

# Introduction to Cryptography

TEK 4500 (Fall 2023)

Problem Set 3

## Problem 1.

Read Chapter 5 in [BR] (Sections 5.6 and 5.8 can be skipped. The proofs in Section 5.7 can be skipped on first reading, but it is recommended to have a look at it).

## Problem 2.

An encryption scheme secure according to the IND-CPA definition is not required to hide the length of the plaintext. This is captured in the formal IND-CPA security experiment (Fig. 1) by having the encryption oracle  $\mathcal{E}$  encrypt a random string of *equal* length to the input message  $M$  in the case  $b = 1$  (see Line 1 of the  $\mathcal{E}(\cdot)$  oracle).

- a) Suppose the IND-CPA experiment was modified to *not* select  $R$  at random from  $\{0, 1\}^{|M|}$  but instead from the entire message space  $\mathcal{M}$ . Show how you can break *any* encryption scheme  $\Sigma = (\text{KeyGen}, \text{Enc}, \text{Dec})$  that doesn't hide the plaintext length in this variant of IND-CPA. For simplicity, you can assume that the message space consists of all bit strings up to some length, say 1000 bits, i.e.,  $\mathcal{M} = \bigcup_{i=1}^{1000} \{0, 1\}^i$ .
- b) How can you make an encryption scheme IND-CPA secure even against the modified definition above? Again, for simplicity you can assume that  $\mathcal{M} = \bigcup_{i=1}^{1000} \{0, 1\}^i$ .

## Problem 3.

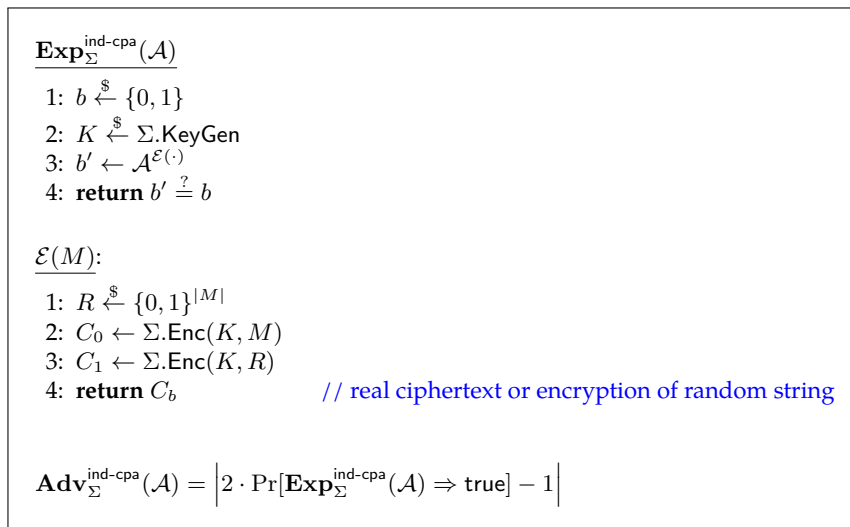
The defining equations for CBC encryption are

$$C_0 \stackrel{\$}{\leftarrow} \{0, 1\}^n \tag{1}$$

$$C_i \leftarrow E_K(M_i \oplus C_{i-1}) \quad i = 1, \dots, \ell \tag{2}$$

where  $E$  is a block cipher with  $n$  bit blocks.

- a) Write the corresponding equations for CBC decryption.
- b) Suppose you have a CBC-encrypted ciphertext  $C = C_0 \| C_1 \| C_2 \| \dots \| C_m$  and you flip one bit in block  $C_j$ . What happens to the decrypted message  $M$ ?



**Figure 1:** IND-CPA security experiment.

- c) Suppose you have a CBC-encrypted ciphertext  $C = C_0 \| C_1 \| C_2 \| \dots \| C_m$  and you drop one of the blocks  $C_j$ . What happens to the decrypted message  $M$ ?
- d) Answer **b)** and **c)** again, but this time assuming the encryption scheme is CTR\$.

