TER5010 MAS

Lecture 4 : SR1

Exercise : Curve fitting USL to data

Question 1

a) Given the USL

$$R(N) = C \frac{N}{1 + \alpha(N-1) + \beta N(N-1)}$$

where R is the performance measure
N is the number of processes/agents
C is a scalar
α is contention (limited resources)
β is coherency (lack of communication)

and the following two data sets

N	R_1	R_2
0	0	0
15	7	20
30	24	50
60	22	65
100	25	30
125	29	26
150	26	15

What numerical parameter values would be a reasonable assumption for "manually" fitting a USL curve to the data? Assume $C=1$ and α and β are unknowns.

In this case we are asked to plot the data and "guess" a system function that would approximate the data.

In order to do so we also need to "guess" parameters of the function.

Lets use Excel to plot data set 1 and graph the USL using initial "guesses" of the α and β parameters

$\alpha = 0{,}03$ Based on slide 15
$\beta = 0$ from lecture 4

and end up with

$\alpha = 0{,}025$
$\beta = 0{,}00003$

as reasonable parameters

Initial "guesses" for data set 2

$\alpha = -0{,}03$ based on slid 17
$\beta = 0{,}0005$ from lecture 4

which is reasonable

The Excel file is "Manual USL.xlsx".

b) Why must we use non-linear regression methods for curve fitting the USL? And why does this complicate the analysis?

The USL is not a linear function in parameters (of type $y_i = x_i^T \beta + \varepsilon_i$) making it non-linear in some form. This is seen from the inverse relation of $1/(\alpha + \beta)$ which makes it difficult to transform into a linar model of type $y_i = \alpha N_i + \beta N_i + \varepsilon_i$

Typically one would minimise the errors between data and model to obtain a optimal fit. This is called least squares (LS) regression.

LS regression is analytically solvable for linar functions, but not guaranteed for non-lineer regression models.

c) Could you use the Python library SciPy to curve fit the US2 to the given data?

See the "NLS.US2.py" file for the Python program with comments.

d) Could you explain the underlying mathematics for curve fitting the VSL using non-linear least squares (NLS)?

In NLS we want to minimize the sum of all squared errors between measured data and the assumed system function.

$$\varepsilon_i = y_i - f(x_i, \beta) \qquad \varepsilon_i = \text{error}$$

The sum of squared errors is given by the identity

$$(1) \quad S = \sum_{i=1}^{m} \varepsilon_i^2 = \sum_{i=1}^{m} \left(y_i - f(x_i, \beta)\right)^2$$

and optimality is given by

$$(2) \quad \frac{\partial S}{\partial \beta_j} = 0 \qquad \text{for all } j = 1, 2, \ldots, n$$

If we do this for the USL

$$S = \sum_{i=1}^{m} \left(y_i - R(N_i)\right)^2 \qquad f(x_i,\beta) = R(N_i)$$

where $R(N) = \dfrac{N}{1+\alpha(N-1)+\beta N(N-1)} = \dfrac{u}{v}$

we get for minimization

$$(3)\quad \frac{\partial S}{\partial \alpha} = 2\sum_{i=1}^{m}\left[\left(y_i - R(N_i)\right)\frac{\partial\left(y_i - R(N_i)\right)}{\partial \alpha}\right]$$

$$= 2\sum_{i=1}^{m}\left[\left(y_i - R(N_i)\right)\frac{\partial R(N_i)}{\partial \alpha}\right] = 0$$

where

$$\frac{\partial R}{\partial \alpha} = \frac{u' \cdot v - v' u}{v^2} = \frac{v - (N-1)N}{v^2}$$

$$= \frac{v - N(N-1)}{v^2}$$

$$(4)\quad \frac{\partial S}{\partial \beta} = 2\sum_{i=1}^{m}\left[(y_i - R(N_i))\,\frac{\partial (y_i - R(N_i))}{\partial \beta}\right]$$

$$= 2\sum_{i=1}^{m}\left[(y_i - R(N_i))\,\frac{\partial R(N_i)}{\partial \beta}\right] = 0$$

where

$$\frac{\partial R}{\partial \beta} = \frac{u' \cdot v - v' u}{v^2} = \frac{v - N(N-1)\cdot N}{v^2}$$

$$= \frac{v - N^2(N-1)}{v^2}$$

Equation 3 and 4 are hard to solve analytically, so we have to use numerical optimization techniques instead for determining system parameters, hence NLS.

$$\beta_j^k \approx B_j^{k+1} + \Delta B_j$$

At each iteration the model

can be linearized by approximation to a first order Taylor polynomial expansion about β^k.

$$(5)\quad f(x_i, \beta) \approx f(x_i, \beta^k) + \sum_{j=1}^{n} \frac{\partial f(x_i, \beta^k)}{\partial \beta_j}(\beta_j - \beta_j^k)$$

$$= f(x_i, \beta^k) + \sum_{j=1}^{n} \frac{\partial f(x_i, \beta^k)}{\partial \beta_j} \Delta\beta^k$$

If we do this we have the following matrix for the updated parameters

$$D = \begin{bmatrix} \Delta\beta_1 \\ \vdots \\ \Delta\beta_n \end{bmatrix} = \begin{bmatrix} \Delta\alpha \\ \Delta\beta \end{bmatrix} \text{ for the VSL}$$

given by the following relation

$$D = H^{-1} A$$

giving $\alpha = \alpha_0 + \Delta\alpha$ and $\beta = \beta_0 + \Delta\beta$

where H for the NLS is

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \qquad H^{-1} = \frac{1}{H_{11}H_{22} - H_{12}H_{21}} \begin{bmatrix} H_{22} & -H_{21} \\ -H_{12} & H_{11} \end{bmatrix}$$

where

$$H_{11} = \sum_{i=1}^{m} \frac{\partial R}{\partial \alpha_0} \cdot \frac{\partial R}{\partial \alpha_0} = \sum_{i=1}^{m} \left(\frac{\partial R}{\partial \alpha_0}\right)^2$$

$$H_{22} = \sum_{i=1}^{m} \frac{\partial R}{\partial \beta_0} \frac{\partial R}{\partial \beta_0} = \sum_{i=1}^{m} \left(\frac{\partial R}{\partial \beta_0}\right)^2$$

$$H_{12} = H_{21} = \sum_{i=1}^{m} \left(\frac{\partial R}{\partial \alpha_0}\right)\left(\frac{\partial R}{\partial \beta_0}\right)$$

and

$$A_1 = \sum_{i=1}^{m} \frac{\partial R}{\partial \alpha_0} \left(y_i - R(N_i, \alpha_0, \beta_0)\right)$$

$$A_2 = \sum_{i=1}^{m} \frac{\partial R}{\partial \beta_0} \left(y_i - R(N_i, \alpha_0, \beta_0)\right)$$

See the "NLS_USL.xlsx" for a Excel spreadsheet of the analytical approach to nonlinear regression of USL.