Introduction to Cryptography

TEK 4500 (Fall 2022)
Problem Set 2

Problem 1.

Read Chapter 3 and Chapter 4 (Sections 4.8–4.10 can be skipped) in [BR].

Problem 2.

Before AES was standardized in 2001, one of the most popular block ciphers was the Data Encryption Standard. Unlike AES, DES only have a key size of 56 bits and a block size of 64 bits.

a) Suppose you have access to a really powerful computer which runs at 10 GHz and is capable of performing a full DES encryption on a single clock cycle. How long would it take to brute-force a 56-bit DES key using this computer? Assume you have a number of known plaintext-ciphertext pairs available.

b) Technically, DES actually takes in a 64-bit key, it's just that it ignores[1] every eight bit. Thus the key is effectively only 56 bit. Now suppose *all* of the key bits were used by DES. How long would the attack take now?

c) After an upgrade, the computer is modified to instead perform a single AES-128 encryption on every clock cycle. How long would it take to brute force an AES-128 key using this computer? Give your answer in years.

d) How old is the universe?

e) How many of these machines would you have to use to brute-force the AES-128 key within one year? The average cost of electricity in Norway in 2020 was 20.7 øre per kWh. Suppose one machine uses about 4000 kWh annually. What would it cost to run your brute-force attack? Compare with the world's yearly gross product.

[1] *Actually*, these bits are meant to be used for parity-checking of the key, but for security purposes this is equivalent to simply ignoring them.

$\underline{\mathbf{Exp}_F^{\mathsf{prf}}(\mathcal{A})}$

1: $b \leftarrow \{0,1\}$
2: $F_0 \xleftarrow{\$} \mathsf{Func}[\mathsf{in}, \mathsf{out}]$
3: $K \xleftarrow{\$} \mathcal{K}$
4: $F_1 \leftarrow F_K$
5: $b' \leftarrow A^{F_b(\cdot)}$
6: **return** $b' \stackrel{?}{=} b$

$\mathbf{Adv}_F^{\mathsf{prf}}(\mathcal{A}) = \left|2 \cdot \Pr[\mathbf{Exp}_F^{\mathsf{prf}}(\mathcal{A}) \Rightarrow \mathsf{true}] - 1\right|$

Figure 1: PRF security experiment for a function $F\colon \mathcal{K} \times \{0,1\}^{\mathsf{in}} \to \{0,1\}^{\mathsf{out}}$.

Problem 3. [Problem 6.8 in [Ros]]

Suppose F is a secure PRF with input length in, and we want to use it to construct another PRF G which has a longer input length. Below are some approaches that don't work ("$\|$" denotes string concatenation, e.g. $101\|01 = 10101$). For each one, describe a successful distinguishing attack and compute its PRF-advantage (i.e., what is $\mathbf{Adv}_G^{\mathsf{prf}}(\mathcal{A})$, where $\mathcal{A}$ is the adversary that runs your attack?):

a) $G(K, X\|X') = F(K, X)\|F(K, X')$, where X and X' are each in bits long.

b) $G(K, X\|X') = F(K, X) \oplus F(K, X')$, where X and X' are each in bits long.

c) $G(K, X\|X') = F(K, X) \oplus F(K, X \oplus X')$, where X and X' are each in bits long.

d) $G(K, X\|X') = F(K, 0\|X) \oplus F(K, 1\|X')$, where X and X' are each $in - 1$ bits long.

Problem 4.

