# Lecture 1 – Introduction to cryptography

**TEK4500**

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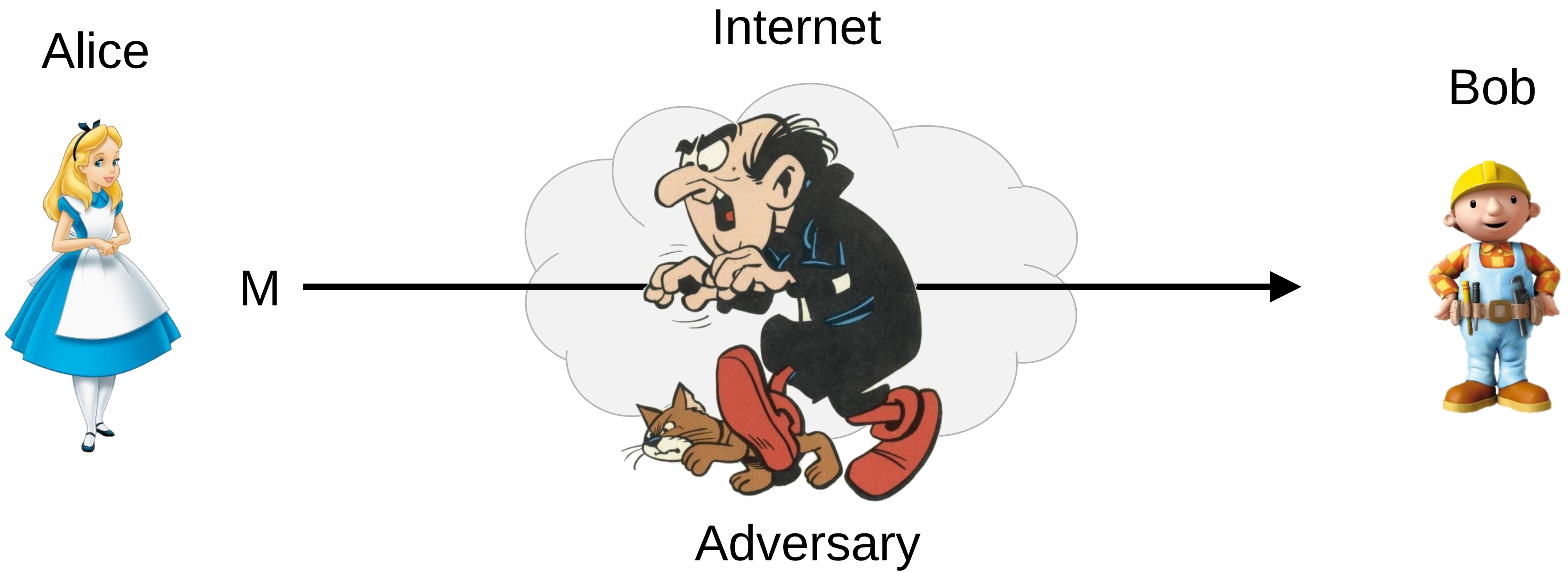
# What is cryptography?



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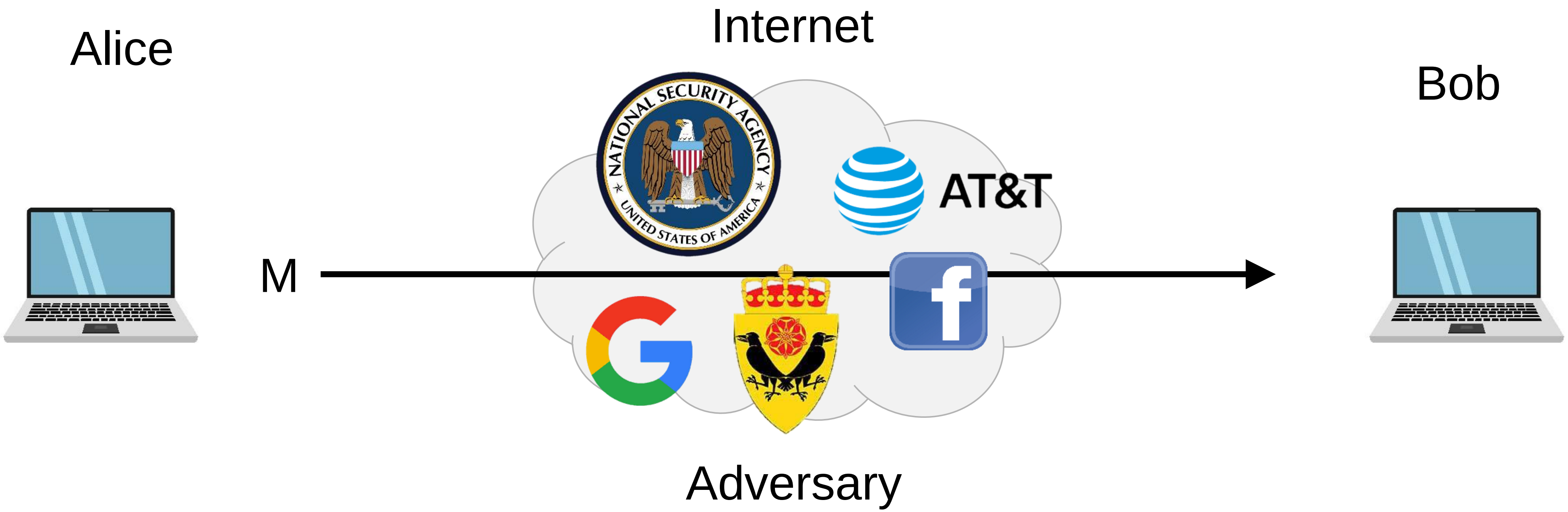

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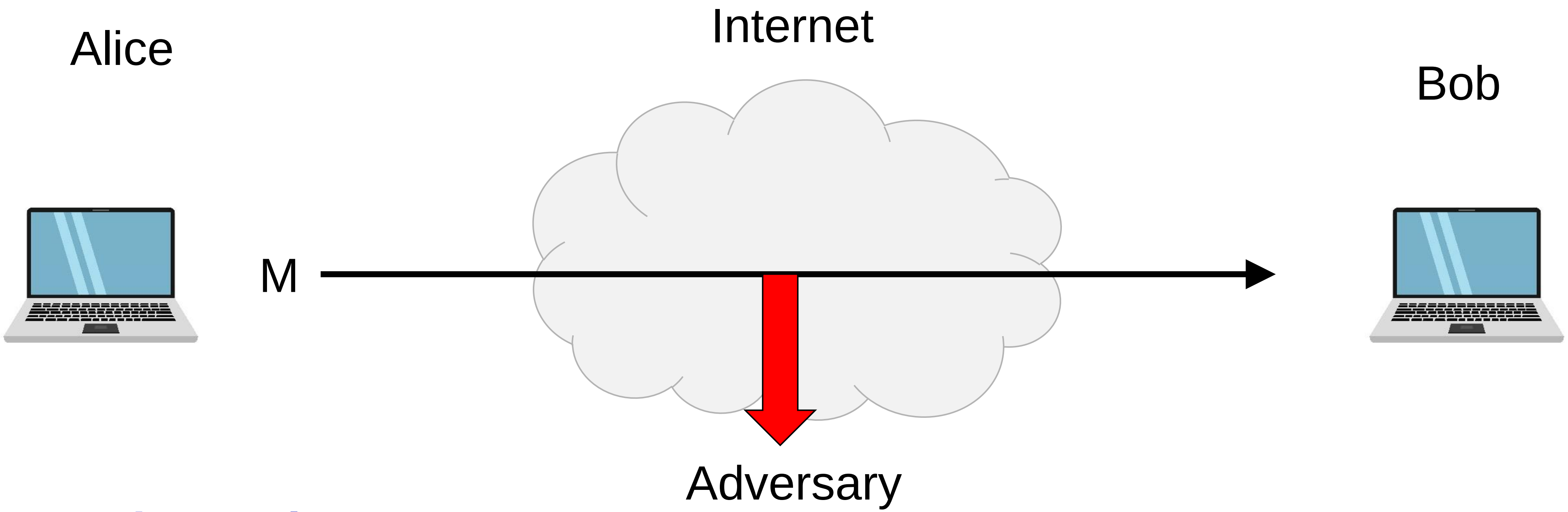
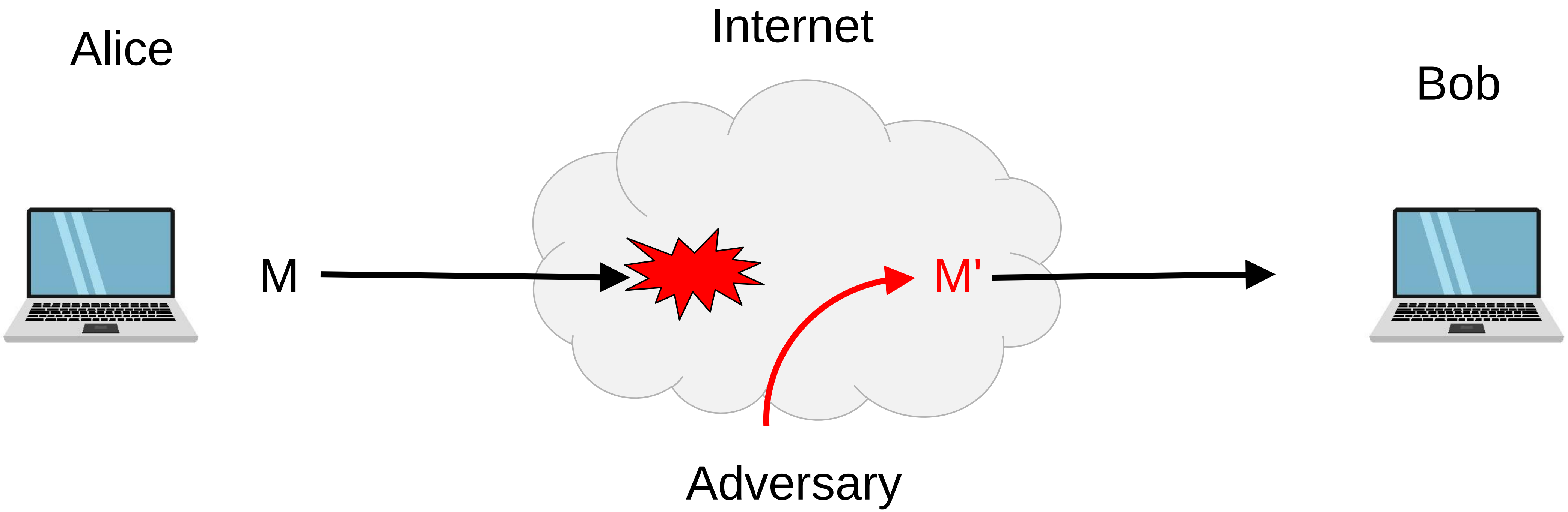



**Security goals:**

- **Data privacy:** adversary should not be able to read message M

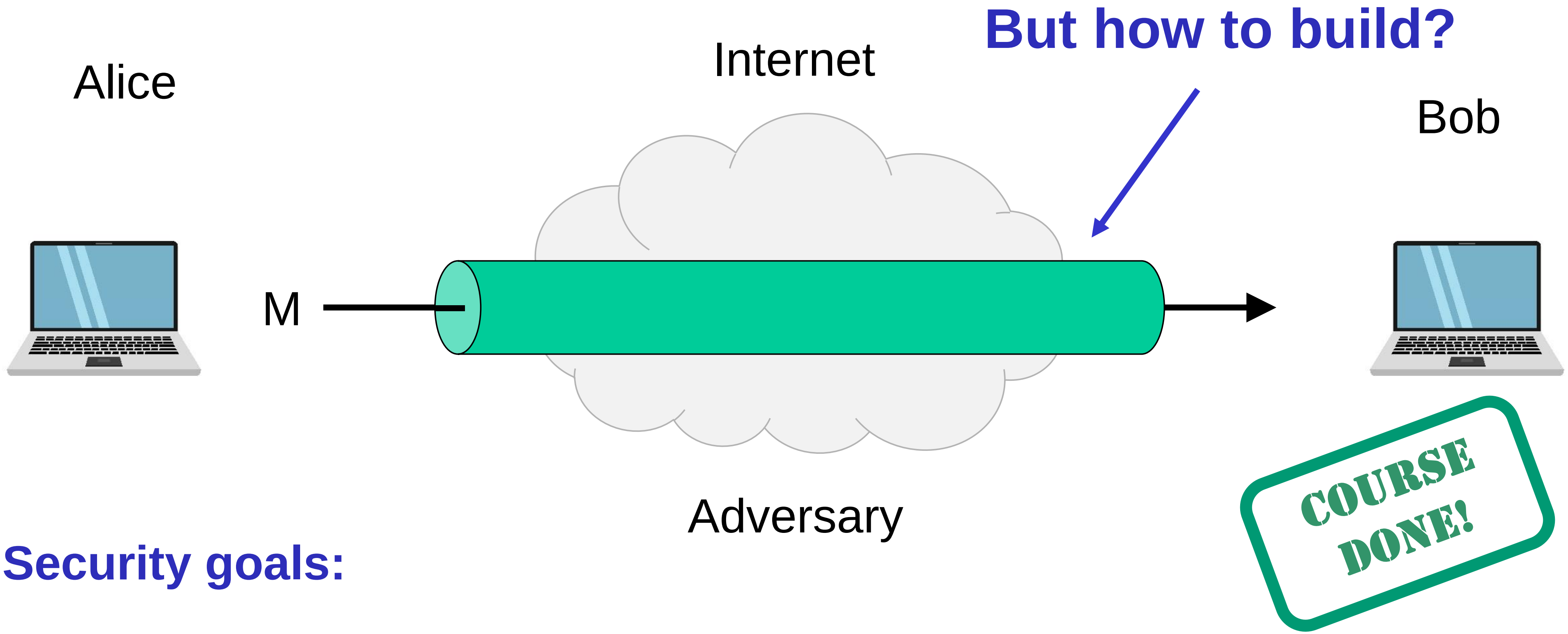
# What is cryptography?




