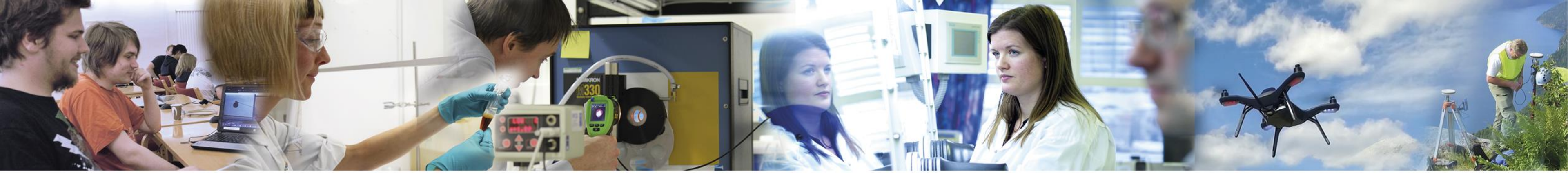


UiO : Department of Technology Systems
University of Oslo

TEK5010/9010 - Multiagent systems 2023
Lecture 10

Cooperative game theory

Jonas Moen

Highlights lecture 10 – Cooperative game theory*

- Cooperative games – forming coalitions
  - Characteristic function
  - The core
  - The Shapley value
- Simple games – 'yes/no' games
  - Weighted voting games
- Coalition structure formation – 'central planner'

*Wooldridge, 2009: chapter 13

Forming coalitions

It can be claimed that agents in Prisoner's dilemma are prevented from cooperation due to:

- Binding agreements are not possible. (The issue of trust.)
- Utility is given directly to individuals as a result of individual action.

If we drop these assumptions we can open up for modelling coalitions of cooperating agents or cooperative games.

Cooperative games

Cooperative game theory attempts to answer two basic questions:

1. Which coalition will be formed by self-interested rational agents?
2. How is the utility divided among members in this coalition?

Cooperative games

Given a set of N agents

$$Ag = \{1,2, \dots, N\}$$

that can form coalitions $C, C', C_1, \dots$

$C \in Ag$ is a subset of Ag

$C = Ag$ is the *Grand Coalition*

$C = \{i\}$ is singelton coalition of one singel agent i

Cooperative games

A cooperative game (or a coalitional game) is a pair

$$G = \langle Ag, v \rangle$$

where Ag is the set of agents

$v: \mathbf{2}^{Ag} \longrightarrow \mathbb{R}$ is the characteristic function

The characteristic function assigns a numerical value to all possible coalitions. $v(C) = k$ means that the value k is assigned to coalition C and distributed among its members.

Cooperative games

Note:

- The characteristic function is given
- The game does not explicitly say how to distribute utility among members

Cooperative games

The cooperation lifecycle – Three stages of cooperative action, [Sandholm et *al*., 1999]

