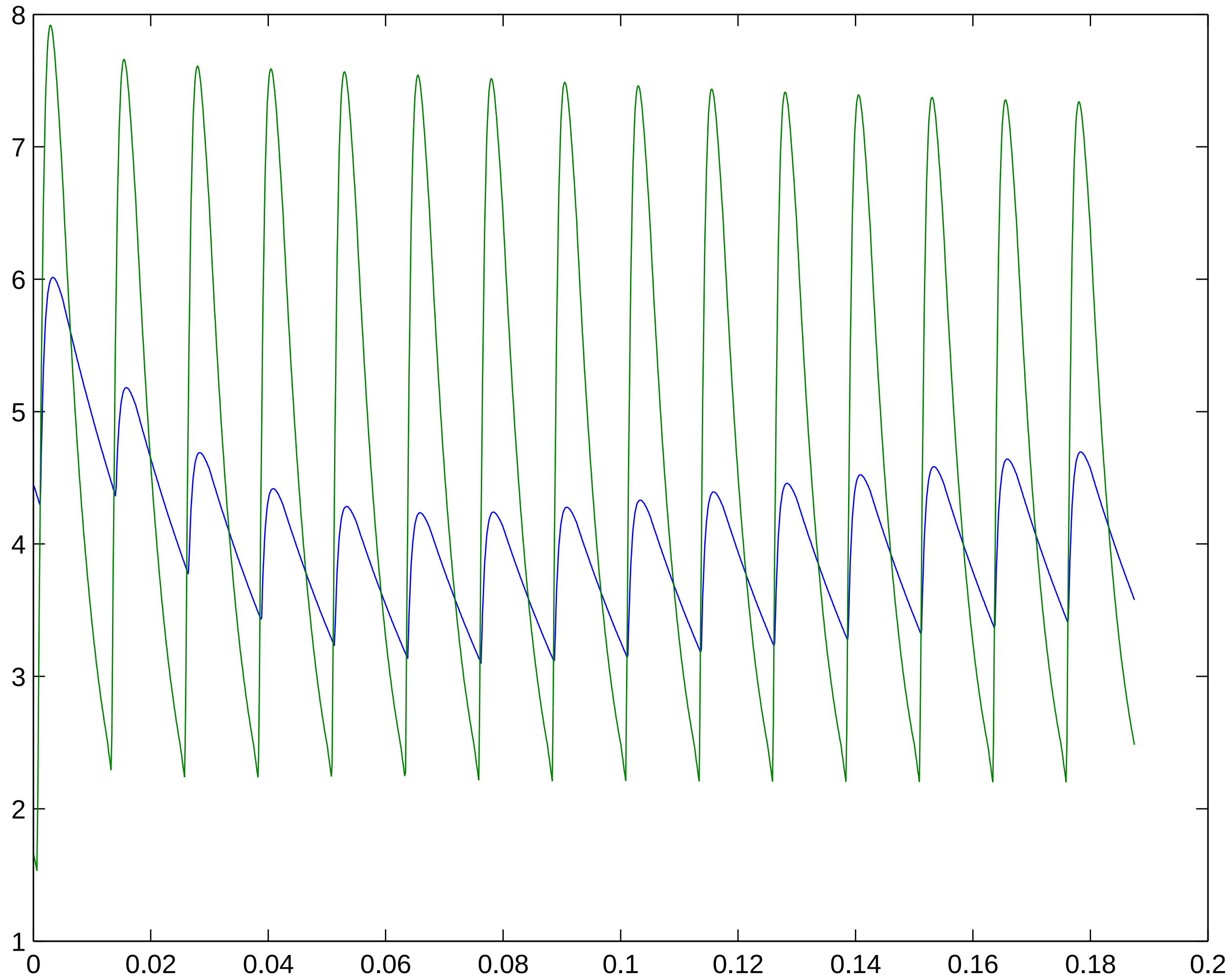
Reduced in-flow to the LV causes reduced filling and thereby reduced LV pressure and arterial pressure.

