Solutions to exercises from Lecture 8 Non-cooperative game theory

TEK5010 Multiagent systems 2020

Question 1

a) Identify all (pure strategy) Nash, PO & SO

(A) "PD"

i \ j	D	C
D	2,2 *	4,1 +
C	1,4 +	3,3 + □

Nash (D,D)

PO (C,D) v (D,C) v (C,C)

SO (C,C)

* Nash , + PO , □ SO

(B) "MP"

		i	
		H	T
j	H	$\underline{1}, -1$ $+$ $\square$	$-1, \underline{1}$ $+$ $\square$
	T	$-1, \underline{1}$ $+$ $\square$	$\underline{1}, -1$ $+$ $\square$

No NaM

All outcomes are PO

All outcomes are SO

* NaM, + PO, $\square$ SO

(C) "BS"

i \ j	O	F
O	□ $\underline{3}, \underline{2}$ * +	0, 0
F	0, 0	□ $\underline{2}, \underline{3}$ * +

Nash $(O,O) \vee (F,F)$

PO $(O,O) \vee (F,F)$

SO $(O,O) \vee (F,F)$

* Nash , + PO , □ SO

↳ What is Nash's theorem?

Every game in which every player has a finite set of possibilities has a Nash equilibrium in mixed strategies.

Often MSNE is hard to compute!

Mixed strategy Nash equilibrium:

Find optimal probabilities of playing the various strategies. How often do you play a particular strategy?

A mixed strategy over $(s_1, s_2, \ldots, s_M)$ strategies is to find a probability distribution $(p_1, p_2, \ldots, p_M)$ of playing the strategies (so opponents become indifferent in their choice of strategy)

( ) Find MSNE in the games

A

		j: q	j: $(1-q)$
		D	C
i: p	D	2,2	4,1
i: $(1-p)$	C	1,4	3,3

p is probability of i playing D
$(1-p)$ is probability of i playing C
q is probability of j playing D
$(1-q)$ is probability of j playing C

$\Rightarrow$ $p+(1-p)=1$ $q+(1-q)=1$

$\Rightarrow$ What is p and q if

$$E_i(U_D)=E_i(U_C) \wedge E_j(U_D)=E_j(U_C)$$

( ) Find MSNE in the games

(A)

		j: q	j: $(1-q)$
i		D	C
p	D	2,2	4,1
$(1-p)$	C	1,4	3,3

I $E_i(u_D) = E_i(u_C) \Rightarrow u_p = u_{(1-p)}$

II $E_j(u_D) = E_j(u_C) \Rightarrow u_q = u_{(1-q)}$

II $u_q = p2 + (1-p)4 = u_{(1-q)} = p \cdot 1 + (1-p)3$

$\Rightarrow 2p + 4 - 4p = p + 3 - 3p$

$2p - 4p - p + 3p = 3 - 4$

$0p = -1 \quad \emptyset$

( ) Find MSNE in the games

A

		j: q	$(1-q)$
i		D	C
p	D	2,2	4,1
$(1-p)$	C	1,4	3,3

I $E_i(U_D) = E_i(U_C)$ $\quad U_p = U_{(1-p)}$

II $E_j(U_D) = E_j(U_C)$ $\quad U_q = U_{(1-q)}$

I $q2 + (1-q)4 = q1 + (1-q)3$

$2q + 4 - 4q = q + 3 - 3q$

$2q - 4q - q + 3q = 3 - 4$

$0q = -1 \quad \varnothing$

No MSNE, but pure strategy exist so Nash's theorem holds $\square$

B

$$
\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{j} \\
 & & q & 1-q \\
 & & H & T \\
\hline
i \quad p & H & 1,-1 & -1,1 \\
\hline
1-p & T & -1,1 & 1,-1 \\
\hline
\end{array}
$$

I $\quad E_i(U_H) = E_i(U_T)$

II $\quad E_j(U_H) = E_j(U_T)$

I $\quad U_p = q\cdot 1 + (1-q)(-1) = U_{1-p} = q(-1) + (1-q)1$

$$q - 1 + q = -q + 1 - q$$

$$q + q + \cancel{q} + \cancel{q} = 1 + 1$$

$$4q = 2 \quad \Rightarrow \quad \underline{q = \frac{1}{2}}$$

B

j

i		q	1-q
		H	T
p	H	1,-1	-1,1
1-p	T	-1,1	1,-1

I $E_i(u_H) = E_i(u_T)$

II $E_j(u_H) = E_j(u_T)$

II $p(-1) + (1-p)1 = p1 + (1-p)(-1)$

$-p + 1 - p = p - 1 + p$

$-p - p - p - p = -1 - 1$

$-4p = -2 \Rightarrow \underline{p = \frac{1}{2}}$

MSNE

$\underline{p = \frac{1}{2} \wedge q = \frac{1}{2}}$

c)

j

i			q	$1-q$
			O	F
	p	O	3,2	0,0
	$1-p$	F	0,0	2,3

I $E_i(u_O) = E_i(u_F)$

II $E_j(u_O) = E_j(u_F)$

I $3q + (1-q)\cdot 0 = 0\cdot q + (1-q)2$

$3q = 2 - 2q$

$3q + 2q = 2$

$5q = 2 \Rightarrow \underline{\underline{q = \frac{2}{5}}}$

(c)

		j: q	$1-q$
i		O	F
p	O	3,2	0,0
$1-p$	F	0,0	2,3

I $\quad E_i(u_O) = E_i(u_F)$

II $\quad E_j(u_O) = E_j(u_F)$

II $\quad 2p + (1-p)0 = 0p + (1-p)3$

$2p = 3 - 3p$

$5p = 3$

$p = \frac{3}{5}$

MSNE

$p = \frac{3}{5} \wedge q = \frac{2}{5}$

Summary of the 3 games

(A) "PD"

		q: D	q: C
p	D	*	†
	C	†	† □

No MSNE?

- In PD it is not possible to reach a rational outcome that is also SO
- Nash's theorem holds due to pure NE

(B) "MP"

		q: H	q: T
p	H	† □	† □
	T	† □	† □

No pure NE?

MSNE $p = \frac{1}{2} \wedge q = \frac{1}{2}$

- Need MSNE to solve MP

(C) "BS"

		q: O	q: F
p	O	* † □	
	F		* † □

Multiple pure NE?

MSNE $p = \frac{3}{5} \wedge q = \frac{2}{5}$

- Need MSNE to chose between multiple pure NE?

* Pure Nash, † PO, □ SO

I $\quad E_i(u_C) = E_i(u_D)$

II $\quad E_j(u_C) = E_j(u_D)$

			j: q	$1-q$
			D	C
i	p	D	$u_{DD_{i,j}}$	$u_{DC_{i,j}}$
	$1-p$	C	$u_{CD_{i,j}}$	$u_{CC_{i,j}}$

II

$$
\begin{aligned}
E_j &= q\,E_j(u_C) + (1-q)\,E_j(u_D) \\
&= q\left(p\,u_{DD} + (1-p)\,u_{CD}\right) + (1-q)\left(p\,u_{DC} + (1-p)\,u_{CC}\right) \\
&= q\,U + (1-q)\,U \qquad \text{if } E_j(u_C) = E_j(u_D) = U \\
&= U\left(q + (1-q)\right) \\
&= \underline{U} \quad \text{for alle } q\,!
\end{aligned}
$$