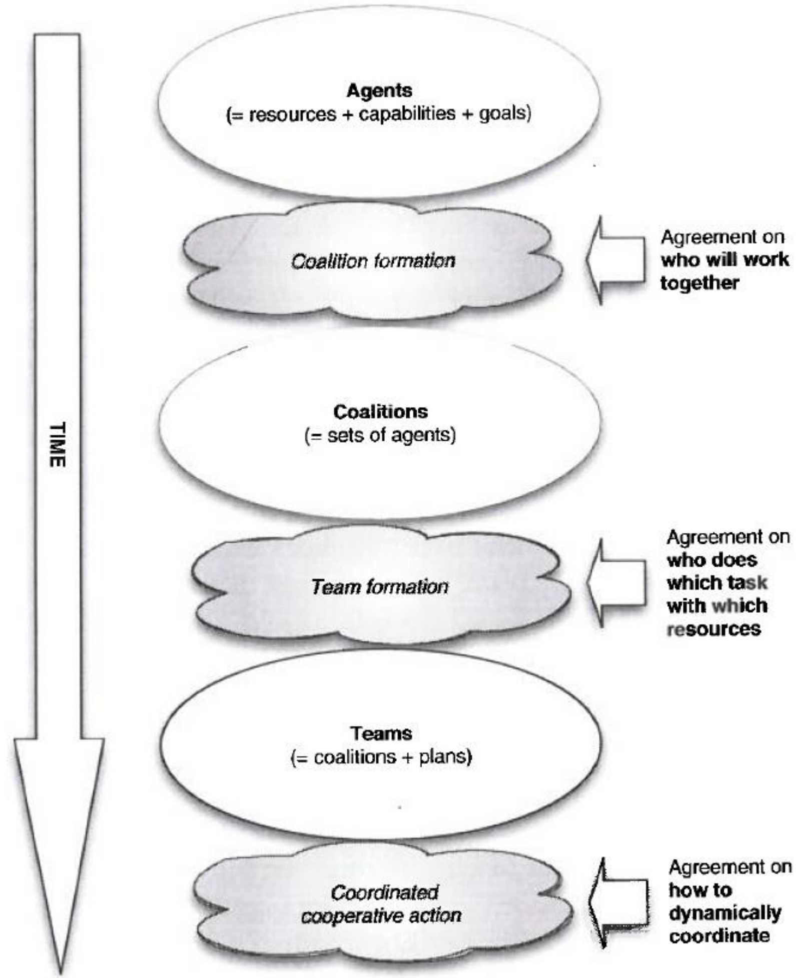


Image: Figure 13.1, Wooldridge 2009

The cooperation lifecycle

1. Coalition structure generation
   Agents maximize utility by looking at the characteristic function of all possible coalitions. Which coalition is stable? This is answered by using the notion of the *core*.

2. Solving the optimization problem for each coalition
   This is assumed to be solved given the utility of the characteristic function.

The cooperation lifecycle

3. Dividing the utility of the solution for each coalition
   The *Shapley value* is used for ‘fair’ distribution of the utility among coalition members.

The core – coalitional stability

We say that the core is the set of outcomes for a coalition, that no other coalition objects to.

Formally, a coalition $C' \subseteq Ag$ objects to an outcome $\mathbf{x}\langle x_1, \ldots, x_N \rangle$ of the coalition C if there exist some outcome $\mathbf{x}'\langle x_1', \ldots, x_N' \rangle$ in C' such that $x_i' > x_i$ for all $i \in C'$. Meaning x_i' is strictly better for all i. The outcomes satisfies the characteristic function v.

$$v(C) = \sum_{i \in C} x_i$$

The core – coalitional stability

The coalition C is stable if the core is non-empty.

The core – coalitional stability

Example of a cooperative game $G = \langle Ag, v \rangle$ where

$$Ag = \{1,2\}$$
$$v(\{1\}) = 5$$
$$v(\{2\}) = 5$$
$$v(\{1,2\}) = 20$$

The core – coalitional stability

The grand coalition of game $G = \langle Ag, v \rangle$ is stable for outcomes

$v(\{1,2\}) = \langle 0,20 \rangle = 20 \rightarrow v(\{1\}) = 5$: agent 1 objects

$v(\{1,2\}) = \langle 1,19 \rangle = 20 \rightarrow v(\{1\}) = 5$: agent 1 objects

$$\left.\begin{array}{c} v(\{1,2\}) = \langle 5,15 \rangle = 20 \\ \vdots \\ v(\{1,2\}) = \langle 15,5 \rangle = 20 \end{array}\right\} \text{The Core}$$

$v(\{1,2\}) = \langle 20,0 \rangle = 20 \rightarrow v(\{2\}) = 5$: agent 2 objects

The core – coalitional stability

The grand coalition of game $G = \langle Ag, v \rangle$ is stable for outcomes

$$\left.\begin{array}{c} v(\{1,2\}) = \langle 5,15 \rangle = 20 \\ \vdots \\ v(\{1,2\}) = \langle 15,5 \rangle = 20 \end{array}\right\} \text{The Core}$$

How to divide the utility among coalition members?