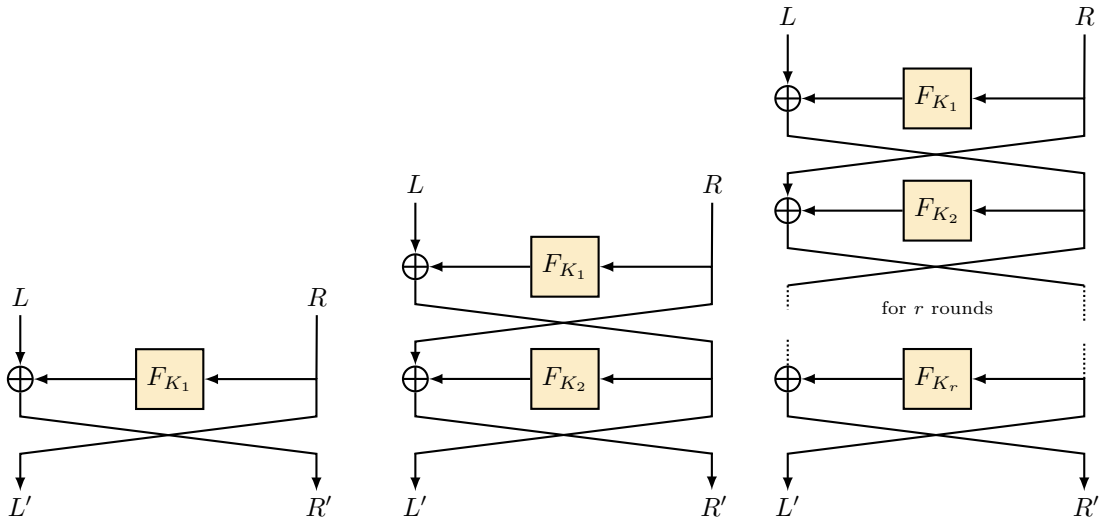
Suppose $F : \{0,1\}^{128} \times \{0,1\}^{128} \to \{0,1\}^{128}$ is a secure PRF. For each of the following constructions of a *new* PRF from F, decide whether it is also a secure PRF. If you think it's not, describe an attack, else, indicate why the new construction is also secure.

a) $G(K, X) = \begin{cases} 0^{128}, & \text{if } K = 0^{128} \\ F_K(X), & \text{otherwise} \end{cases}$

b) $G(K, X) = \begin{cases} 0^{128}, & \text{if } X = 0^{128} \\ F_K(X), & \text{otherwise} \end{cases}$

c) $G(K, X) = F(K, X) \oplus 1^{128}$

d) $G(K, X) = F(K, X) \oplus C$, where $C \in \{0,1\}^{128}$ is a *fixed* and *public* (and thus known to the adversary) hard-coded string of some arbitrary value.



(**a**) One-round. (**b**) Two-rounds. (**c**) r rounds.

Figure 2: Feistel network.

Problem 5.

In this problem we'll look at a way of turning *PRFs* into *PRPs*. The construction is called a *Feistel network* (after Horst Feistel) and is shown in Fig. 2. In detail, a Feistel network converts a PRF $F : \{0,1\}^k \times \{0,1\}^n \to \{0,1\}^n$ into a PRP $E : \{0,1\}^{r \cdot k} \times \{0,1\}^{2n} \to \{0,1\}^{2n}$, having double the block length and r times longer key, by applying F in r rounds where the first k bits of the key are used in the first round, the next k bits are used in the second round, and so on.

a) Show that the Feistel network turns *any* PRF F into a PRP E. That is, for all keys $K \in \{0,1\}^{r \cdot k}$, show that the function $E_K : \{0,1\}^{2n} \to \{0,1\}^{2n}$ is *invertible*.

b) Let $E^{(1)} : \{0,1\}^{128} \times \{0,1\}^{128} \to \{0,1\}^{128}$ denote the block cipher defined by the one-round Feistel network shown in Fig. 2a, where $F : \{0,1\}^{128} \times \{0,1\}^{64} \to \{0,1\}^{64}$ is the internal round function. Show that $E^{(1)}$ is *not* a secure PRF by demonstrating an attack. What is the PRF-advantage of your attack? That is, what is $\mathbf{Adv}^{\mathsf{prf}}_{E^{(1)}}(\mathcal{A})$, where $\mathcal{A}$ is the adversary that runs your attack?

Hint: What is $E^{(1)}(K_1, 0^{128})$?

c) Let $E^{(2)} : \{0,1\}^{256} \times \{0,1\}^{128} \to \{0,1\}^{128}$ denote the block cipher defined by the two-round Feistel network shown in Fig. 2b. Show that $E^{(2)}$ is *not* a secure PRF by demonstrating an attack. What is the PRF-advantage of your attack?

