TEK 5010 MAS

Lecture 5 : SR2

Exercise : Urn models

Question 1

a) Could you explain the generalized equation for urn models given by

$$\Delta B(B) = 4\left(P_{FB}\left(\frac{B}{N}\right) - \frac{1}{2}\right)\left(\frac{B}{N} - \frac{1}{2}\right)$$

where B is number of robots with opinion σ_B
N is number of robots in swarm
$A = N - B$ is the number of robots with opinion σ_A in a two opinion swarm.
ΔB is change in number of robots of opinion σ_B in one iteration of an urn model (or draw)

P_{FB} is the feedback of the urn model

Ehrenfest : $P_{FB}(\frac{B}{N}) = 0 \Rightarrow \frac{B}{N}^* = \frac{1}{2}$

Eigen : $P_{FB}(\frac{B}{N}) = 1 \Rightarrow \frac{B}{N}^* = 0 \vee \frac{B}{N}^* \; 1$

Hamann : $P_{FB}(\frac{B}{N}) = \frac{3}{4} \sin(\frac{\pi B}{N}) \Rightarrow \frac{B}{N}^* = 0{,}23 \vee \frac{B}{N}^* = 0.72$

If $\Delta B = 0$ we have reached $\frac{B}{N}^*$ equilibrium between the two opinions

↳ What urn model would you employ if all robots were to converge on one of the two ways? Could you calculate the expected change in the ratio assuming initial distribution of 65/35%, 90/10% and 51/49%?

Since we want all robots to go one way we need urn model $\Delta B = 0$ at $\frac{B}{N}^* = 0$ or $\frac{B}{N}^* = 1$

This is the Eigen feedback of $P_{FB} = 1$ giving

$$\Delta B = 4(1 - 0.5)\left(\frac{B}{N} - 0.5\right)$$
$$= 4 \cdot 0.5 \left(\frac{B}{N} - 0.5\right)$$
$$= 2\frac{B}{N} - 1$$

$$\Delta B = 2 \cdot 0.65 - 1 = 0.3$$

We expect that the number of robots with opinion σ_B is increased by 0.3 giving updated B fraction

$$s' = \frac{B'}{N} = \frac{B}{N} + \frac{\Delta B}{N} = 0.65 + \frac{0.3}{10}$$
$$= 0.65 + 0{,}03 = \underline{\underline{0{,}68}}$$

For 90/10% and 51/49% we get

$$\Delta B = 2 \cdot 0.9 - 1 = 0.8 \Rightarrow s' = 0.9 + \frac{0.8}{10} = 0.98$$
$$\Delta B = 2 \cdot 0.51 - 1 = 0{,}02 \Rightarrow s' = 0.51 + \frac{0.02}{10} = 0{,}512$$

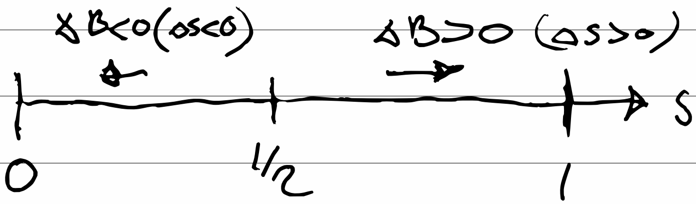
Stability analysis of Eigen

$$\Delta B = 2s - 1$$

$$\Delta B = 0 \Rightarrow s^* = \frac{B^*}{N} = \frac{1}{2}$$

$$\Delta B(s > \tfrac{1}{2}) > 0$$

$$\Delta B(s < \tfrac{1}{2}) < 0$$



which gives $s = 1/2$ as unstable saddle point and only $s^* = 0$ or $s^* = 1$ as equilibrium.

c) What urn model would give a 50/50% split of swarm between the two ways?