**Security goals:**

- **Data privacy:** adversary should not be able to read message M
- **Data integrity:** adversary should not be able to modify message M
- **Data authenticity:** message M really originated from Alice

# Ideal solution: secure channels




**Security goals:**

- **Data privacy:** adversary should not be able to read message M ✓
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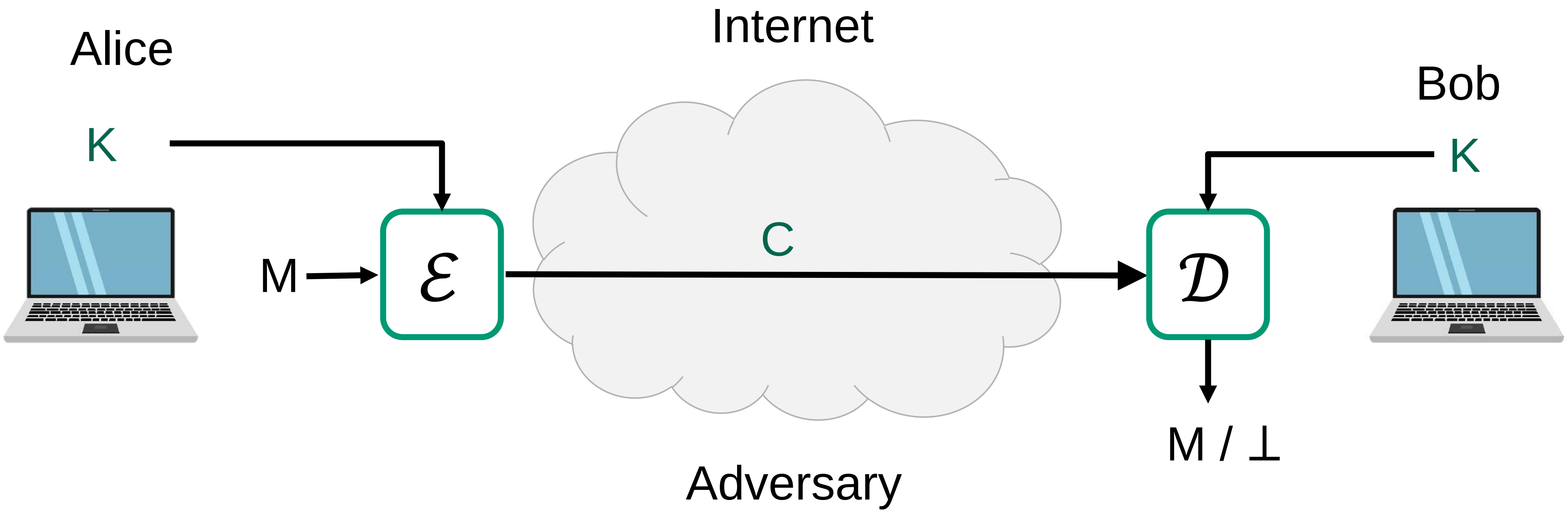
- **Data authenticity:** message M really originated from Alice ✓

# Creating secure channels: encryption schemes

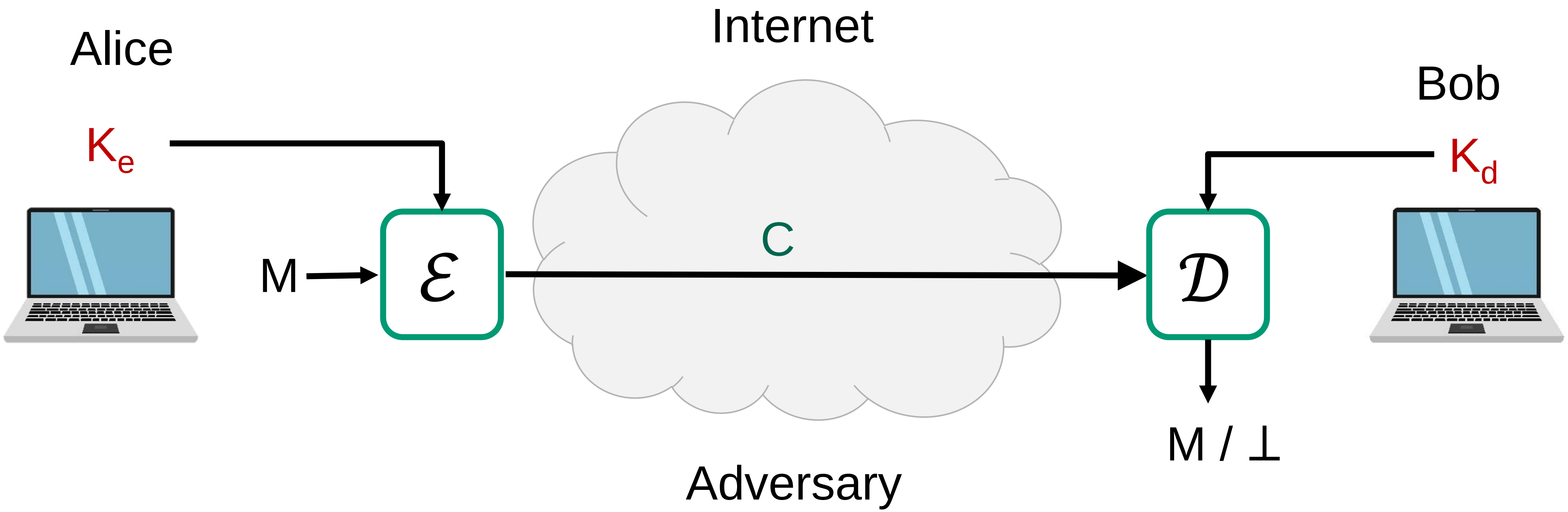



$\mathcal{E}$ : encryption algorithm (public)

**K :** encryption / decryption key (secret)

$\mathcal{D}$ : decryption algorithm (public)

# Creating secure channels: encryption schemes




$\mathcal{E}$ : encryption algorithm (public)

$\mathcal{D}$ : decryption algorithm (public)

$K_e$ : encryption key (public)

$K_d$ : decryption key (secret)

# Basic goals of cryptography

| | **Message privacy** | **Message integrity / authentication** |
|---|---|---|
| **Symmetric keys** | Symmetric encryption | Message authentication codes (MAC) |
| **Asymmetric keys** | Asymmetric encryption (a.k.a. public-key encryption) | Digital signatures |

# Basic goals of cryptography

| | Message privacy | Message integrity / authentication |
|---|---|---|
| **Symmetric keys** | Symmetric encryption | Message authentication codes (MAC) |
| **Asymmetric keys** | Asymmetric encryption (a.k.a. public-key encryption) | Digital signatures |

# Some notation

- $\in$ – "element in"
  - $3 \in \{1,2,3,4,5\}$
  - $7 \notin \{1,2,3,4,5\}$
- $\{0,1\}^n$ – set of all bitstrings of length $n$
  - $000, 010, 110 \in \{0,1\}^3$
- $\{0,1\}^*$ – set of all bitstrings of *finite* length
  - $1, 1001, 10, 10001101000001 \in \{0,1\}^*$
- $F : \mathcal{X} \rightarrow \mathcal{Y}$ – function from set $\mathcal{X}$ to set $\mathcal{Y}$
  - $F : \{0,1\}^5 \rightarrow \{0,1\}^3$
  - $G : \{A, B, C, D\} \rightarrow \{0,1,2, \ldots\}$

- $\forall$ – "for all"
  - "$\forall X \in \{0,1\}^4$ ... " = "for all bitstrings of length 4…"

**…independent, and uniformily distributed…**

- $X \leftarrow 5$ – "assign value 5 … "
- $X \xleftarrow{\$} \mathcal{X}$ – "assign $X$ a *random* value from set $\mathcal{X}$"

# Symmetric encryption – syntax

$$\Pi = (\mathcal{E}, \mathcal{D})$$

$$\mathcal{E} : \mathcal{K} \times \mathcal{M} \rightarrow \mathcal{C}$$
$$\mathcal{E}(K, M) = \mathcal{E}_K(K) = C$$

$$\mathcal{D}: \mathcal{K} \times \mathcal{C} \rightarrow \mathcal{M}$$
$$\mathcal{D}(K, C) = \mathcal{D}_K(C) = M$$

| | | |
|---|---|---|
| $\mathcal{K} = \{0,1\}^{128}$ | $\mathcal{M} = \{0,1\}^*$ | $\mathcal{C} = \{0,1\}^*$ |
| $\mathcal{K} = \{0,1\}^{128}$ | $\mathcal{M} = \{A, B, \dots, Z\}$ | $\mathcal{C} = \{A, B, \dots, Z\}$ |
| $\mathcal{K} = \{0,1\}^{128}$ | $\mathcal{M} = \{\text{YES}, \text{NO}\}$ | $\mathcal{C} = \{0,1\}^*$ |
| $\mathcal{K} = \{1, \dots, p\}$ | $\mathcal{M} = \{A, B, \dots, Z\}$ | $\mathcal{C} = \{0,1\}^*$ |

**Correctness requirement:**

$$\forall K \in K, \forall M \in M:$$

$$\mathcal{D}\big(K, \mathcal{E}(K, M)\big) = M$$

**Valid encryption scheme:**

$$\mathcal{E}_K(M) = M$$
$$\mathcal{D}_K(C) = C$$

# Symmetric encryption – security

$\Pi = (\mathcal{E}, \mathcal{D})$

$\mathcal{E} : \mathcal{K} \times \mathcal{M} \rightarrow \mathcal{C}$

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**Correctness requirement:**

$\forall K \in K, \forall M \in M:$

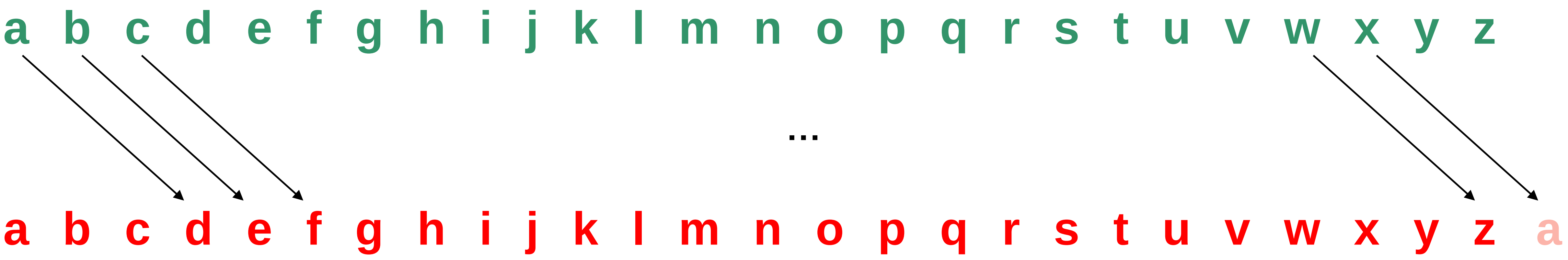
$$\mathcal{D}\big(K, \mathcal{E}(K, M)\big) = M$$

**Possible privacy security goals:**

- Hard to recover $K$ from $C$
- Hard to recover $M$ from $C$
- Hard to learn any bit of $M$ from $C$
- Hard to learn parity of $M$ from $C$
- …

# Historical encryption algorithms

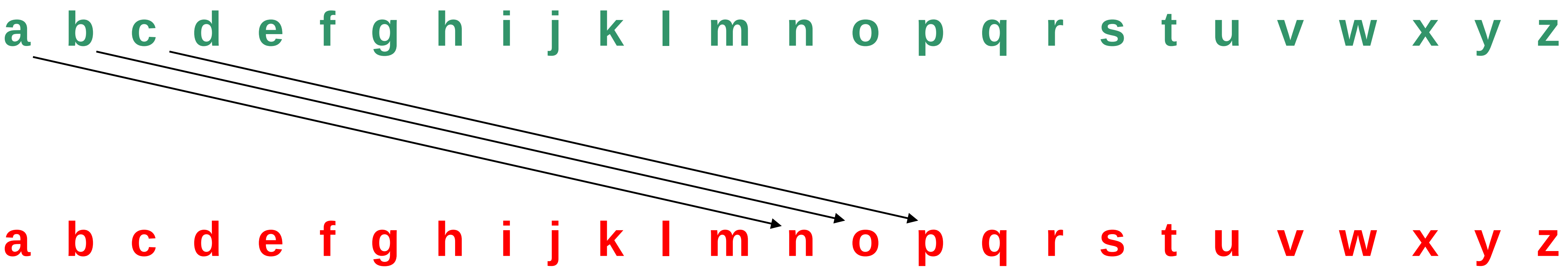
# Ceasar cipher




in the far distance a helicopter skimmed down between the roofs, hovered for an instant like a bluebottle, and darted away again with a curving flight. It was the police patrol, snooping into people's windows

lq wkh idu glvwdqfh d kholfrswhu vnlpphg grzq ehwzhhq wkh urriv, kryhuhg iru dq lqvwdqw olnh d eoxherwwoh, dqg gduwhg dzdb djdlq zlwk d fxuylqj ioljkw. Lw zdv wkh srolfh sdwuro, vqrrslqj lqwr shrsoh'v zlqgrzv

# Ceasar cipher (ROT-13)




in the far distance a helicopter skimmed down between the roofs, hovered for an instant like a bluebottle, and darted away again with a curving flight. It was the police patrol, snooping into people's windows

va gur sne qvfgnapr n uryvpbcgre fxvzzrq qbja orgjrra gur ebbsf, ubirerq sbe na vafgnag yvxr n oyhrobggyr, naq qnegrq njnl ntnva jvgu n pheivat syvtug. Vg jnf gur cbyvpr cngeby, fabbcvat vagb crbcyr'f jvaqbjf

# Ceasar cipher

- a $\leftrightarrow$ 0
- b $\leftrightarrow$ 1
- c $\leftrightarrow$ 2
- d $\leftrightarrow$ 3
- e $\leftrightarrow$ 4

$\vdots$

- z $\leftrightarrow$ 25

$$C \leftarrow M + 3 \pmod{26}$$

# ROT-13

- a ↔ 0
- b ↔ 1
- c ↔ 2
- d ↔ 3
- e ↔ 4

⋮

- z ↔ 25

$$C \leftarrow M + 13 \pmod{26}$$

$$M \leftarrow C - 13 \pmod{26}$$

$$\mathcal{E} : \mathcal{K} \times \mathcal{M} \rightarrow \mathcal{C}$$

$$\mathcal{D} : \mathcal{K} \times \mathcal{C} \rightarrow \mathcal{M}$$

$$\mathcal{K} = \{\ \}$$

$$\mathcal{M} = \{0,1,2,\ldots,25\}$$

$$\mathcal{C} = \{0,1,2,\ldots,25\}$$

# ROT-K

- a ↔ 0
- b ↔ 1
- c ↔ 2
- d ↔ 3
- e ↔ 4

⋮

- z ↔ 25

$$C \leftarrow M + K \pmod{26}$$

$$\mathcal{E} : \mathcal{K} \times \mathcal{M} \rightarrow \mathcal{C}$$

$$\mathcal{D} : \mathcal{K} \times \mathcal{C} \rightarrow \mathcal{M}$$

$$\mathcal{K} = \{0,1,2, \dots, 25\}$$

$$\mathcal{M} = \{0,1,2, \dots, 25\}$$

$$\mathcal{C} = \{0,1,2, \dots, 25\}$$

# Attacking ROT-K

$|\mathcal{K}| = 26$

$C$ = va gur sne qvfgnapr n uryvpbcgre…

| $K$ | $M$ |
|---|---|
| 0 | va gur sne qvfgnapr n uryvpbcgre.… |
| 1 | wb hvs tof rwghobqs o vszwqcdhsf … |
| 2 | xc iwt upg sxhipcrt p wtaxrdeitg … |
| 3 | yd jxu vqh tyijqdsu q xubysefjuh … |
| ⋮ | |
| 12 | hm sgd ezq chrszmbd z gdkhbnosdq |
| **13** | **in the far distance a helicopter…** |
| 14 | jo uif gbs ejtubodf b ifmjdpqufs… |
| 25 | uz ftq rmd puefmzoq m tqxuoabfqd… |