Hint: it is possible to obtain a very high PRF-advantage by making two oracle queries in the PRF experiment $\mathbf{Exp}^{\mathsf{prf}}_{E^{(2)}}(\mathcal{A})$.

d) Bonus: What can you say about the 3-round Feistel network $E^{(3)}$?

Problem 6.

The DES block cipher introduced in Problem 2 is based on the Feistel network. Suppose DES was only using a *single* round and suppose you have access to two plaintext-ciphertext pairs (X, Y), (X', Y') (in particular, $X = L_0 \| R_0$ and $Y = L_1 \| R_1$; $X' = L'_0 \| R'_0$ and $Y' = L'_1 \| R'_1$). Explain how you can recover the *key* K_1 of this one-round version of DES.

Hint 1: Unlike in Problem 5b), you should now exploit the *concrete* round function $F : \{0,1\}^{48} \times \{0,1\}^{32} \to \{0,1\}^{32}$ used inside DES. The following amount of detail about the DES round function is sufficient to answer this question (refer to Fig. 3):

- E expands 32 bits to 48 bits by copying the 32 input bits to 32 different positions in the output, and then duplicating certain bits of the input in the remaining 16 positions of the output.
- $S_1, \ldots, S_8$ are the DES S-boxes. By design, each S-box is a 4-to-1 function that maps 6 bits to 4 bits.
- P shuffles the 32 input bits around.

Hint 2: Some trial-and-error of candidate keys is necessary. However, it should be possible to obtain K_1 by trying about $4^{48/6} = 2^{16}$ candidate keys. Notice that this is much less than the possibly 2^{48} keys you would have to try by brute-force.

Problem 7.

A crucial component of round functions used in DES and AES is their S-boxes. The S-boxes are the only non-linear parts of DES and AES. Recall that a function F is linear if $F(A + B) = F(A) + F(B)$ for all inputs A, B. In this exercise you are asked to validate that the first S-box of DES, S_1, is indeed non-linear by computing the output values for a set of input values. In particular, show that $S_1(X_1) \oplus S_1(X_2) \neq S_1(X_1 \oplus X_2)$ for:

a) $X_1 = 000000, X_2 = 000001$

b) $X_1 = 111111, X_2 = 100000$

c) $X_1 = 101010, X_2 = 010101$

The definition of the S_1 S-box can be found here.

Extra: Write a script (e.g. in Python) that checks whether S_1 is non-linear for *all* inputs. Do the same for the other DES S-boxes and the AES S-box. Values for the DES and AES S-boxes can be found online, e.g., here (DES) and here (AES).

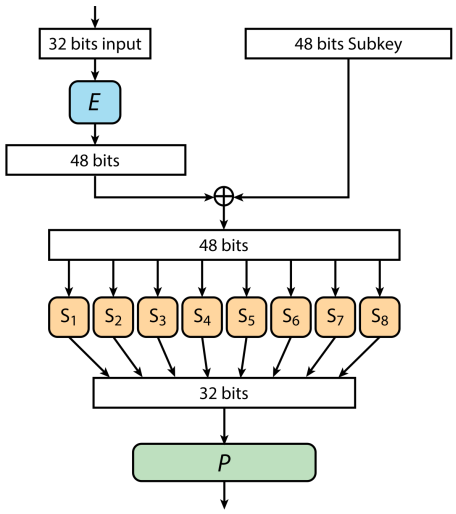


Figure 3: DES round function F.

References

[BR] Mihir Bellare and Phillip Rogaway. *Introduction to Modern Cryptography*. https://web.cs.ucdavis.edu/~rogaway/classes/227/spring05/book/main.pdf.

[PP] Christof Paar and Jan Pelzl. *Understanding Cryptography - A Textbook for Students and Practitioners*. Springer, 2010.

[Ros] Mike Rosulek. *The Joy of Cryptography*, (draft Feb 6, 2020). https://web.engr.oregonstate.edu/~rosulekm/crypto/crypto.pdf.