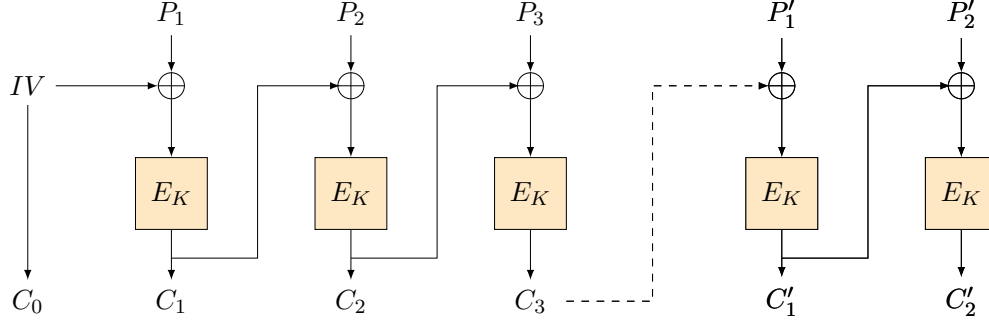
**Problem 4.** [Problem 5.5 in [BR]]

There's a variant of CBC mode called *CBC-Implicit* where the IV for the first message is chosen at random (like normal CBC), but the IV for each subsequent message is taken from the last block of the previous ciphertext (see Fig. 2). The scheme is probabilistic and stateful. Show that CBC-Implicit is insecure by giving a simple and efficient adversary that breaks it in the IND-CPA sense.

**Note:** Curiously, CBC-Implicit was how CBC was implemented in TLS 1.0. This was patched in TLS 1.1 and higher.

**Problem 5.** [Problem 3.21 in [KL07]]

Let  $\Sigma_1 = (\text{KeyGen}_1, \text{Enc}_1, \text{Dec}_1)$  and  $\Sigma_2 = (\text{KeyGen}_2, \text{Enc}_2, \text{Dec}_2)$  be two encryption schemes for which it is known that at least one is IND-CPA secure. The problem is that you don't know which one is IND-CPA secure and which one may not be. Show how to construct an encryption scheme  $\Sigma$  that is *guaranteed* to be IND-CPA secure as long as at least one of  $\Sigma_1$  or  $\Sigma_2$  is IND-CPA secure. Give a high-level justification for why your scheme is secure.



**Figure 2:** CBC-Implicit mode-of-operation illustrated for two messages  $M_1 = P_1 || P_2 || P_3$  and  $M_2 = P'_1 || P'_2$ . The IV used for the second message is the last block of the previous ciphertext, i.e.,  $C_3$ .

**Problem 6.**

Show that the CBC and CTR modes-of-operation are *not* IND-CCA secure by describing concrete attacks and calculating their IND-CCA advantages. The IND-CCA security experiment is given in Fig. 3.

$\text{Exp}_{\Sigma}^{\text{ind-cca}}(\mathcal{A})$	$\mathcal{E}(M):$	$\mathcal{D}(C):$
1: $b \xleftarrow{\$} \{0, 1\}$	1: $R \xleftarrow{\$} \{0, 1\}^{ M }$	// Asking for the decryption of a $C$
2: Ciphertexts $\leftarrow []$	2: $C_0 \leftarrow \Sigma.\text{Enc}(K, M)$	// received from a previous call to the
3: $K \xleftarrow{\$} \Sigma.\text{KeyGen}$	3: $C_1 \leftarrow \Sigma.\text{Enc}(K, R)$	// $\mathcal{E}$ oracle is not allowed
4: $b' \leftarrow \mathcal{A}^{\mathcal{E}(\cdot), \mathcal{D}(\cdot)}$	4: Ciphertexts.add( $C_b$ )	1: <b>if</b> $C \in \text{Ciphertexts}$ <b>then</b>
5: <b>return</b> $b' \stackrel{?}{=} b$	5: <b>return</b> $C_b$	2: <b>return</b> $\perp$
		3: <b>return</b> $\Sigma.\text{Dec}(K, D)$

**Figure 3:** IND-CCA security experiment.

**Problem 7.** [Problem 7.9 in [Ros]]

Suppose  $E$  is a secure block cipher with block length  $n$ . Let  $\Sigma = (\text{KeyGen}, \text{Enc}, \text{Dec})$  be the encryption scheme defined as follows:

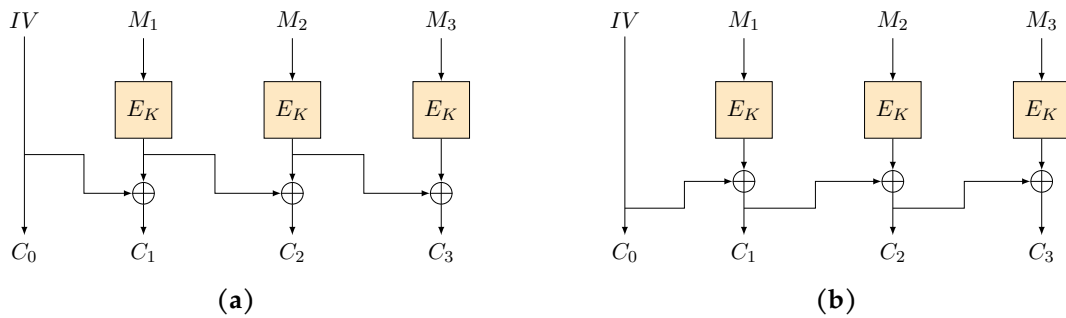
$\Sigma.\text{KeyGen}$ : 1: $K \xleftarrow{\$} \{0, 1\}^n$ 2: <b>return</b> $K$	$\Sigma.\text{Enc}(K, M_1 \  M_2)$ : 1: $R \xleftarrow{\$} \{0, 1\}^n$ 2: $S \leftarrow E_K(R \oplus M_1)$ 3: $T \leftarrow E_K(R \oplus M_1 \oplus E_K(M_1) \oplus M_2)$ 4: <b>return</b> $R, S, T$
---	--