The Shapley value

The Shapley value is based on the idea that agents should get the average marginal contribution it makes to a coalition, estimated over all possible positions that it would enter the coalition.

The Shapley value is the unique value that satisfies the 'fairness' axioms.

The Shapley value

The Shapley value of agent i is given by:

$$Sh_i = \frac{1}{|Ag|!} \sum_{o \in \prod(Ag)} \mu_i\big(C_i(o)\big)$$

Where $\prod(Ag)$ is set of all possible ordering of coalition C

o is an ordering of an coalition

$\mu_i(C) = v(C \cup \{i\}) - v(C)$, given that $C \subseteq Ag \backslash \{i\}$.

is the marginal contribution of agent i to C

The Shapley value

The Shapley value satisfies 3 ‘fairness’ axioms:

1. Symmetry
   Agents that make the same contribution to the coalition should get the same utility.

   Formally, if $\mu_i(C) = \mu_j(C)$ for $C \subseteq Ag \backslash \{i, j\}$ then i and j are interchangeable and $sh_i = sh_j$.

The Shapley value

The Shapley value satisfies 3 ‘fairness’ axioms:

2. Dummy player
   Agents that do not contribute to coalitions should only receive what they can earn on their own.

   If $\mu_i(C) = v(\{i\})$ for $C \subseteq Ag \backslash \{i\}$ then $sh_i = v(\{i\})$.

The Shapley value

The Shapley value satisfies 3 ‘fairness’ axioms:

3. Additivity
   Agents that play two games get the sum of the two games. The agent does not benefit from playing the game more than once.

The Shapley value

The Shapley value satisfies 3 ‘fairness’ axioms:

3. Additivity

   If $G_1 = \langle Ag, v_1 \rangle$ and $G_2 = \langle Ag, v_2 \rangle$ with $sh_{i,1}$ and $sh_{i,2}$ for player $i \in Ag$ then $G_{1+2} = \langle Ag, v_{1+2} \rangle$ such that $v_{1+2}(C) = v_1(C) + v_2(C)$ then $sh_{i,1+2} = sh_{i,1} + sh_{i,2}$.

The Shapley value

Example of a cooperative game $G = \langle Ag, v \rangle$ where

$$Ag = \{1,2,3\}$$
$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 5$$
$$v(\{1,2\}) = v(\{1,3\}) = 10$$
$$v(\{2,3\}) = 20$$
$$v(\{1,2,3\}) = 25$$

The Shapley value

Permutations of {1,2,3} Marginal contribution of {1,2,3}

$(1,2,3)$ $\rightarrow (5,5,15)$

$(1,3,2)$

$(2,1,3)$

$(2,3,1)$

$(3,1,2)$

$(3,2,1)$

The Shapley value

Permutations of {1,2,3}	Marginal contribution of {1,2,3}
$(1,2,3)$	$\rightarrow (5,5,15)$
$(1,3,2)$	$\rightarrow (5,15,5)$
$(2,1,3)$	
$(2,3,1)$	
$(3,1,2)$	
$(3,2,1)$	

The Shapley value

Permutations of {1,2,3}	Marginal contribution of {1,2,3}
$(1,2,3)$	$\rightarrow (5,5,15)$
$(1,3,2)$	$\rightarrow (5,15,5)$
$(2,1,3)$	$\rightarrow (5,5,15)$
$(2,3,1)$	$\rightarrow (5,5,15)$
$(3,1,2)$	$\rightarrow (5,15,5)$
$(3,2,1)$	$\rightarrow (5,15,5)$

The Shapley value

Permutations of {1,2,3} Marginal contribution of {1,2,3}

$(1,2,3) \rightarrow (5,5,15)$

$(1,3,2) \rightarrow (5,15,5)$

$(2,1,3) \rightarrow (5,5,15)$

$(2,3,1) \rightarrow (5,5,15)$

$(3,1,2) \rightarrow (5,15,5)$

$(3,2,1) \rightarrow (5,15,5)$

Shapley value $\Rightarrow \left(\frac{30}{6}, \frac{60}{6}, \frac{60}{6}\right)$

The Shapley value

The Shapley value is hard to compute and represent for large number of players (exponential in N).