# Substitution cipher

| a | b | c | d | e | f | g | h | i | j | k | l | m | n | o | p | q | r | s | t | u | v | w | x | y | z |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ↕ | ↕ | ↕ | ↕ | | | | | | | | | | | … | | | | | | | | | | | ↕ |
| s | x | d | y | w | q | f | m | j | k | o | i | l | g | z | b | e | n | t | u | c | p | a | r | v | h |

$$|\mathcal{K}| = 26! \approx 10^{26} \approx 2^{88}$$

in the far distance a helicopter skimmed down between the roofs, hovered for an instant like a bluebottle, and darted away again with a curving flight. It was the police patrol, snooping into people's windows

jg umw qsn yjtusgdw s mwijdzbuwn tojllwy yzag xwuawwg umw nzzqt, mzpwnwy qzn sg jgtusgu ijow s xicwxzuuiw, sgy ysnuwy sasv sfsjg ajum s dcnpjgf qijfmu. ju ast umw bzijdw bsunzi, tgzzbjgf jguz bwzbiw't ajgyzat

# Substitution cipher – formal syntax

- $\Sigma = \{a, b, c, \dots, z\}$
- $\mathcal{M} = \Sigma^*$
- $\mathcal{C} = \Sigma^*$
- $\mathcal{K}$ = all permutations on $\Sigma = \{\pi : \Sigma \to \Sigma \mid \pi \text{ a permutation}\}$
- $\pi \in \mathcal{K}$

$\mathcal{E} : \mathcal{K} \times \mathcal{M} \to \mathcal{C}$

$\mathcal{D} : \mathcal{K} \times \mathcal{C} \to \mathcal{M}$

| σ | a | b | c | d | e | f | g | h | … |
|---|---|---|---|---|---|---|---|---|---|
| $\pi(\sigma)$ | $o$ | $y$ | $e$ | $z$ | $p$ | $u$ | $g$ | $t$ | … |

- $M = feed$
- $C = \mathcal{E}(\pi, M) = \pi(f)\pi(e)\pi(e)\pi(d) = uppz$
- $\mathcal{D}(\pi, C) = \pi^{-1}(u)\pi^{-1}(p)\pi^{-1}(p)\pi^{-1}(z) = feed$

# Attacking the substitution cipher

$$|\mathcal{K}| = 26! \approx 10^{26} \approx 2^{88}$$

jg umw qsn yjtusgdw s mwijdzbuwn
tojllwy yzag xwuawwg umw nzzqt,
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xicwxzuuiw, sgy ysnuwy sasv sfsjg ajum
s dcnpjgf qijfmu. ju ast umw bzijdw
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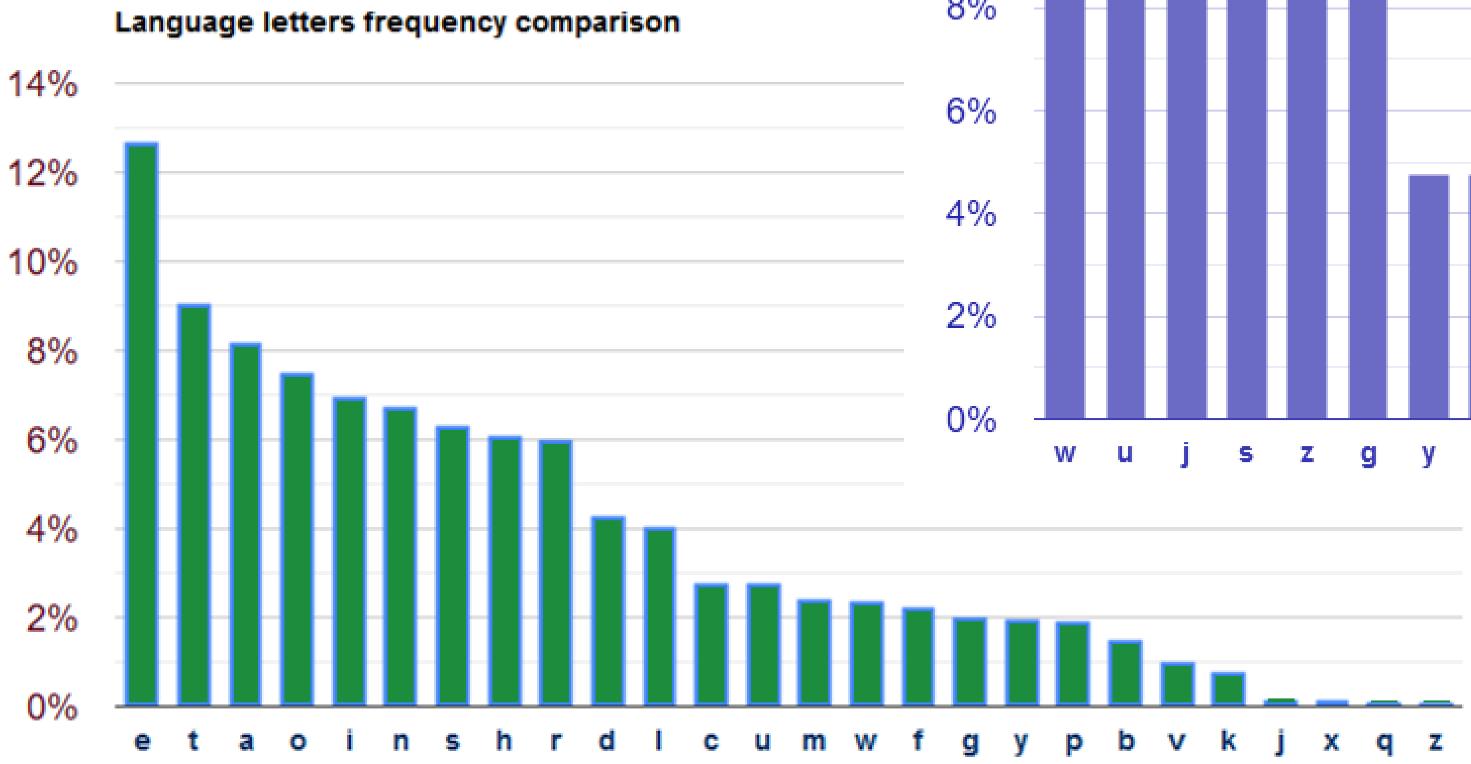
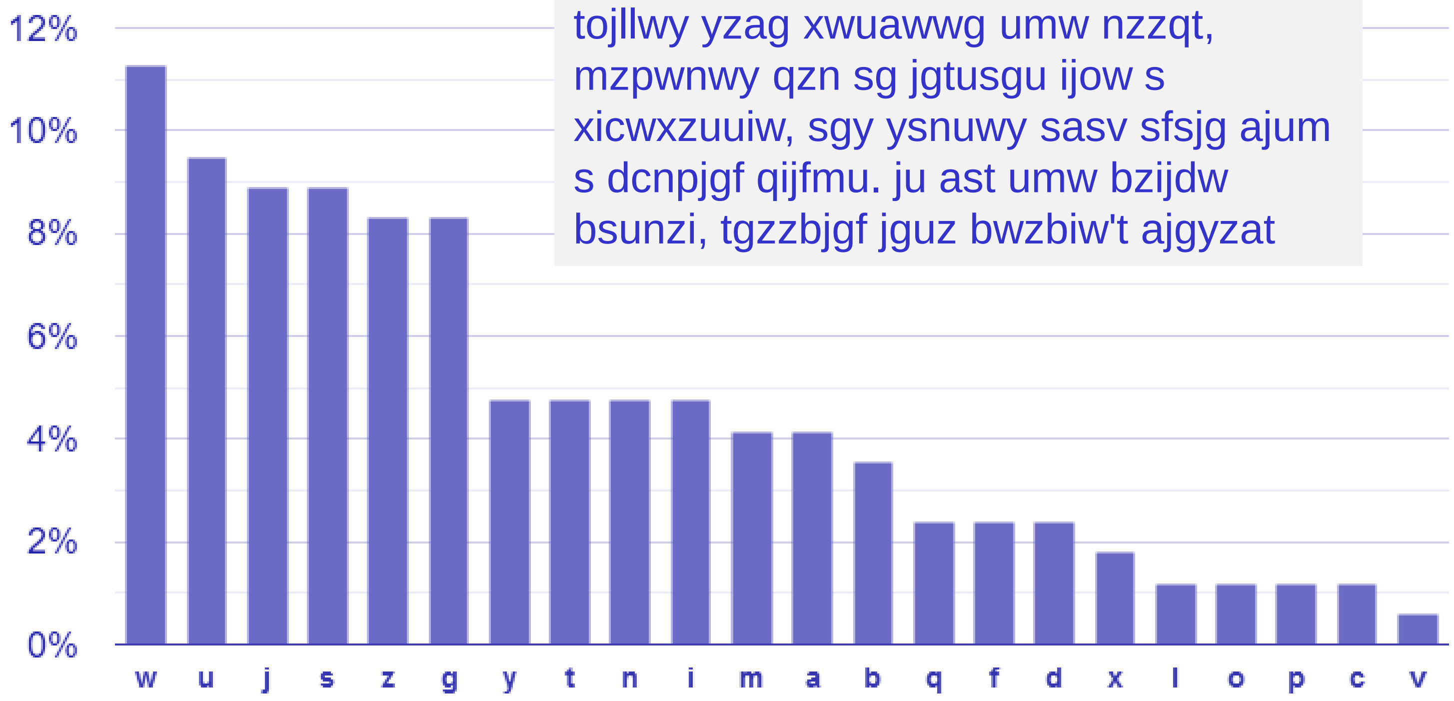
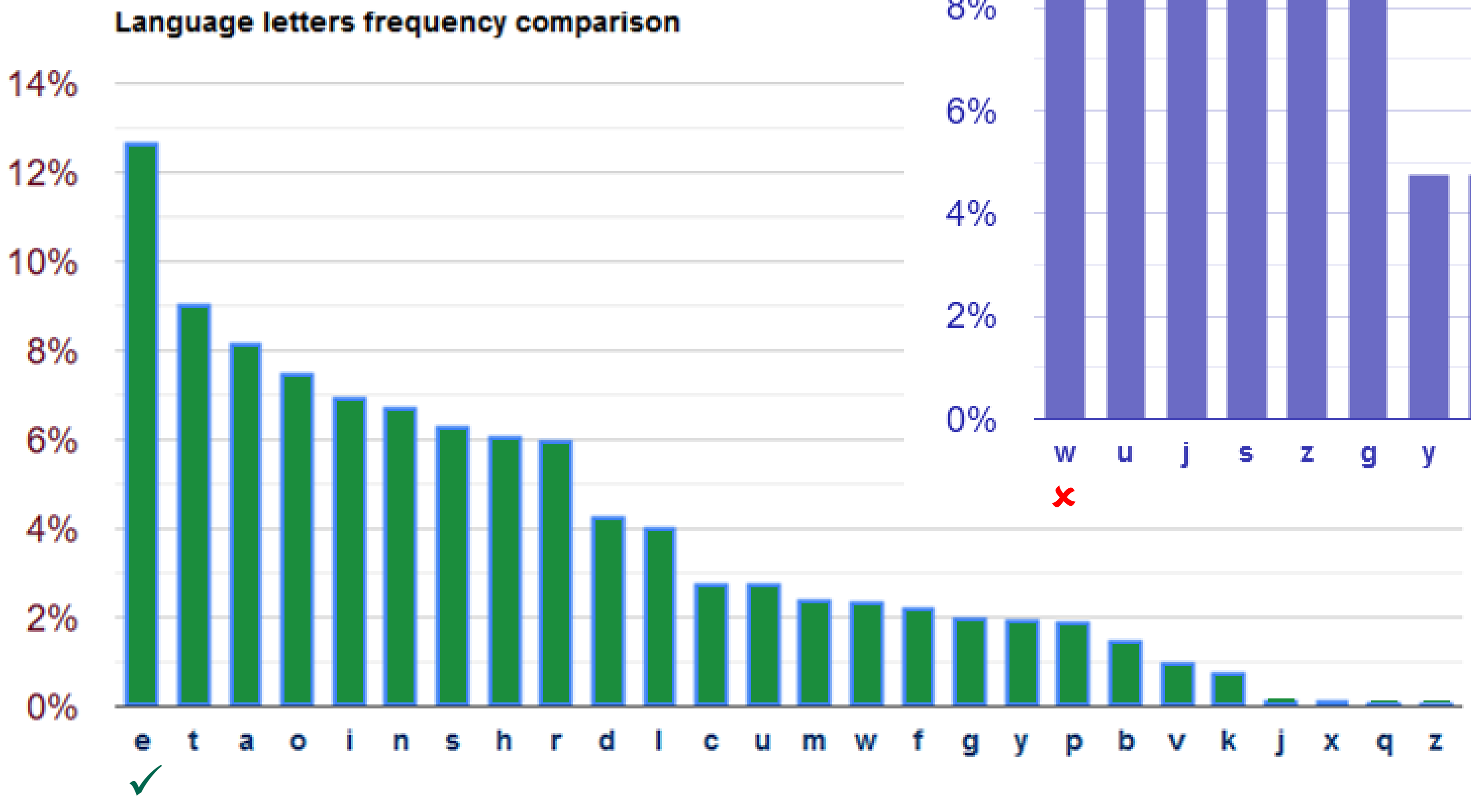


# Attacking the substitution cipher

$$|\mathcal{K}| = 26! \approx 10^{26} \approx 2^{88}$$

jg ume qsn yjtusgde s meijdzbuen tojlley yzag xeuaeeg ume nzzqt, mzpeney qzn sg jgtusgu ijoe s xicexzuuie, sgy ysnuey sasv sfsjg ajum s dcnpjgf qijfmu. ju ast ume bzijde bsunzi, tgzzbjgf jguz bezbie't ajgyzat