Here we want stable $s^* = 1/2$ which is the Ehrenfest model of $p_{FB} = 0$

We get the following expression

$$\Delta B(B) = 4(0-0.5)\left(\frac{B}{N} - 0.5\right)$$

$$= -2\left(\frac{B}{N} - 0.5\right)$$

$$= -\frac{2B}{N} + 1$$

$$\Delta B(0.65) = -2 \cdot 0.65 + 1 = -0.3$$

$$s' = \frac{B'}{N} + \frac{\Delta B}{N} = 0.65 - \frac{0.3}{10} = 0.62$$

$$= \underline{0.62}$$

$$\Delta B(0.90) = -2 \cdot 0.90 + 1 = -0.8$$

$$s' = 0.90 - \frac{0.8}{10} = \underline{0.82}$$

$$\Delta B(0.51) = -2 \cdot 0.51 + 1 = -0.02$$

$$s' = 0.51 - \frac{0.02}{10} = \underline{0.508}$$

Note that $|\Delta B(0.51)| < |\Delta B(0.65)| < |\Delta B(0.9)|$

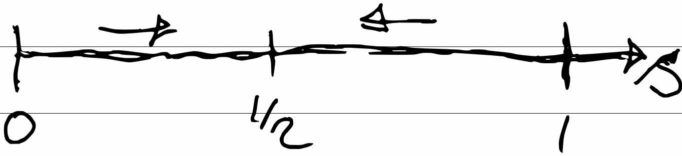
Stability analysis of Ehrenfest

$\Delta B = -2s + 1$

$\Delta B = 0 \quad \Rightarrow \quad s^* = \frac{B^*}{n} = \frac{1}{2}$

$\Delta B(s > \frac{1}{2}) < 0$

$\Delta B(s < \frac{1}{2}) > 0$



which gives $s^* = 1/2$ as stable equilibrium point

d) Or if we want 23/77 % split between the two ways?

In this case we can use the Hamann swarm model

$$p_{FB} = \frac{3}{4} \cdot \sin\left(\pi \frac{B}{n}\right) = \frac{3}{4} \sin \pi s$$

This will give

$$\Delta B(B) = 4\left(\frac{3}{4}\sin\left(\pi\frac{B}{N}\right) - \frac{1}{2}\right)\left(\frac{B}{N} - \frac{1}{2}\right)$$

$$\Delta B = (3\sin(\pi s) - 2)\left(s - \frac{1}{2}\right)$$

For 65/35 % distribution

$$\Delta B = (3 \cdot \sin(\pi\, 0.65) - 2)\left(0.65 - \frac{1}{2}\right)$$

$$\approx 0.6744 \cdot 0.15 = 0.10$$

$$s' = \frac{B'}{N} = \frac{B + \Delta B}{N} = 0.65 + \frac{0.10}{10} = \underline{0.66}$$

For 90/10 % and 51/49 % we get

$$\Delta B(0.9) = (3 \cdot \sin(\pi\, 0.9) - 2)(0.9 - 0.5)$$

$$= (-1.0688) \cdot 0.4 = -0.427$$

$$s' = 0.9 - \frac{0.427}{10} = \underline{0.857}$$

$$\Delta B(0.51) = (3 \cdot \sin(\pi \cdot 0.51) - 2)(0.51 - 0.5)$$

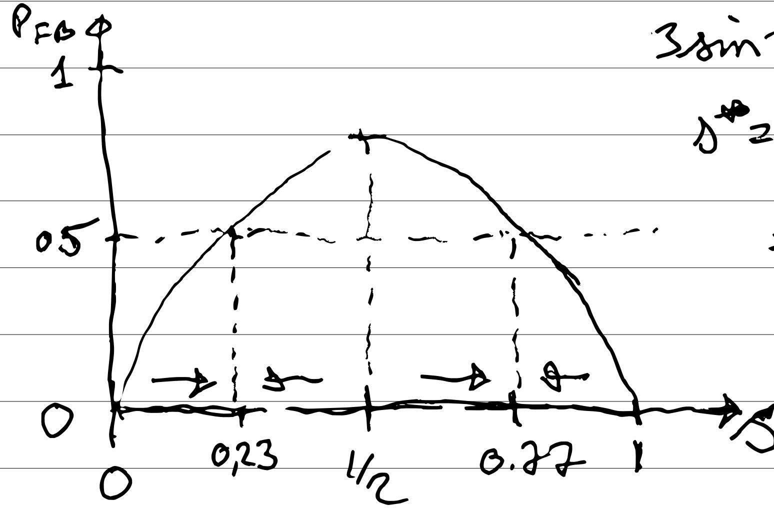
$$= 0.9985$$

$$s' = 0.51 + \frac{0.9985}{10} = \underline{0.609}$$

Stability analysis of Hamann

$$\Delta B(s) = (3\sin(\pi s) - 2)(s - 1/2)$$

$$\Delta B = 0 \Rightarrow s^* = 1/2 \vee 3\sin\pi s^* - 2 = 0$$



$$3\sin\pi s^* = 2$$

$$s^* = \frac{\sin^{-1}(2/3)}{\pi}$$

$$= 0.23$$

which gives $s^* = 1/2$ as unstable saddle point and $s^* = 0{,}23 \vee s^* = 0{,}77$ as stable equilibrium points.

e) Optional: Could you simulate the end state of the different algorithms?

See the "UrnModels.py" file for a Python program with comments.