Is it possible to represent the different coalition permutations in a more succinct and tractable way?

1. Induced subgraph representation is succinct but not complete
2. Marginal contribution nets are complete and succinct

Induced subgraph

The characteristic function can be defined by an undirected, weighted graph, in which nodes in the graph are members of Ag and the edges are the weight $w_{i,j}$ of the edge from node i to node j in the graph [Deng and Papadimitriou, 1994].

Induced subgraph

To compute the characteristic value of a coalition $C \subseteq Ag$ we simply sum the weights $w_{i,j}$ over all the edges in the graph whose components are all contained in C:

$$v(C) = \sum_{\{i,j\} \subseteq C} w_{i,j}$$

Induced subgraph

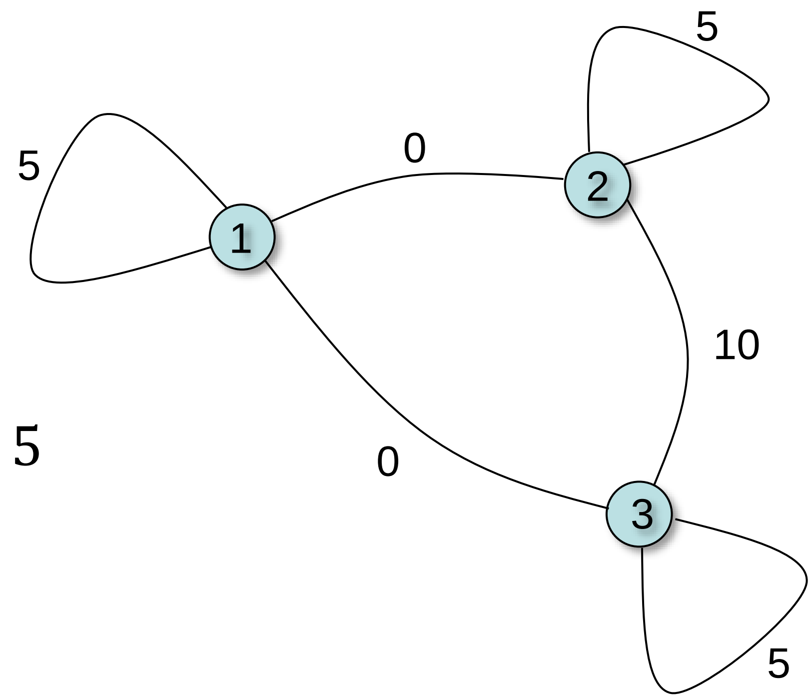
The Shapley value of each player is then given by:

$$sh_i = \frac{1}{2}\sum_{j \neq i} w_{i,j}$$

This comes directly from the symmetry axiom, where agents are interchangeable in a two player game.

Induced subgraph

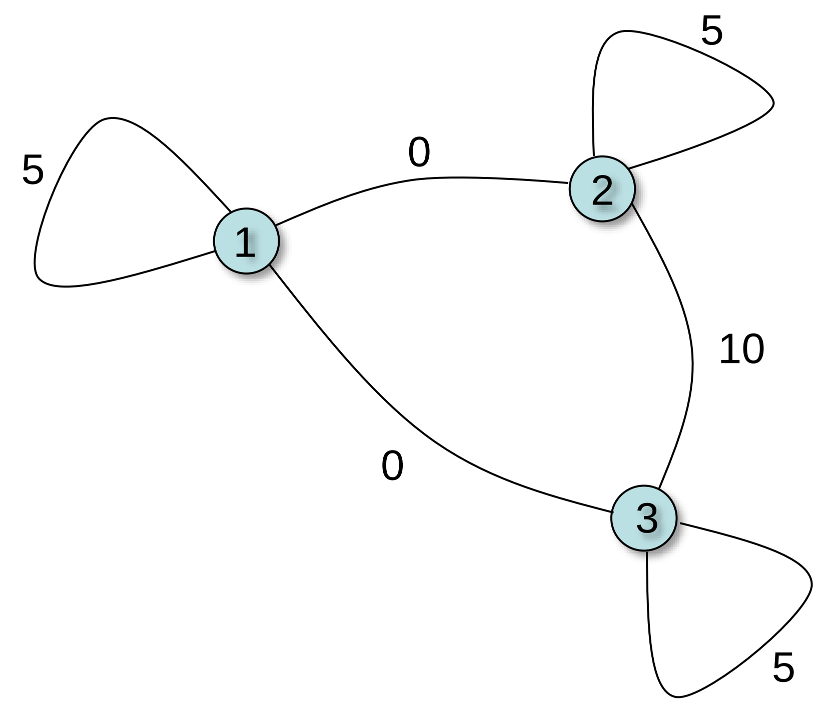
Example:



$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 5$$
$$v(\{1,2\}) = v(\{1,3\}) = 10,$$
$$v(\{2,3\}) = 20$$
$$v(\{1,2,3\}) = 25$$

Induced subgraph

Example:



$$sh_1 = 5 + 0 + 0 = 5$$

$$sh_2 = 5 + 0 + \frac{10}{2} = 10$$

$$sh_3 = 5 + \frac{10}{2} + 0 = 10$$

Marginal contribution nets

A marginal contribution net is an extension to the induced subgraph. The characteristic function of a game is represented by a set of rules rs:

$$rs_C = \{\phi \rightarrow x \in rs | C| = \phi\}$$

where $\phi \rightarrow x$ is a rule

C is a coalition

$C| =$ means that the rule applies to coalition C

Marginal contribution nets

The characteristic function v_{rs} associated with the rule set rs is defined as follows:

$$v_{rs}(C) = \sum_{\phi \to x \in rs_C} x$$

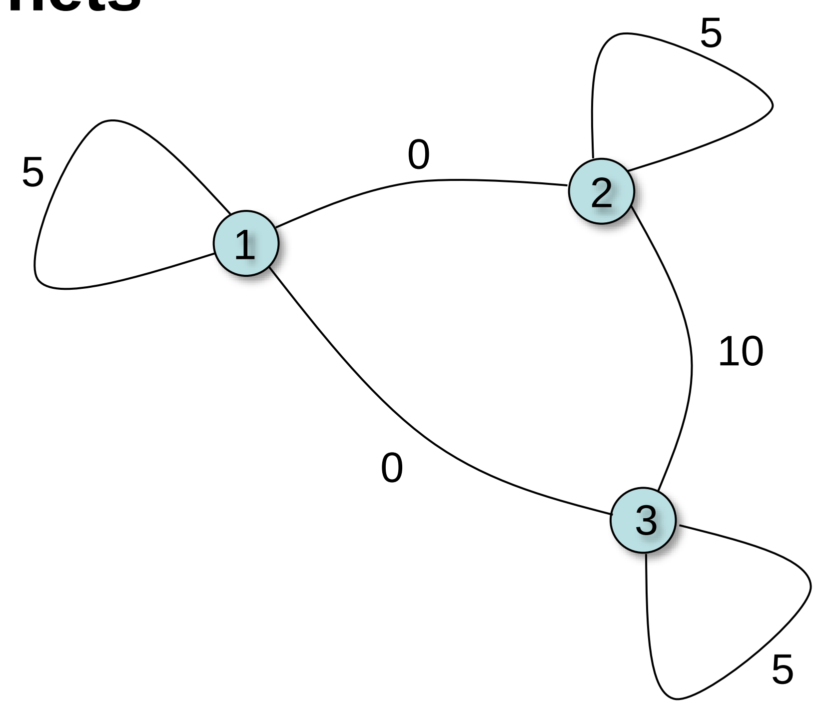
where $\phi \to x$ is a rule in the rule set rs_C