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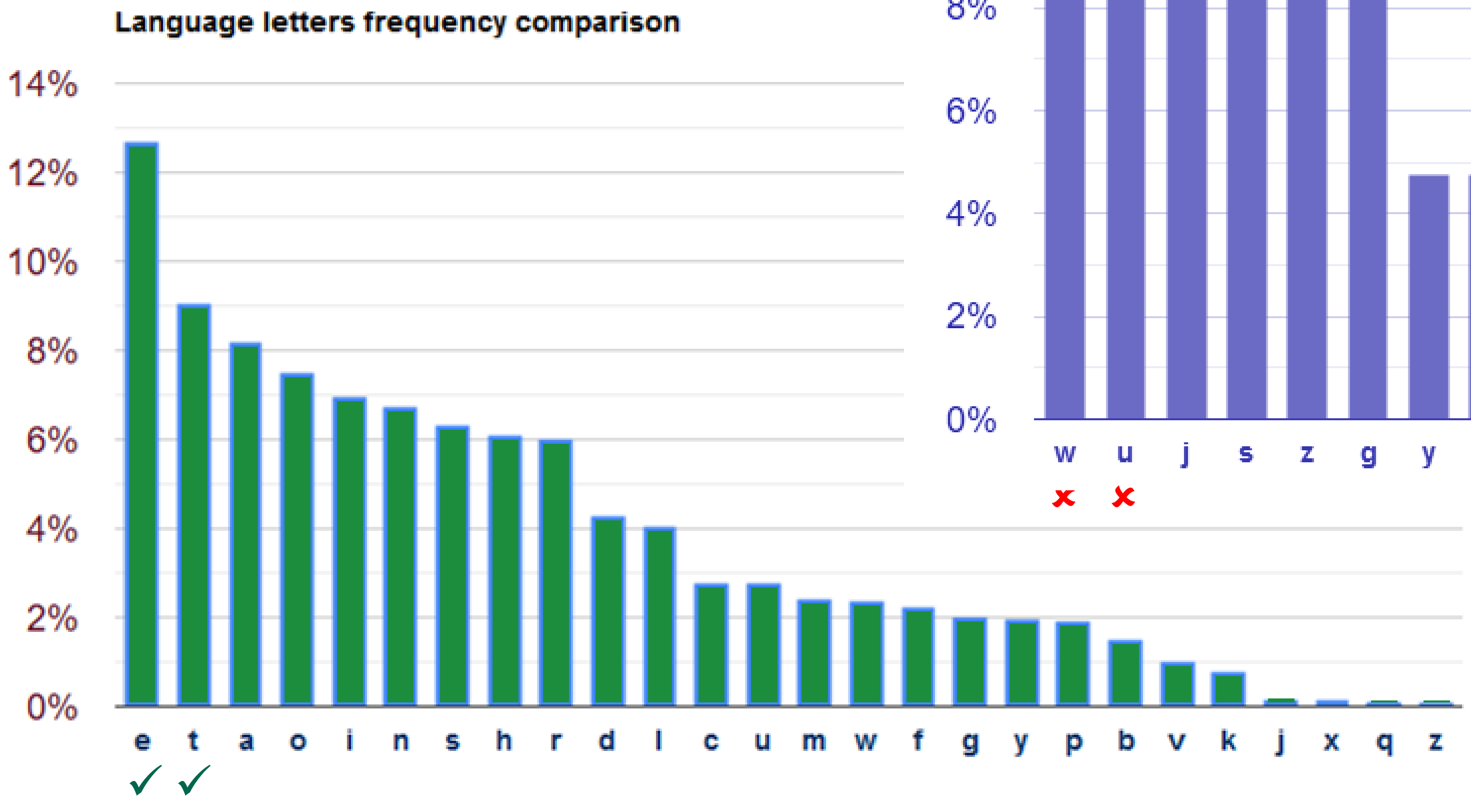
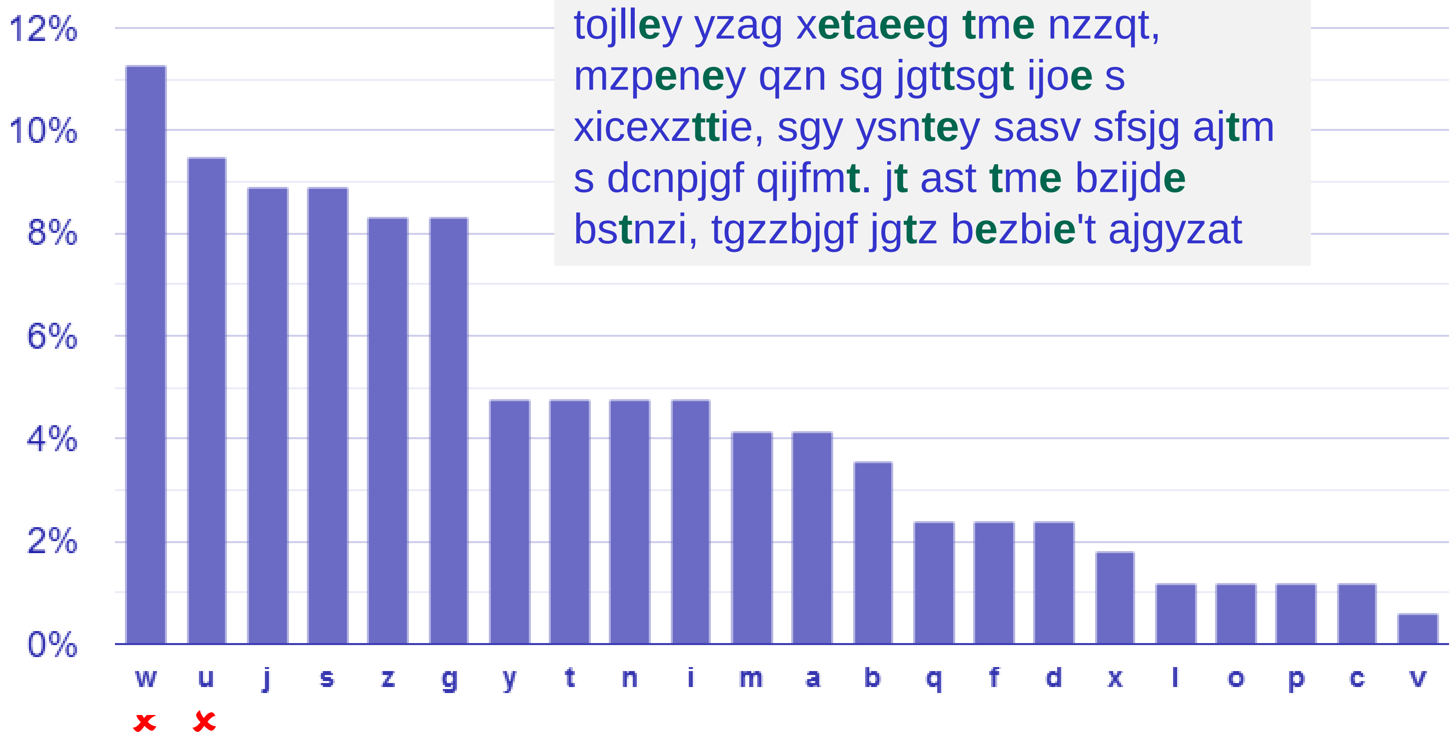


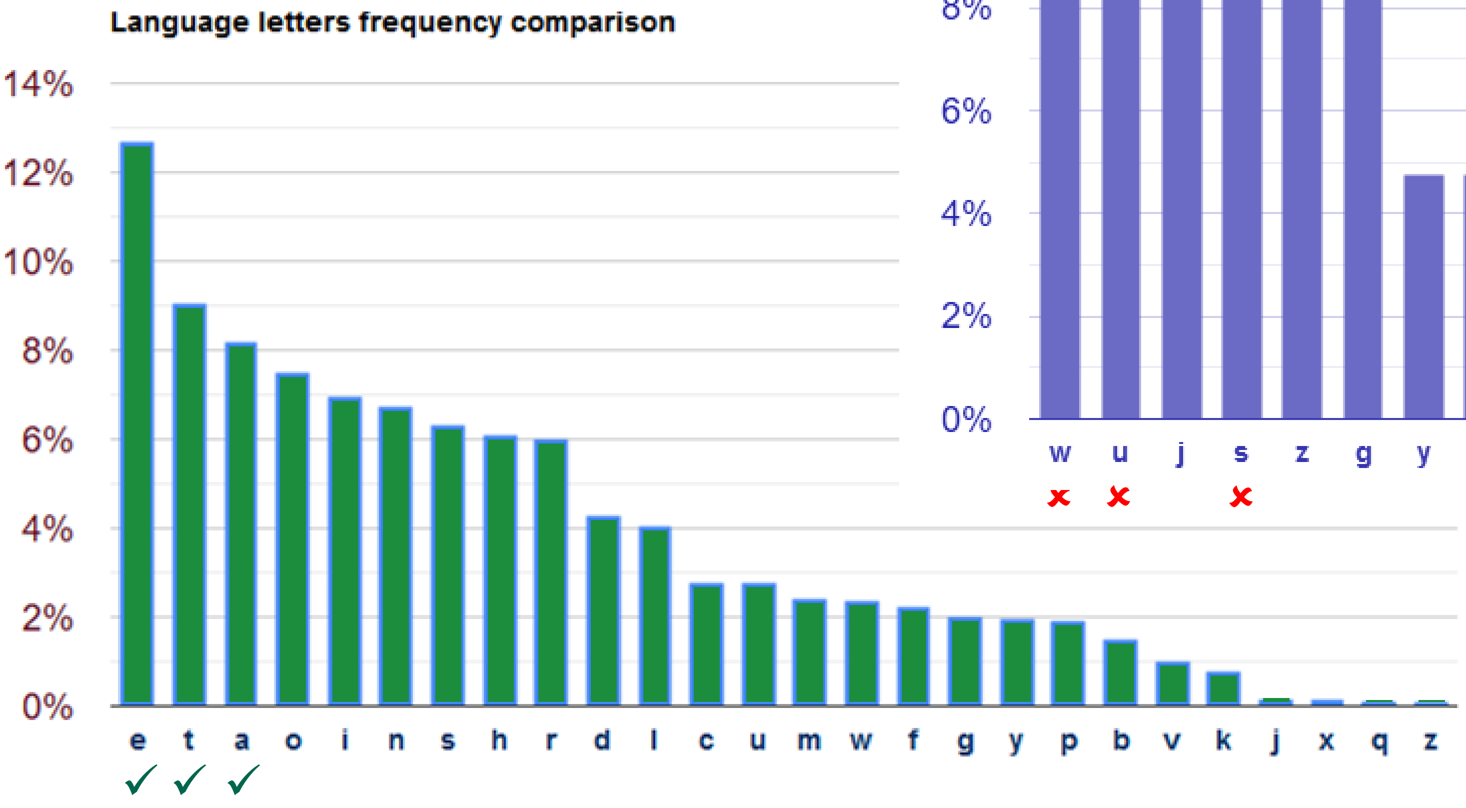




jg tme qsn yjttsgde s meijdzbten
tojlley yzag xetaeeg tme nzzqt,
mzpeney qzn sg jgttsgt ijoe s
xicexzttie, sgy ysntey sasv sfsjg ajtm
s dcnpjgf qijfmt. jt ast tme bzijde
bstnzi, tgzzbjgf jgtz bezbie't ajgyzat

# Attacking the substitution cipher

$|\mathcal{K}| = 26! \approx 10^{26} \approx 2^{88}$

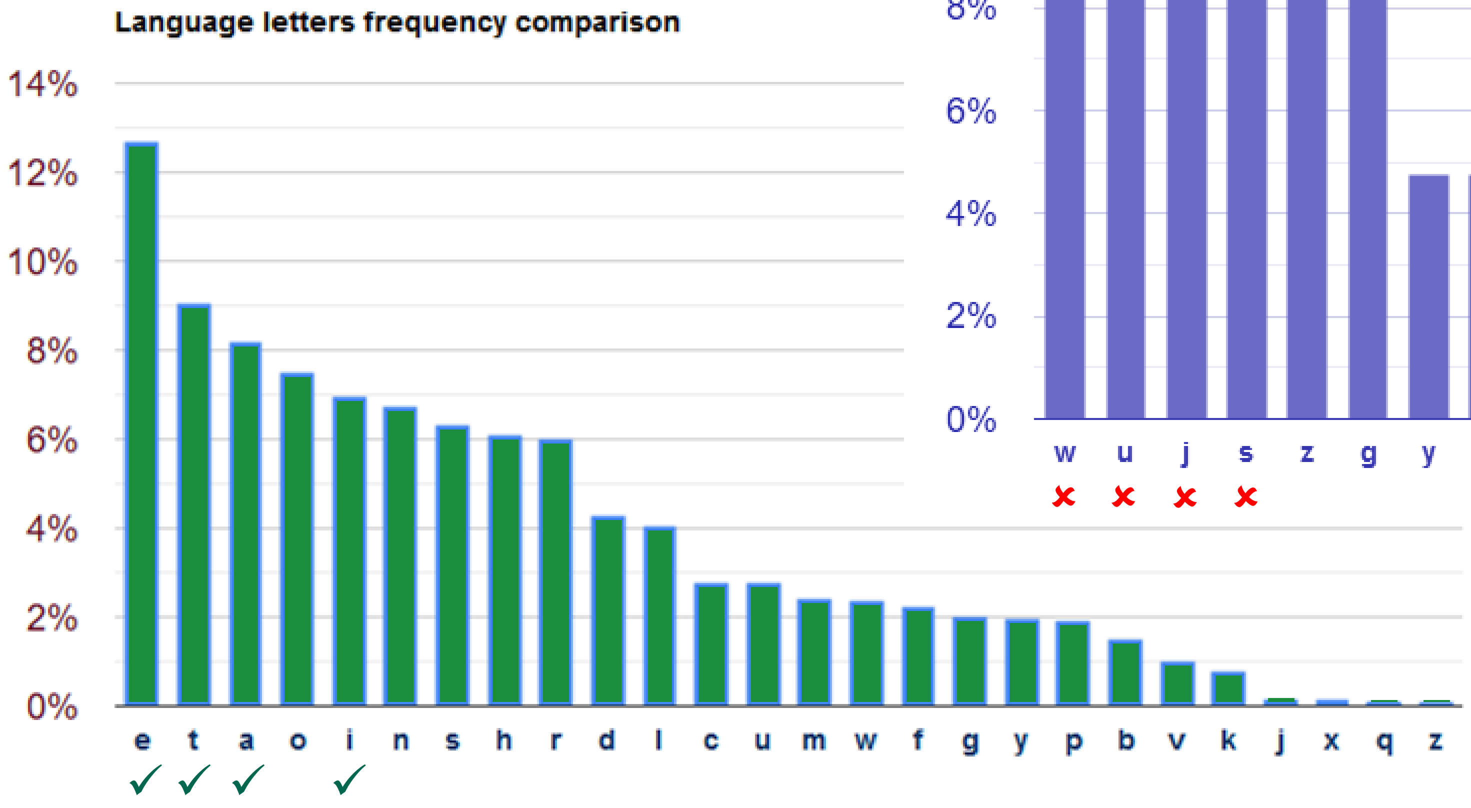
jg tme qan yjttagde a meijdzbten tojlley yzag xetaeeg tme nzzqt, mzpeney qzn ag jgttagt ijoe a xicexzttie, agy yantey aaav afajg ajtm a dcnpjgf qijfmt. jt aat tme bzijde batnzi, tgzzbjgf jgtz bezbie't ajgyzat