- a) What is the key space, message space, and ciphertext space of  $\Sigma$ ?
- b) Describe the decryption algorithm of  $\Sigma$ .
- c) Show that  $\Sigma$  is *not* IND-CPA secure.

**Problem 8.** [Problem 8.8 in [Ros]]

Below are two block cipher modes of operation using a block cipher  $E$  with block length  $n$  and where the IV is chosen randomly from  $\{0, 1\}^n$ . For each of the modes:

- Describe the corresponding decryption procedure.
- Is the mode IND-CPA secure? If yes, give a high-level justification, if no, give an attack and describe its IND-CPA advantage.



**Figure 4:** Ciphers for Problem 8.

**Problem 9.** (Optional):

Bellare and Rogaway [BR] are using a definition of IND-CPA security which is slightly different from the one we used in class. In BR's security experiment the adversary is again tasked with distinguishing between two worlds, but this time the adversary always submits *two* (equal length) messages  $(M_0, M_1)$  to the encryption oracle. In the "Left" world the adversary always gets back an encryption of the left input  $M_0$ , while in the "Right" world it always gets back an encryption of the right input  $M_1$ . This is shown in Fig. 5. For the purposes of this exercise let's call BR's definition Left-or-Right (LoR) security, and the one we used in class (defined in Fig. 1) Real-or-Random (RoR) security.

$\text{Exp}_{\Sigma}^{\text{lor-ind-cpa}}(\mathcal{A})$ <ol style="list-style-type: none"><li>1: <math>b \leftarrow \{0, 1\}</math></li><li>2: <math>K \xleftarrow{\\$} \Sigma.\text{KeyGen}</math></li><li>3: <math>b' \leftarrow \mathcal{A}^{\mathcal{E}^{\text{LoR}}(\cdot, \cdot)}</math></li><li>4: <b>return</b> <math>b' \stackrel{?}{=} b</math></li></ol> $\mathcal{E}^{\text{LoR}}(M_0, M_1):$ <ol style="list-style-type: none"><li>1: <b>if</b> <math> M_0  \neq  M_1 </math> <b>then</b> // Adversary's cheating! Suppress output</li><li>2: <b>return</b> <math>\perp</math></li><li>3: <b>return</b> <math>\Sigma.\text{Enc}(K, M_b)</math> // Either encryption of <math>M_0</math> or <math>M_1</math></li></ol> $\text{Adv}_{\Sigma}^{\text{lor-ind-cpa}}(\mathcal{A}) = \left  2 \cdot \Pr[\text{Exp}_{\Sigma}^{\text{lor-ind-cpa}}(\mathcal{A}) \Rightarrow \text{true}] - 1 \right $
--

**Figure 5:** The Left-or-Right IND-CPA security experiment defined in [BR].

You will now show that these two definitions are in fact equivalent. That is, an encryption scheme having LoR security also has RoR security, and vice versa.

a) Show that LoR security implies RoR security.

**Hint:** Show instead the equivalent contrapositive statement: RoR *insecurity* implies LoR *insecurity*. That is, suppose there was some algorithm  $\mathcal{A}$  that could break an encryption scheme  $\Sigma$  according to the RoR definition given in Fig. 1, then use this  $\mathcal{A}$  to construct *another* adversary  $\mathcal{B}$  that breaks  $\Sigma$  according to the LoR definition in Fig. 5.