Marginal contribution nets

Example:

$$Ag = \{1,2,3\}$$

$1 \rightarrow 5$
$2 \rightarrow 5$
$3 \rightarrow 5$
$2 \wedge 3 \rightarrow 10$

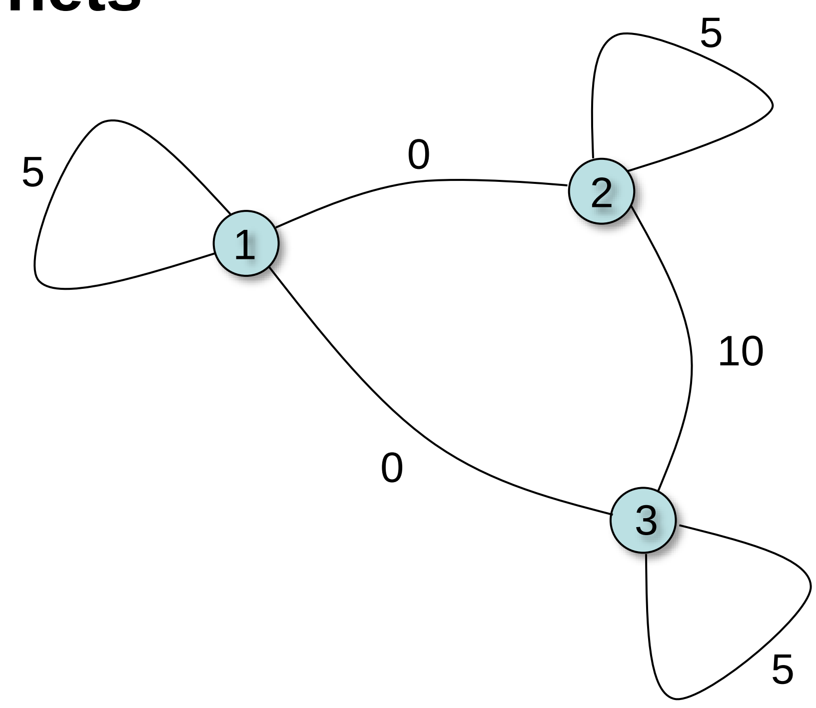


Marginal contribution nets

Example:

$$1 \rightarrow 5$$
$$2 \rightarrow 5$$
$$3 \rightarrow 5$$
$$2 \wedge 3 \rightarrow 10$$

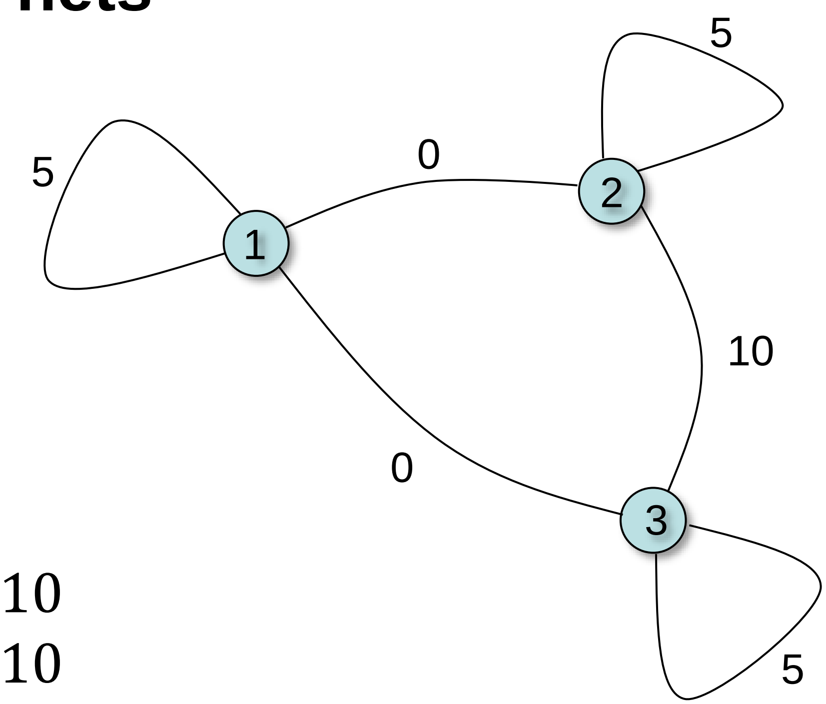
$$v(\{1\}) = 5$$
$$v(\{2\}) = 5$$
$$v(\{3\}) = 5$$