# Attacking the substitution cipher

$$|\mathcal{K}| = 26! \approx 10^{26} \approx 2^{88}$$

ig tme qan yittagde a meiidzbten toilley yzag xetaeeg tme nzzqt, mzpeney qzn ag igttagt iioe a xicexzttie, agy yantey aaav afaig aitm a dcnpigf qiifmt. it aat tme bziide batnzi, tgzzbigf igtz bezbie't aigyzat

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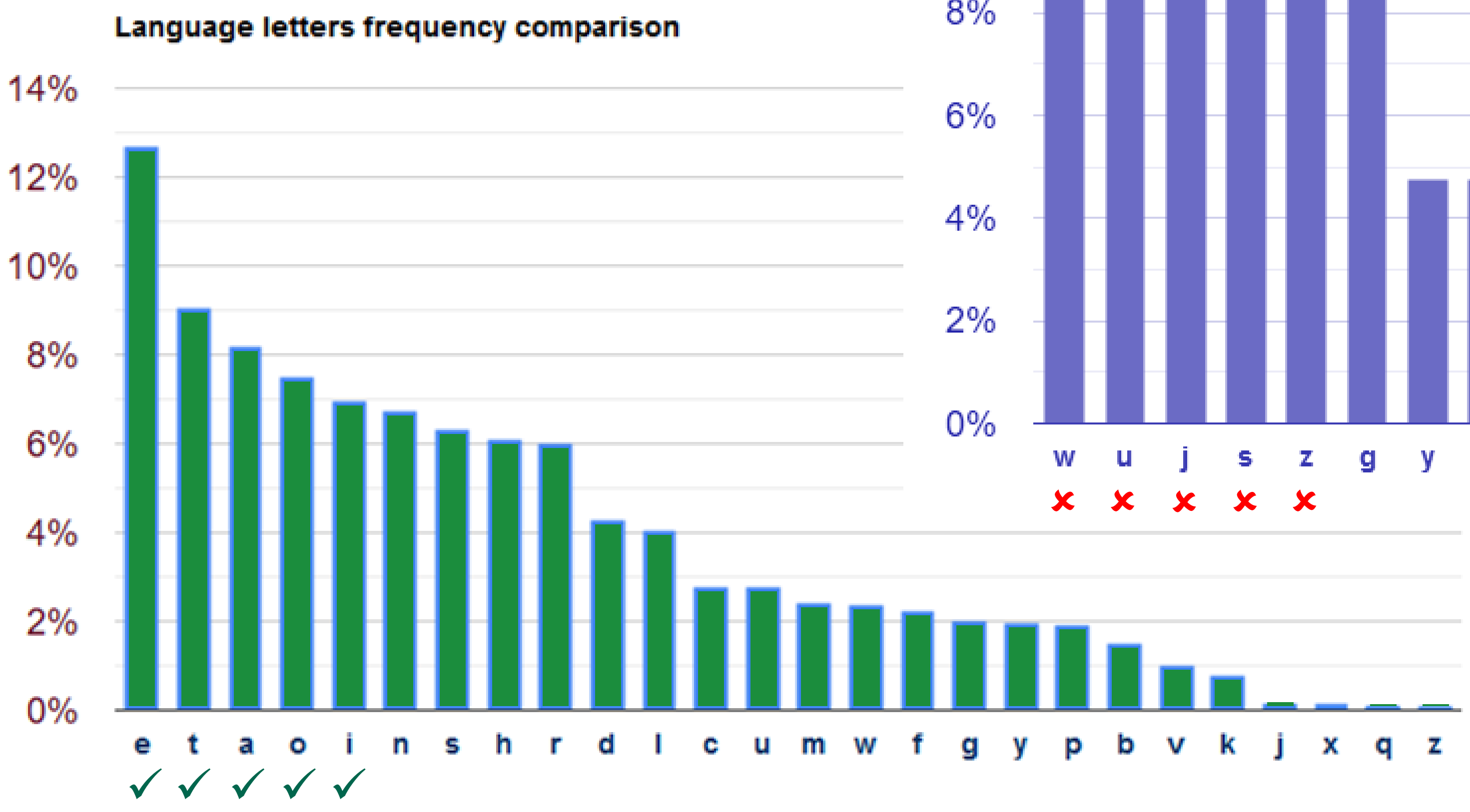


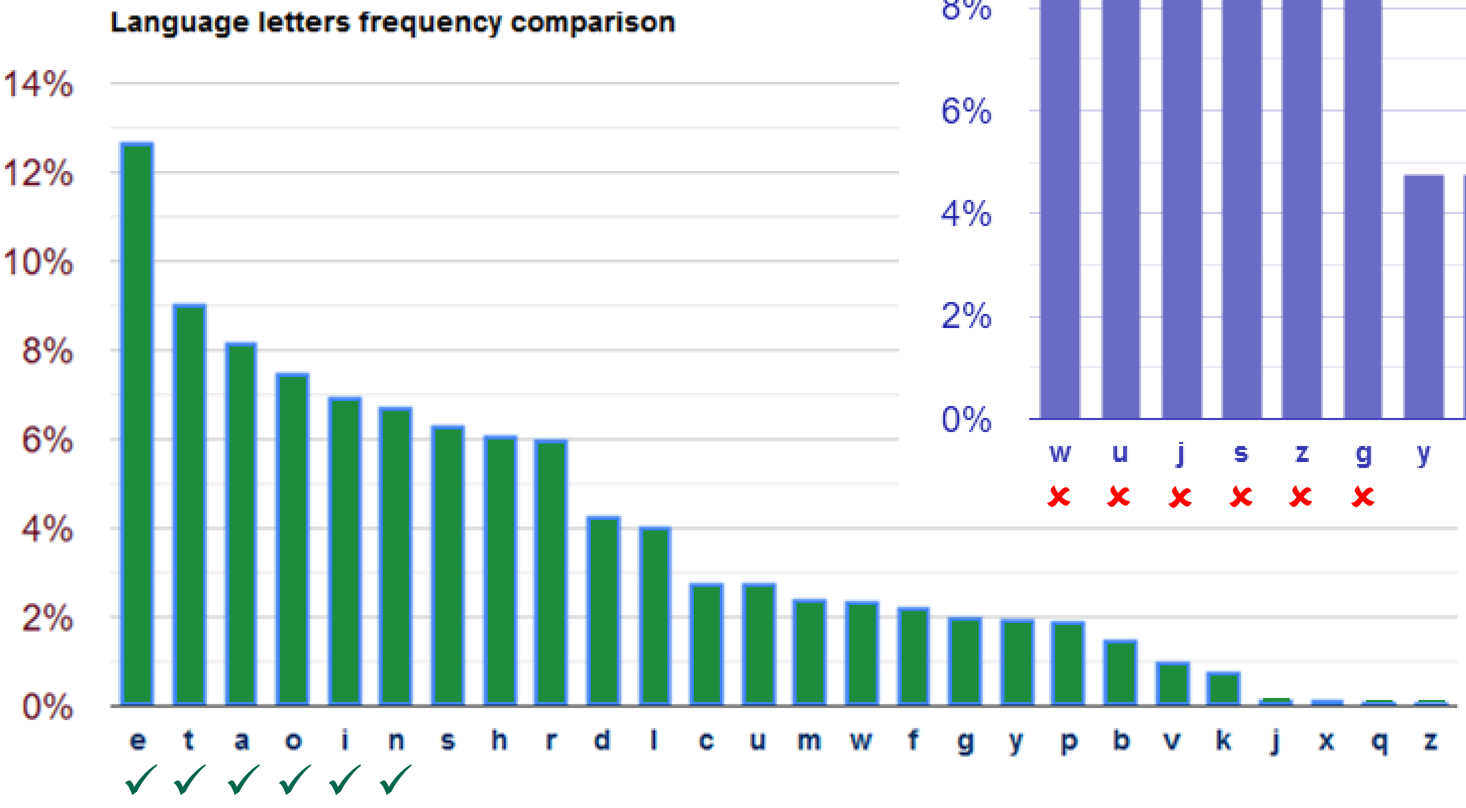


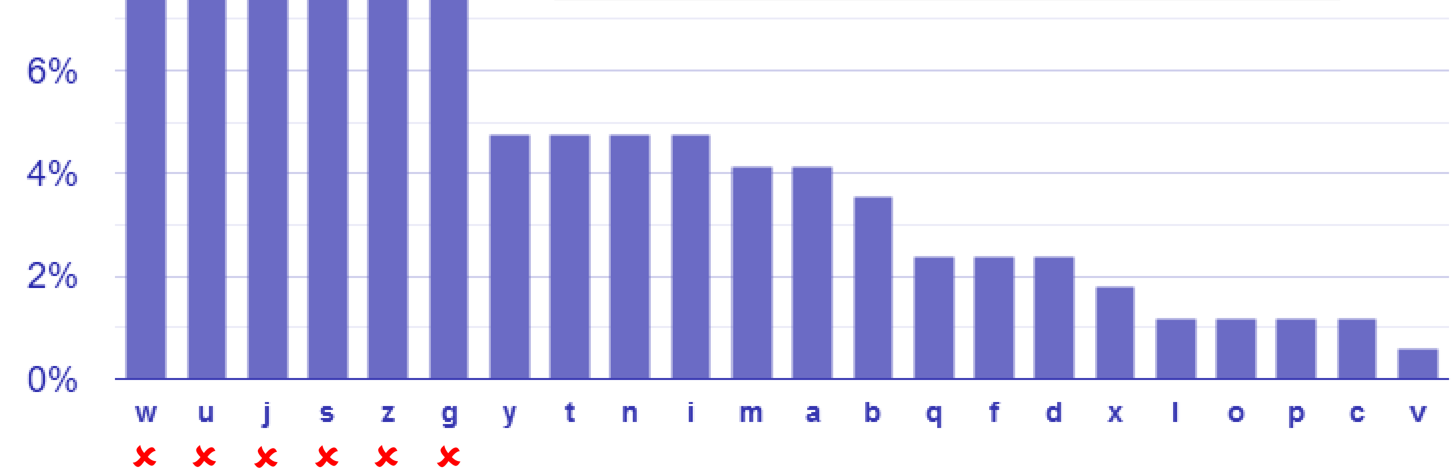
ig tme qan yittagde a meiidobten toilley yoag xetaeeg tme nooqt, mopeney qon ag igttagt iioe a xicexottie, agy yantey aaav afaig aitm a dcnpigf qiifmt. it aat tme boiide batnoi, tgoobigf igto beobie't aigyoat

# Attacking the substitution cipher

$$|\mathcal{K}| = 26! \approx 10^{26} \approx 2^{88}$$

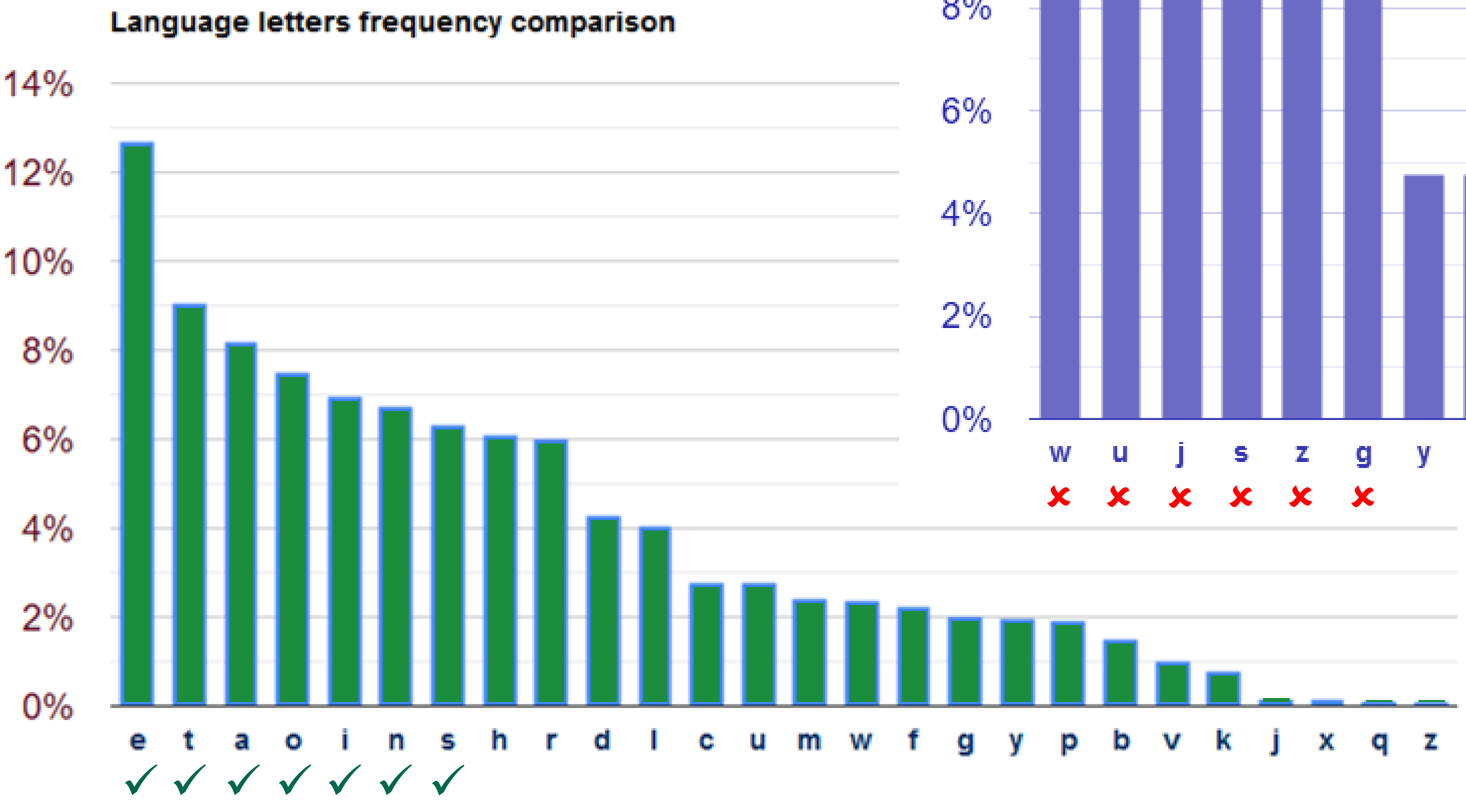
in tme qan yittande a meiidobten toilley yoan xetaeen tme nooqt, mopeney qon an inttant iioe a xicexottie, any yantey aaav afain aitm a dcnpinf qiifmt. it aat tme boiide batnoi, tnoobinf into beobie't ainyoat