**Hint:** Think of yourself as being the adversary  $\mathcal{B}$ : you're playing in the LoR experiment and have access to an encryption oracle  $\mathcal{E}^{\text{LoR}}(\cdot, \cdot)$  that takes in *two* inputs and returns an encryption of either the left or the right input. Additionally, you also (by hypothesis) have access to some algorithm  $\mathcal{A}$  which thinks it plays in the RoR experiment. How  $\mathcal{A}$  is represented is not important; you can simply think of it as some piece of code that

you can run on your own machine. Once you start running  $\mathcal{A}$  it will begin outputting *one* and *one* message, and expect back a response which is either a real encryption or a “fake” encryption of some random data. That is,  $\mathcal{A}$  expects to have access to an encryption oracle  $\mathcal{E}^{\text{RoR}}(\cdot)$ . Eventually,  $\mathcal{A}$  will output some bit  $b_{\text{RoR}}$  as its guess of which world it thinks it’s in. How can you answer  $\mathcal{A}$ ’s queries so that it believes that it’s interacting with the  $\mathcal{E}^{\text{RoR}}(\cdot)$  oracle of the actual RoR experiment (Fig. 1)?

**Hint:** Use your access to the  $\mathcal{E}^{\text{LoR}}(\cdot, \cdot)$  oracle: when  $\mathcal{A}$  makes a query  $M$ , submit this as the “left” input to  $\mathcal{E}^{\text{LoR}}(\cdot, \cdot)$ . What should you use as the “right” input? Return the response from the  $\mathcal{E}^{\text{LoR}}(\cdot, \cdot)$  oracle back to  $\mathcal{A}$ .

**Hint:** Let your output in the LoR experiment simply equal  $\mathcal{A}$ ’s output, i.e.,  $b_{\text{LoR}} = b_{\text{RoR}}$ . If  $\mathcal{A}$  has RoR advantage  $\text{Adv}_{\Sigma}^{\text{ror-ind-cpa}}(\mathcal{A})$ , what’s your LoR advantage  $\text{Adv}_{\Sigma}^{\text{lor-ind-cpa}}(\mathcal{B})$ ?

b) Show that RoR security implies LoR security.

**Hint:** Prove the contrapositive: LoR *insecurity* implies RoR *insecurity*.

**Hint:** Suppose some algorithm  $\mathcal{A}$  is able to break  $\Sigma$  according to the LoR definition of Fig. 5. Create an RoR adversary  $\mathcal{B}$ , having access to an encryption oracle  $\mathcal{E}^{\text{RoR}}(\cdot)$ , that uses  $\mathcal{A}$  to break  $\Sigma$  according to the RoR definition of Fig. 1.

**Hint:** Imagine yourself in  $\mathcal{B}$ ’s shoes: once you start running  $\mathcal{A}$ , it will begin submitting *pairs* of messages  $(M_0, M_1)$ , expecting to get back an encryption of one of them. How can you answer these queries, given that you only have at your disposal an encryption oracle  $\mathcal{E}^{\text{RoR}}(\cdot)$  which takes a *single* input?

**Hint:** Define an *internal* bit  $b_{\text{sim}}$  that you flip yourself (once). Whenever  $\mathcal{A}$  makes a query  $(M_0, M_1)$ , forward  $M_{b_{\text{sim}}}$  to your encryption oracle  $\mathcal{E}^{\text{RoR}}(\cdot)$  and return the result back to  $\mathcal{A}$ . That is, use the internal bit  $b_{\text{sim}}$  to decide which of  $M_0, M_1$  to forward to the  $\mathcal{E}^{\text{RoR}}(\cdot)$  oracle. When  $\mathcal{A}$  eventually outputs a bit  $b_{\mathcal{A}}$ , what should  $\mathcal{B}$  output to its own RoR experiment?

**Hint:** Make a comparison between  $b_{\mathcal{A}}$  and the internal bit  $b_{\text{sim}}$  that you created yourself. From  $\mathcal{A}$ ’s point-of-view, what does the cases of  $b_{\text{sim}}$  correspond to?

**Hint:** If  $\mathcal{A}$  has LoR advantage  $\text{Adv}_{\Sigma}^{\text{lor-ind-cpa}}(\mathcal{A})$ , then  $\mathcal{B}$ ’s RoR advantage should be:

$$\text{Adv}_{\Sigma}^{\text{ror-ind-cpa}}(\mathcal{B}) \geq \frac{\text{Adv}_{\Sigma}^{\text{lor-ind-cpa}}(\mathcal{A})}{2}$$

## References

- [BR] Mihir Bellare and Phillip Rogaway. *Introduction to Modern Cryptography*. <https://web.cs.ucdavis.edu/~rogaway/classes/227/spring05/book/main.pdf>.
- [KL07] Jonathan Katz and Yehuda Lindell. *Introduction to Modern Cryptography*. Chapman and Hall/CRC Press, 2007.
- [Ros] Mike Rosulek. *The Joy of Cryptography*, (draft Feb 6, 2020). <https://web.engr.oregonstate.edu/~rosulekm/crypto/crypto.pdf>.