Marginal contribution nets

Example:

$$1 \rightarrow 5$$
$$2 \rightarrow 5$$
$$3 \rightarrow 5$$
$$2 \wedge 3 \rightarrow 10$$

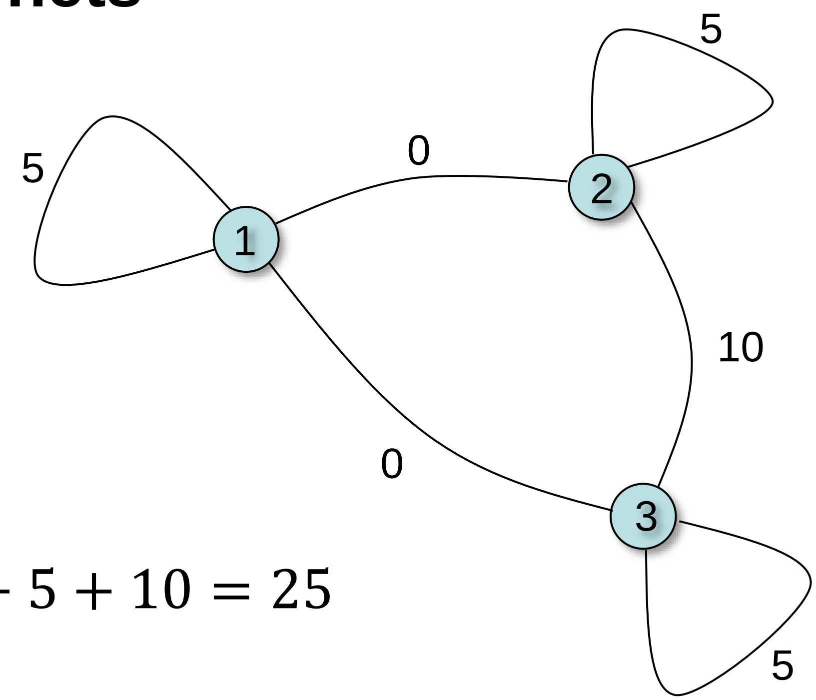


$$v(\{1,2\}) = 5 + 5 = 10$$
$$v(\{1,3\}) = 5 + 5 = 10$$
$$v(\{2,3\}) = 5 + 5 + 10 = 20$$

Marginal contribution nets

Example:

$$1 \rightarrow 5$$
$$2 \rightarrow 5$$
$$3 \rightarrow 5$$
$$2 \wedge 3 \rightarrow 10$$