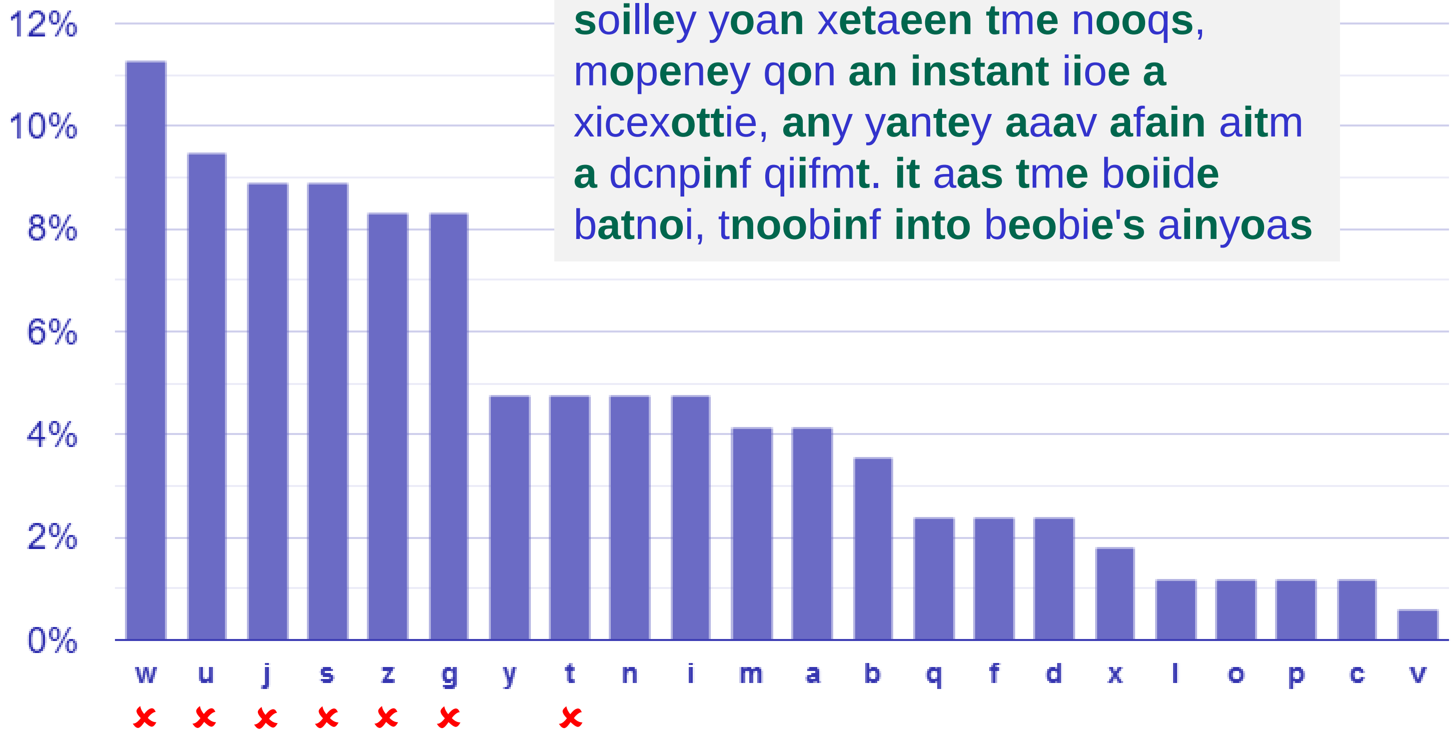


# Attacking the substitution cipher

$|\mathcal{K}| = 26! \approx 10^{26} \approx 2^{88}$

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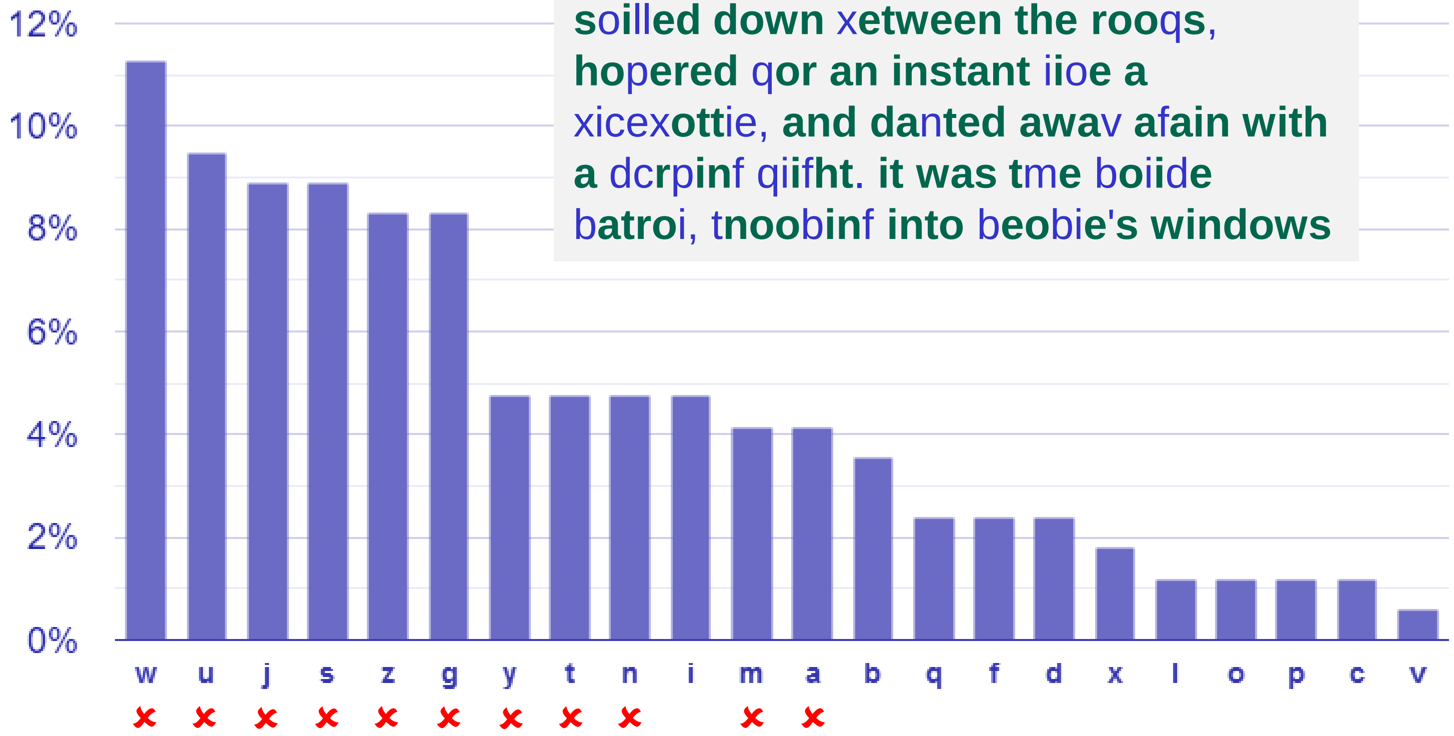
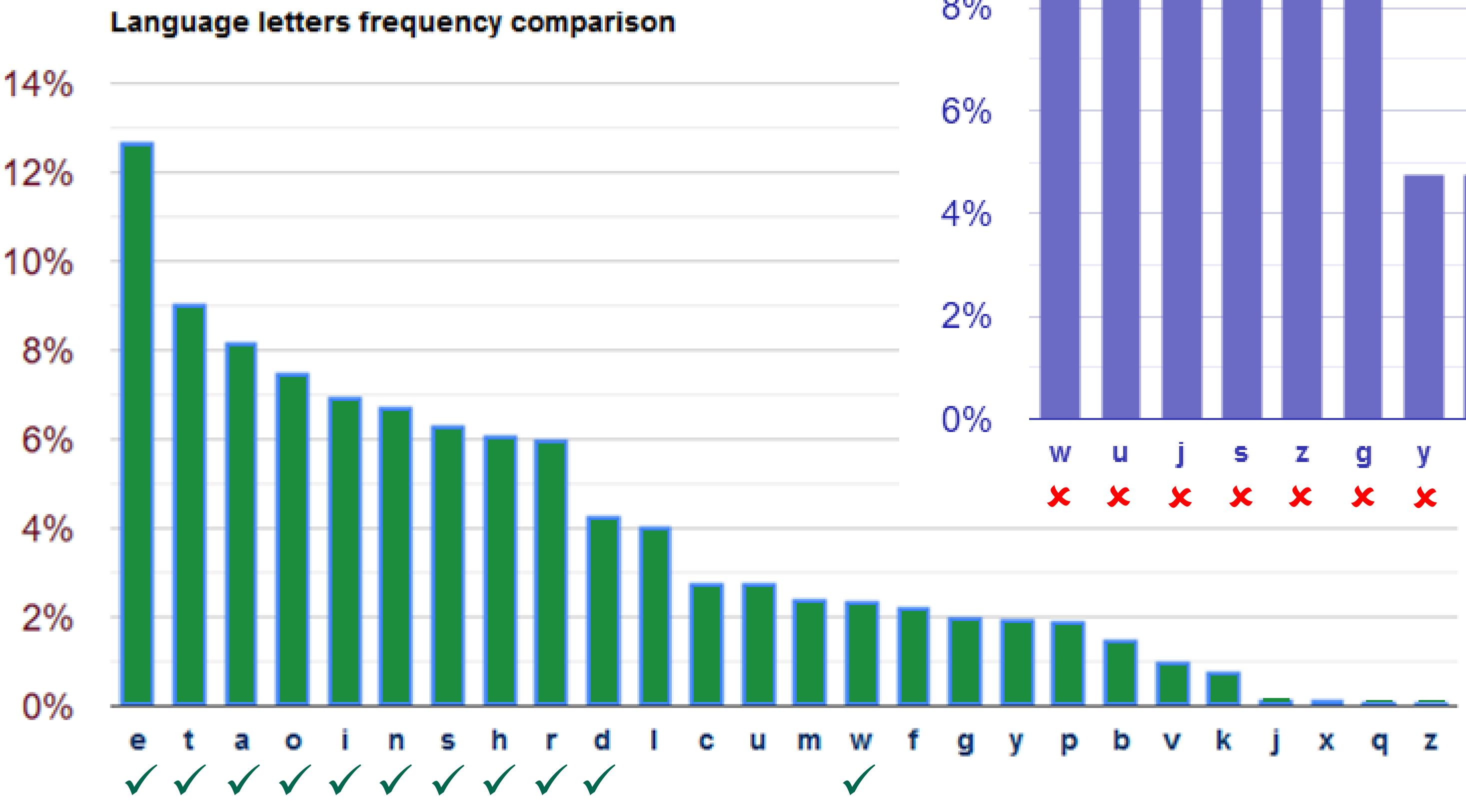
soilley yoan xetaeen tme nooqs,
mopeney qon an instant iioe a
xicexottie, any yantey aaav afain aitm
a dcnpinf qiifmt. it aas tme boiide
batnoi, tnoobinf into beobie's ainyoas

# Attacking the substitution cipher

$$|\mathcal{K}| = 26! \approx 10^{26} \approx 2^{88}$$

in the qan distande a heiidobter soilled down xetween the rooqs, hopered qor an instant iioe a xicexottie, and danted awav afain with a dcrpinf qiifht. it was tme boiide batroi, tnoobinf into beobie's windows

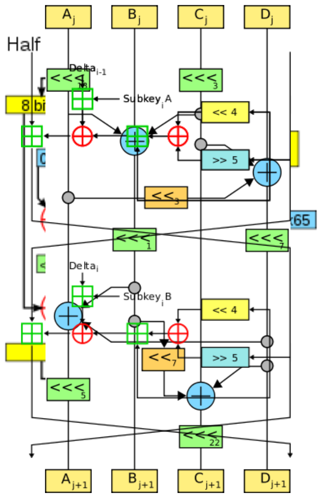


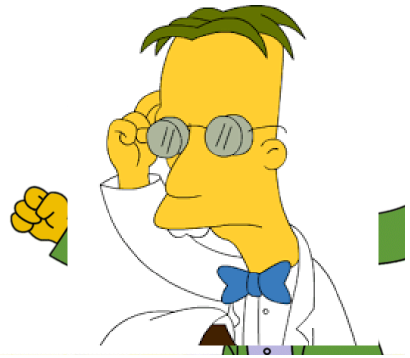



# Conclusions

- Key space must be large enough
- Ciphertext should not reveal letter frequency of the message
- Is this enough?

# Historical approach to crypto development



build → break → fix → break → fix → break → fix ...

# Modern approach

- Trying to make cryptography more a **science** than an **art**
- Focus on **formal definitions** of security (and insecurity)
- Clearly stated **assumptions**
- Analysis supported by mathematical **proofs**
- ... but old fashioned **cryptanalysis** continues to be very important!