RV compliance, pressures and flows



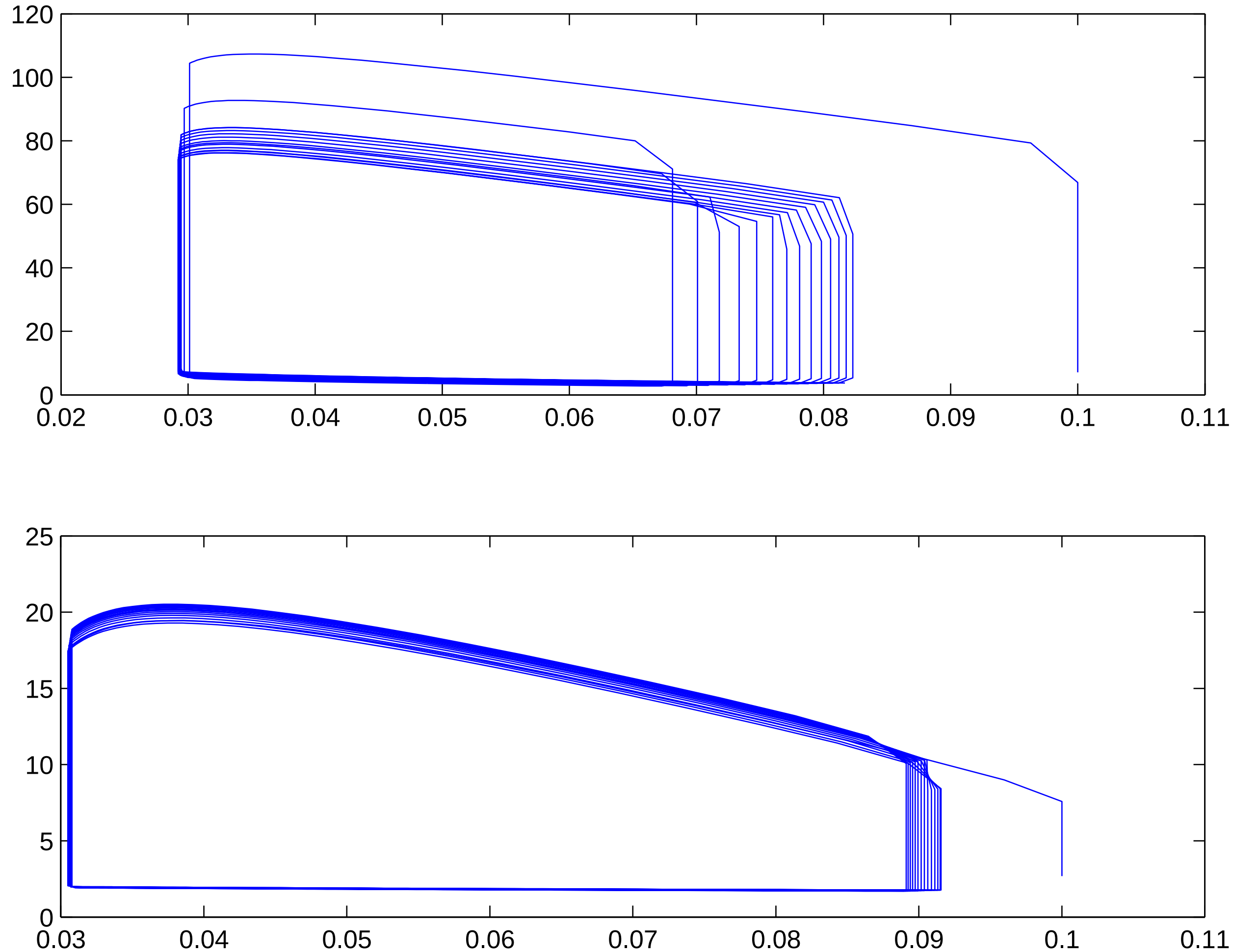
The RV pressure increases because blood is shifted from the systemic circulation to the pulmonary circulation.

Systemic and pulmonary flows



Blood is shifted from the systemic circulation to the pulmonary circulation.