# The one-time-pad (OTP)

$\mathcal{K} = \{0,1\}^n$

$\mathcal{M} = \{0,1\}^n$

$\mathcal{C} = \{0,1\}^n$

$\mathcal{E} : \mathcal{K} \times \mathcal{M} \to \mathcal{C}$

$\mathcal{E}(K, M) = K \oplus M$

$\mathcal{D} : \mathcal{K} \times \mathcal{C} \to \mathcal{M}$

$\mathcal{D}(K, C) = K \oplus C$

$$\begin{array}{rll} & 0101100100 & M \\ \oplus & 1110001101 & K \\ \hline = & 1011101001 & C \end{array}$$

$$\begin{array}{rll} & 1011101001 & C \\ \oplus & 1110001101 & K \\ \hline = & 0101100100 & M \end{array}$$

# The one-time-pad (OTP)

$\mathcal{K} = \{0,1\}^n$

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$\mathcal{D} : \mathcal{K} \times \mathcal{C} \to \mathcal{M}$

$\mathcal{D}(K, C) = K \oplus C$

**Theorem:** The OTP encryption scheme has **one-time perfect privacy**

**Definition (Shannon 1949):** An encryption scheme has **one-time perfect privacy** if for any two $M_1, M_2 \in \mathcal{M}$ and any $C \in \mathcal{C}$

$$\Pr[\mathcal{E}_K(M_1) = C] = \Pr[\mathcal{E}_K(M_2) = C]$$

(probability taken over the random choice $K \stackrel{\$}{\leftarrow} \mathcal{K}$ and the random coins used by $\mathcal{E}$ (if any))

# (One-time) perfect secrecy

- From adversary's POV the ciphertext is *uniformily* distributed over $\mathcal{C}$
  - $C$ cannot give *any* information about $M$!

| **Prob** | $\boldsymbol{K}$ | $\boldsymbol{C = K \oplus 101}$ |
|---|---|---|
| 1/8 | 000 | 101 |
| 1/8 | 001 | 100 |
| 1/8 | 010 | 111 |
| 1/8 | 011 | 110 |
| 1/8 | 100 | 001 |
| 1/8 | 101 | 000 |
| 1/8 | 110 | 011 |
| 1/8 | 111 | 010 |

| **Prob** | $\boldsymbol{K}$ | $\boldsymbol{C = K \oplus 001}$ |
|---|---|---|
| 1/8 | 000 | 001 |
| 1/8 | 001 | 000 |
| 1/8 | 010 | 011 |
| 1/8 | 011 | 010 |
| 1/8 | 100 | 101 |
| 1/8 | 101 | 100 |
| 1/8 | 110 | 111 |
| 1/8 | 111 | 110 |

# Proof of OTP one-time perfect privacy

**Theorem:** The OTP encryption scheme has **one-time perfect privacy**

**Definition:** An encryption scheme has **one-time perfect privacy** if for any $M_1, M_2 \in \mathcal{M}$ and any $C \in \mathcal{C}$

$$\Pr[\mathcal{E}_K(M_1) = C] = \Pr[\mathcal{E}_K(M_2) = C]$$

with probability taken over the random choice $K \xleftarrow{\$} \mathcal{K}$ and the random coins used by $\mathcal{E}$ (if any))

**Fix:** $M_1, M_2, C \in \{0,1\}^n$

**Need to show:** $\Pr[K \oplus M_1 = C] = \Pr[K \oplus M_2 = C]$

$$\Pr[K \oplus M_1 = C] = \Pr[K = M_1 \oplus C] = \Pr[K = Z_1] = \frac{1}{2^n}$$

$$\Pr[K \oplus M_2 = C] = \Pr[K = M_2 \oplus C] = \Pr[K = Z_2] = \frac{1}{2^n}$$

**QED**

# One-time pad – perfect?

- OTP gives perfect privacy…for *one* message
  - What happens if you reuse the same key for two messages?
  - $C_1, C_2, \ C_1 \neq C_2$
  - $C_1 \oplus C_2 = (K \oplus M_1) \oplus (K \oplus M_2) = M_1 \oplus M_2$
- Key is as long as the message
  - What happens if it is shorter?
  - Key management becomes very difficult
  - Sort of defeats the purpose
- Nothing special about XOR: ROT-K also has one-time perfect privacy
  - Why doesn't this contradict what we saw earlier about ROT-K?

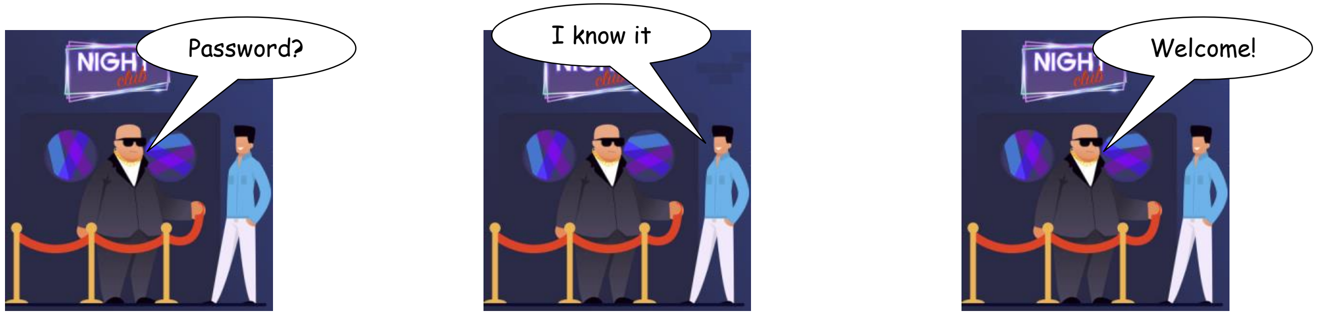
**Theorem:** No encryption scheme can have perfect secrecy if $|\mathcal{K}| < |\mathcal{M}|$

# Outline of course

| | Message privacy | Message integrity / authentication |
|---|---|---|
| **Symmetric keys** | Symmetric encryption | Message authentication codes (MAC) |
| **Asymmetric keys** | Asymmetric encryption (a.k.a. public-key encryption) | Digital signatures |

# Much more to cryptography

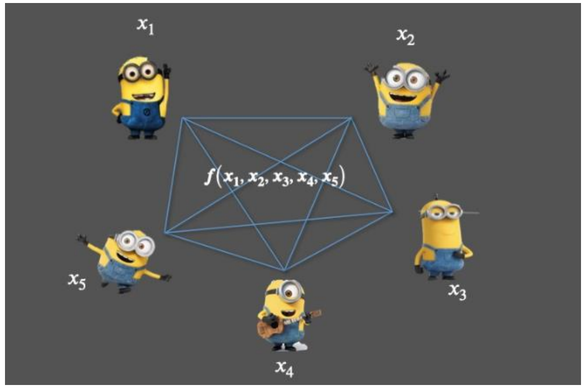
- Zero-knowledge proofs



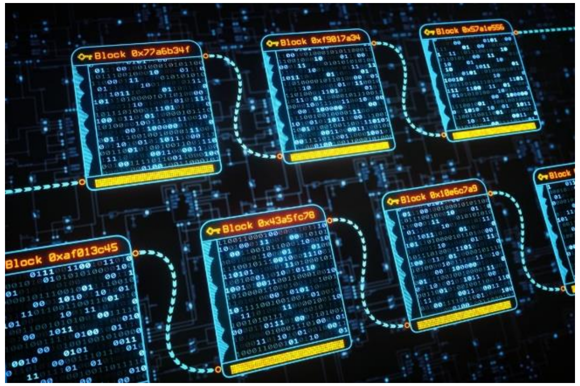

- Fully-homomorphic encryption

$$Enc(K, M_1 + M_2) = Enc(K, M_1) + Enc(K, M_2)$$

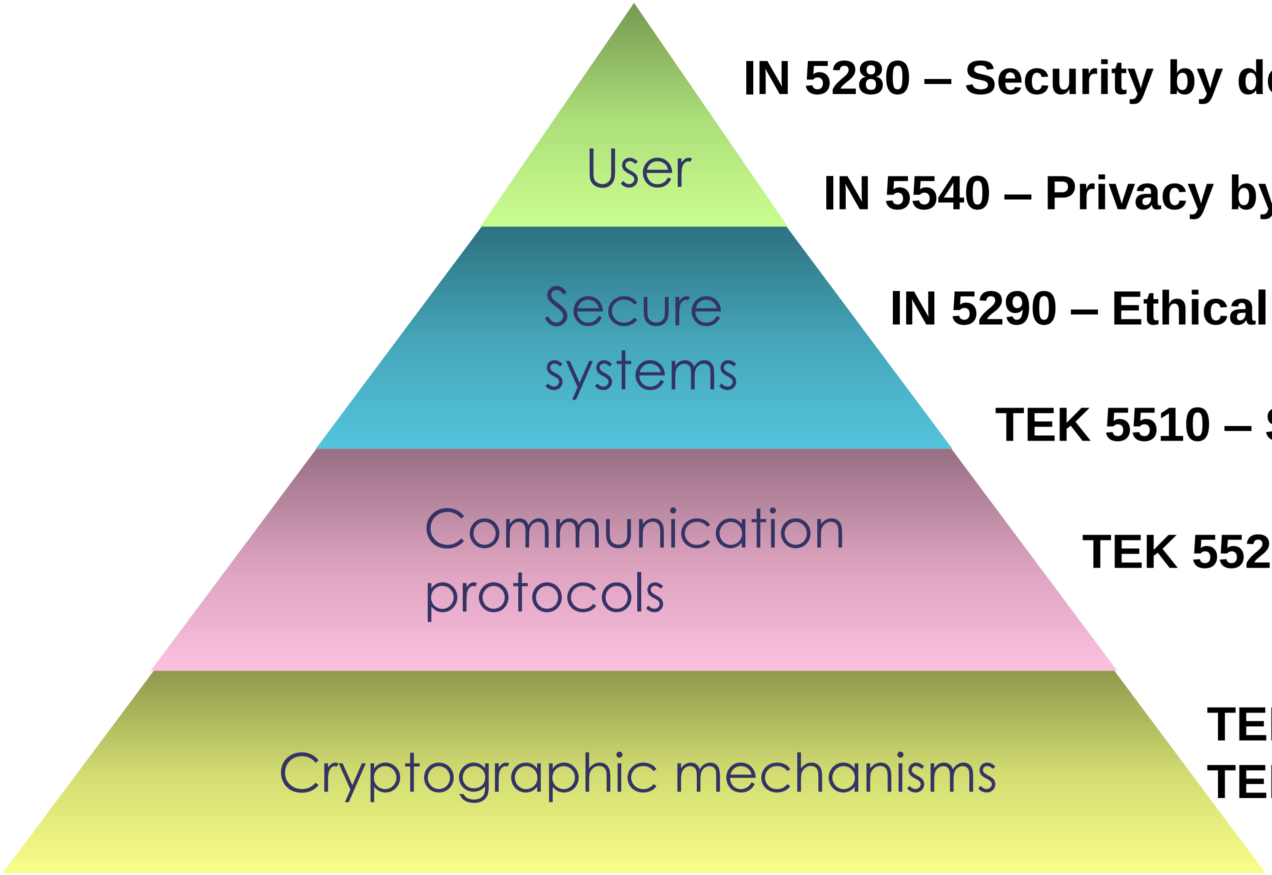
- Multi-party computation




- Blockchain



# The security pyramid




**IN 5280 – Security by design**

**IN 5540 – Privacy by design**

**IN 5290 – Ethical hacking**

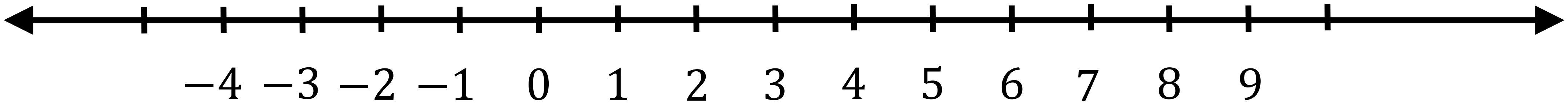
**TEK 5510 – Security in operating systems and software**

**TEK 5520 – Cyber security of industrial systems**

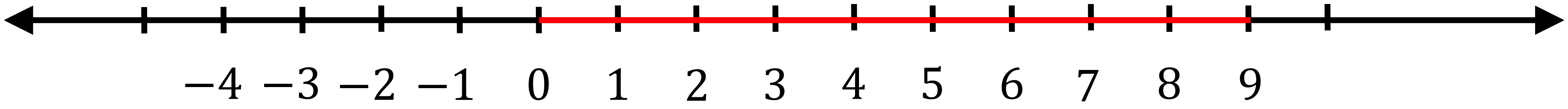
**TEK 4500 – Introduction to cryptography**
**TEK 5550 – Advanced topics in cryptograpy**