Pressure volume loops

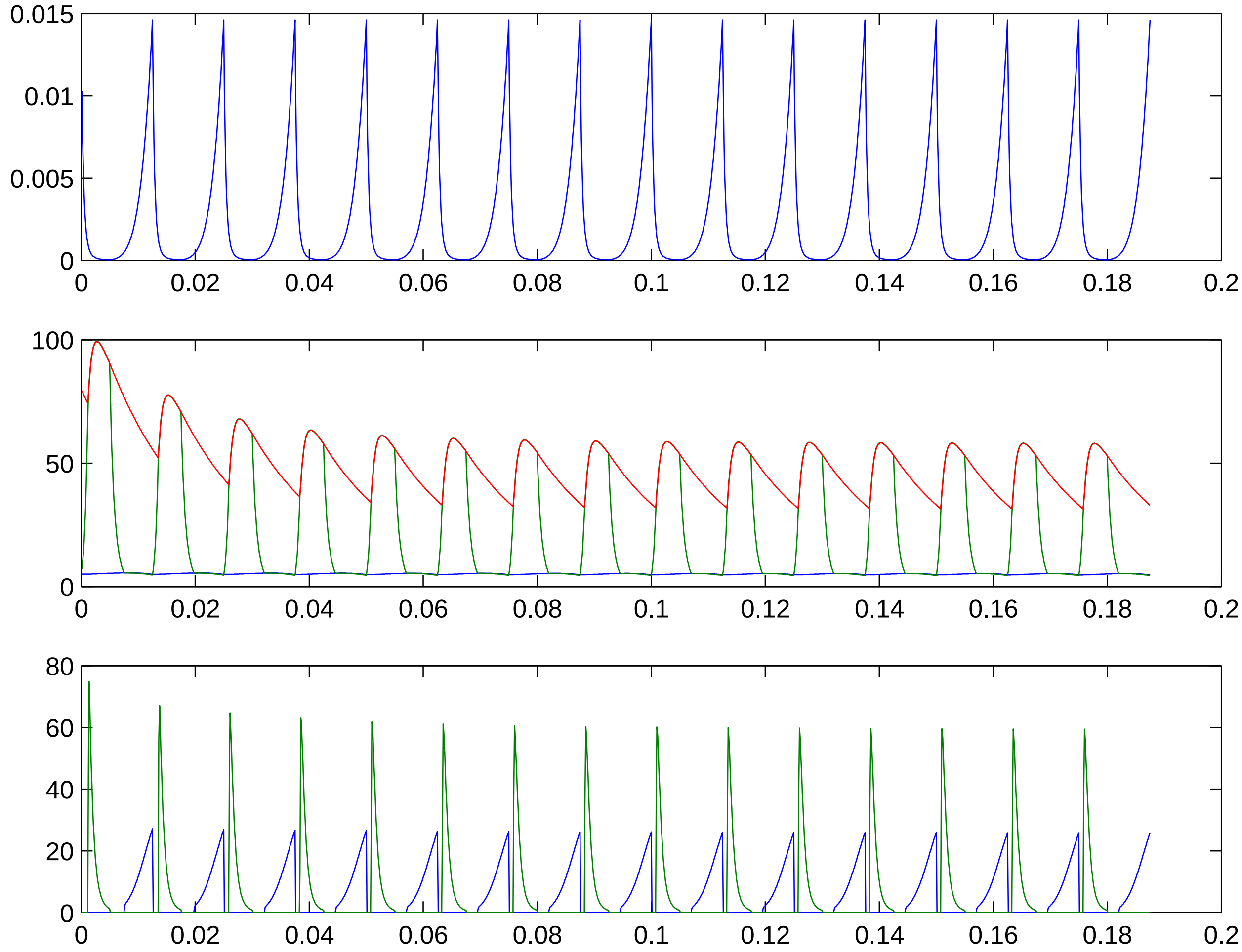


Reduced filling of the LV, slightly higher pressure in the RV.

Reduced systemic resistance

- R_{sys} reduced from 17.5 to 8.5.
- This can be the result of for instance physical activity, when smooth muscle in the circulatory system reduce the resistance to increase blood flow to certain muscles.

LV compliance, pressures and flows



The arterial pressure drops dramatically. This is not consistent with what happens during physical activity.

Summary

- Models for the circulatory system can be constructed from very simple components.
- The models are remarkably realistic, but the simple model presented here has some important limitations.
- The models may be extended to include feedback loops through the nervous system.
- The simple components of the model can be replaced by more advanced models. For instance, the varying compliance model for the heart may be replaced by a bidomain and mechanics solver that relates pressures and volumes.