# Modular arithmetic and discrete probability

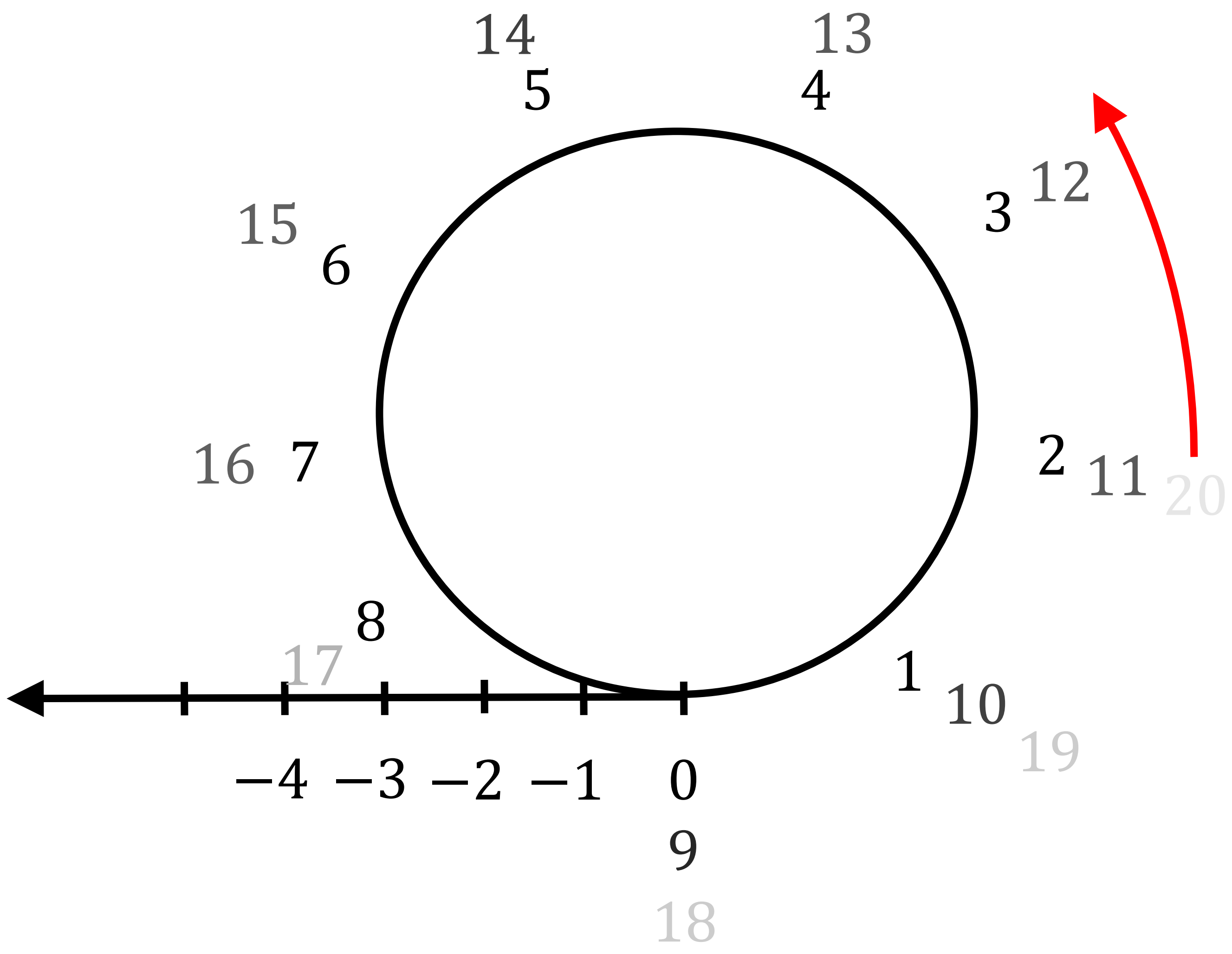
# Modular arithmetic



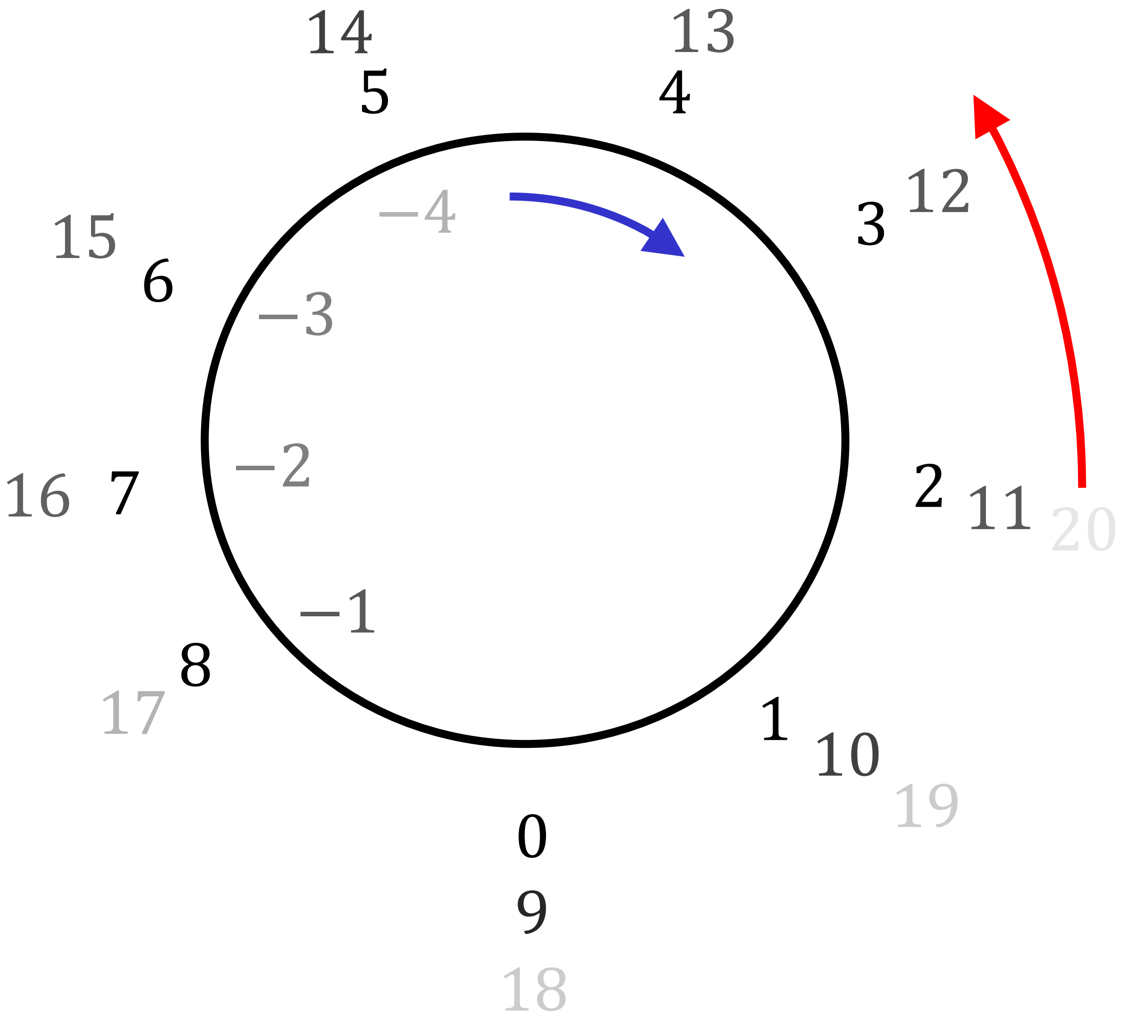
# Modular arithmetic



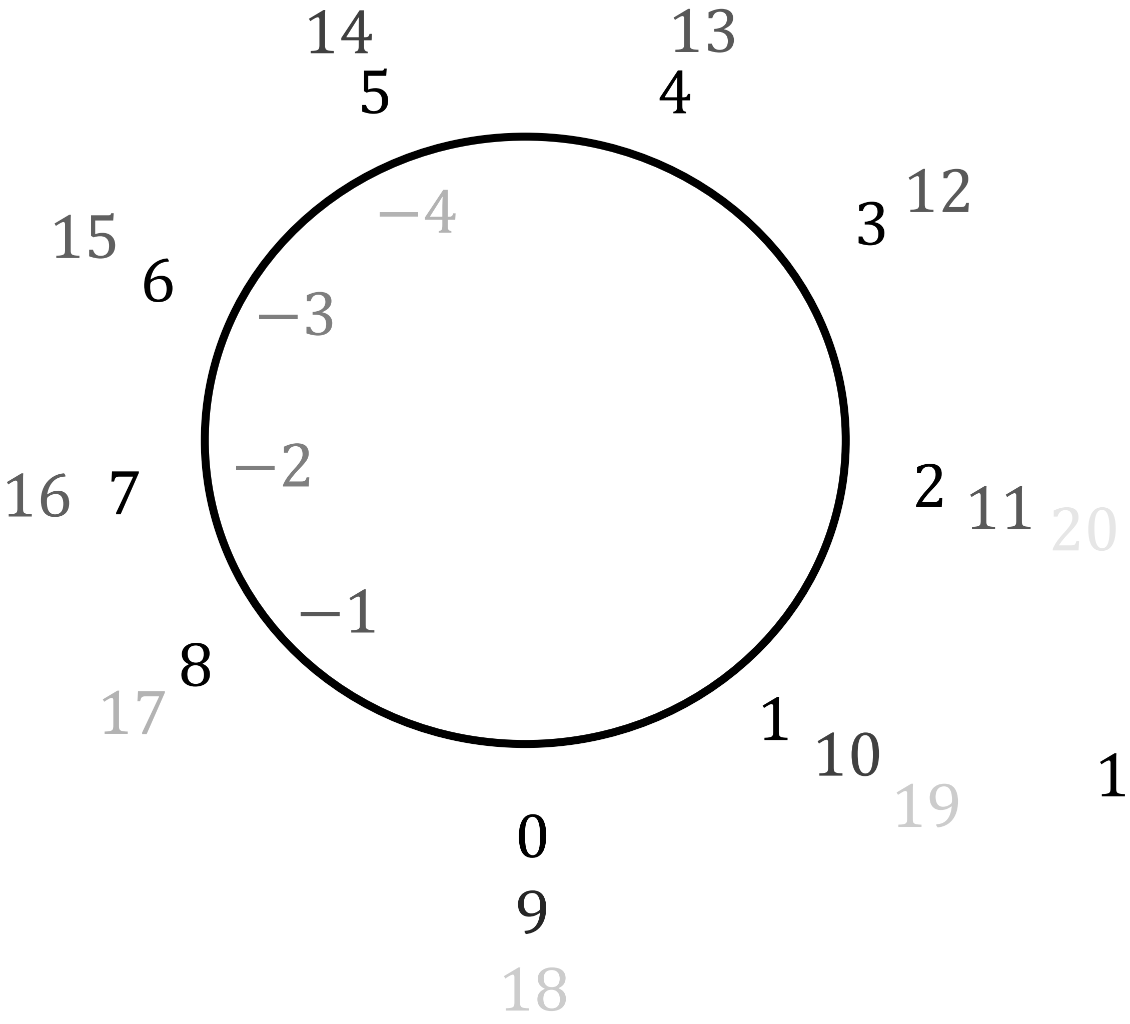
# Modular arithmetic



# Modular arithmetic



# Modular arithmetic




$$1 + 3 = 4$$

$$5 + 8 = 13 \equiv 4 \bmod 9$$

$$5 \cdot 4 = 20 \equiv 2 \bmod 9$$

$$2 - 5 = -3 \equiv 6 \bmod 9$$

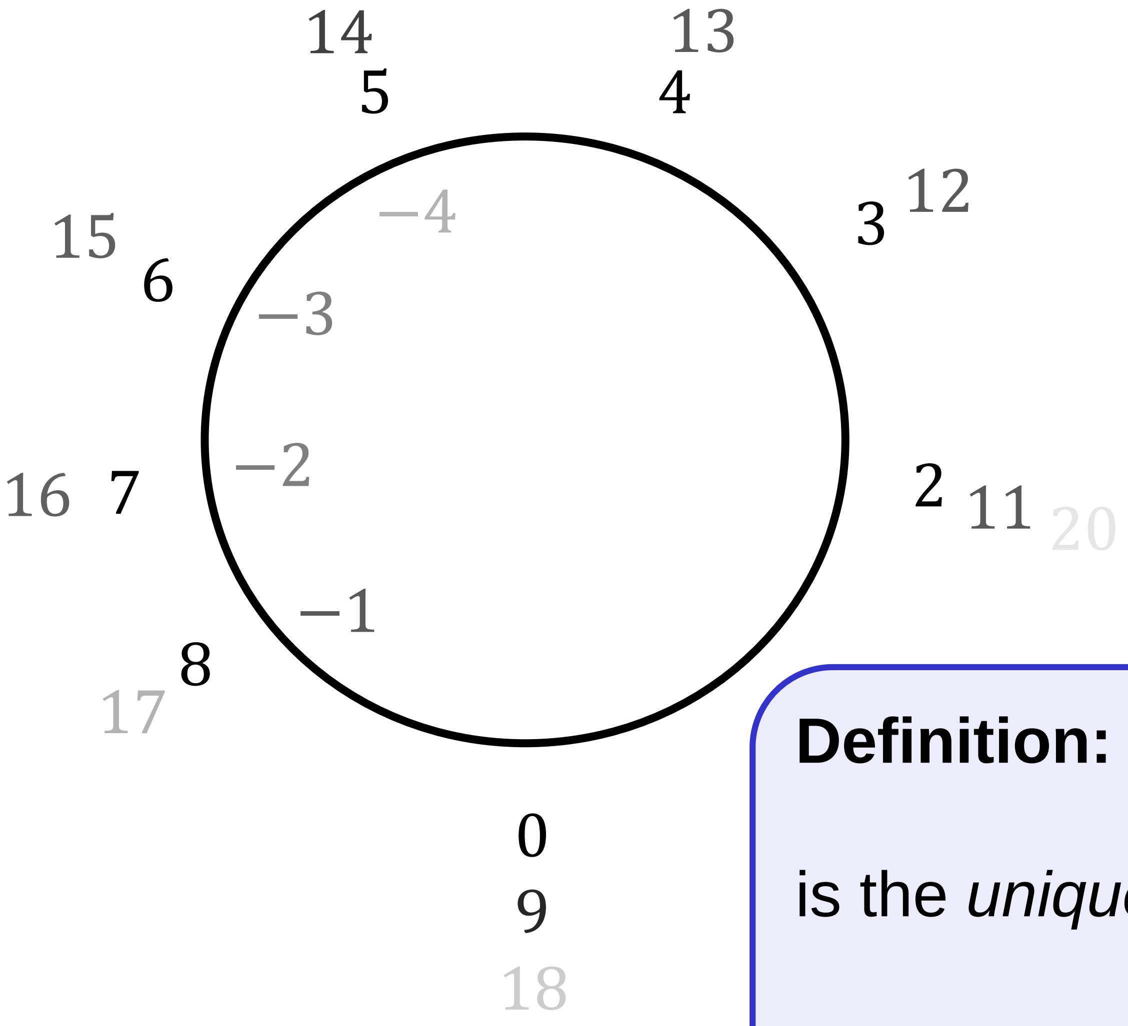
$$2^{10} = 1024 \equiv 7 \bmod 9$$

$$158 = 153 + {\color{red}r} \equiv {\color{red}r} \bmod 9 \equiv 5 \bmod 9$$

$${\color{red}r} < 9$$

$$9 \to 18 \to 27 \to 36 \to \cdots \to \mathbf{153} \to 162$$

# Modular arithmetic

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**Definition:** $r \equiv n \bmod m$

is the *unique* integer $0 \leq r < m$ such that

$$n = q \cdot m + r$$

# Discrete probability

- $\mathcal{X}$ – a finite set (e.g. $\mathcal{X} = \{0,1\}^n$)

**Definition:** A **probability distribution** over $\mathcal{X}$ is a function $\Pr : \mathcal{X} \to [0,1]$ such that

$$\sum_{X \in \mathcal{X}} \Pr[X] = 1$$

- **Examples:**
  - $\mathcal{X} = \{0,1\}^2$ $\Pr[00] = 1/2$ $\Pr[01] = 1/8$ $\Pr[10] = 1/4$ $\Pr[11] = 1/8$
  - Unform distribution: for all $X \in \mathcal{X}$: $\Pr[X] = 1/|\mathcal{X}|$
  - Point distribution at $X_0$: $\Pr[X_0] = 1$ $\forall X \neq X_0$: $\Pr[X] = 0$

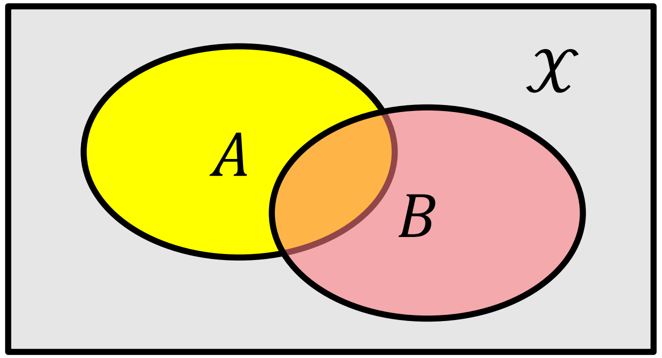
# Discrete probability

- A subset $A \subseteq \mathcal{X}$ is called an **event** and $\Pr[A] = \sum_{X \in A} \Pr[X]$

- **Example**: $\mathcal{X} = \{0,1\}^8$

$$A = \{\text{all } X \text{ in } \mathcal{X} \text{ such that } lsb_2(X) = 11\} \subset \mathcal{X}$$

With the uniform distribution over $\mathcal{X}$, what is $\Pr[A]$? $\Pr[A] = 1/4$




- **Union bound:** For events $A$ and $B$ in $\mathcal{X}$:

$$\Pr[A \cup B] \leq \Pr[A] + \Pr[B]$$

- Events $A$ and $B$ are **independent** if $\Pr[A \text{ and } B] = \Pr[A] \cdot \Pr[B]$

# Randomized algorithms

- Deterministic algorithm: $y \leftarrow A(x)$

- Randomized algorithm:

  $y \leftarrow A(x; r)$ where $r \xleftarrow{\$} \{0,1\}^n$

  $y \xleftarrow{\$} A(x)$