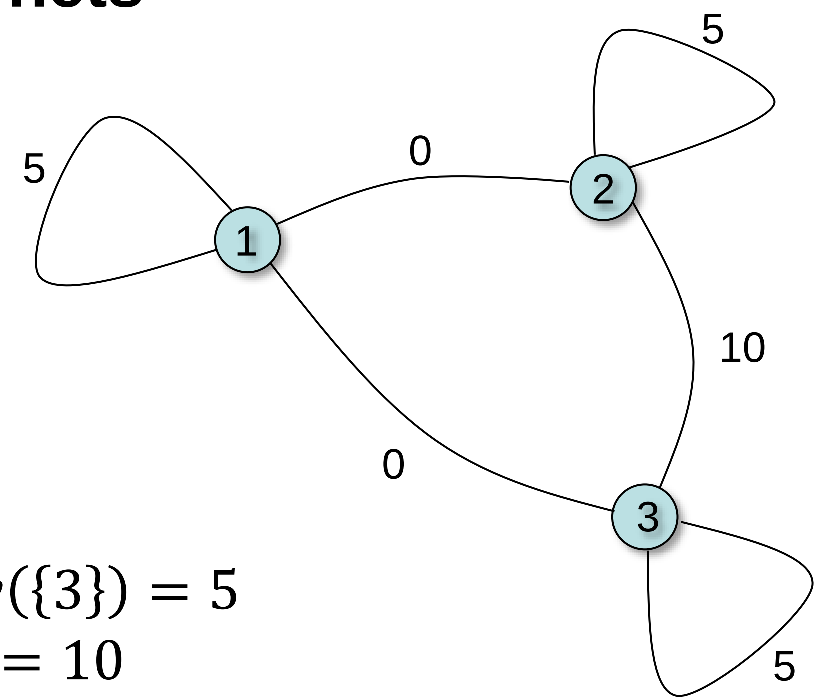


$$v(\{1,2,3\}) = 5 + 5 + 5 + 10 = 25$$

Marginal contribution nets

Example:

$1 \rightarrow 5$
$2 \rightarrow 5$
$3 \rightarrow 5$
$2 \wedge 3 \rightarrow 10$



$v(\{1\}) = v(\{2\}) = v(\{3\}) = 5$
$v(\{1,2\}) = v(\{1,3\}) = 10$
$v(\{2,3\}) = 20, v(\{1,2,3\}) = 25$

Simple games

Simple games are games with coalition values of either 0 (‘losing’) or 1 (‘winning’).

Simple games are representative of ‘yes/no’ voting systems often used in politics.

Which coalition is ‘winning’?

Simple games

Formally, a simple game is a pair of

$$\Upsilon = \langle Ag, W \rangle$$

where $Ag = \{1,2, \dots, N\}$ is N agents/voters

$W \subseteq \mathbf{2}^{Ag}$ is the set of winning coalitions such that

$$v(W) = 1$$

$$v(C \notin W) = 0$$

Simple games

Some properties of simple games:

1. Non-triviality
   There are some winning coaltions, but not all coalitions are winners, $\emptyset \subset W \subset 2^{Ag}$.
2. Monotonicity
   If $C_1 \subseteq C_2$ and $C_1 \in W$ then $C_2 \in W$. Meaning that if C wins then all supersets of C also wins.

Simple games

Some properties of simple games:

3. Zero-sum
   if $C \in W$ then $Ag \backslash C \notin W$
   If coalition C wins then the agents outside C do not win.
4. Empty coalition lose, $\emptyset \notin W$
5. Grand coalition wins, $Ag \in W$

Simple games

Explicitly listing all coalitions will be exponential in number of players.

Weighted voting games are natural extensions of simple games possibly reducing number of players in these games.

Weighted voting games

Weighted voting games are a more concise way of representing many simple games.

For instance, instead of representing 100 US senators explicitly we represent the different ‘blocks’ of voters and evaluate those against the criteria of winning, called the quota.

This can greatly reduce the number of players/voters

Weighted voting games

Given a set of agents $Ag = \{1,2, \dots, N\}$. For each agent i we define a weight w_i and a overall quota q such that every coalition C exceeding quota q is a winning coalition W.

$$v(C) = \begin{cases} 1 \text{ if } \sum_{i \in C} w_i \geq q \\ 0 \end{cases}$$

where v is the characteristic function of coalition C

$q = \frac{|Ag|+1}{2}$ in simple majority voting

Weighted voting games

The weighted voting game can be written on the form

$$\langle q; w_1, w_1, \dots, w_N \rangle$$

where q is the quota
w_i is the weight of player $i \in Ag$

The Shapley value calculates the power of voters in the game.

Weighted voting games

Example from the book

$$\langle 100; 99, 99, 1 \rangle$$

Let us calculate the Shapley values for the Grand coalition in order to determine the power of the 3 voter blocks.

Weighted voting games

How many permutations of $C = \langle 100; 99,99,1 \rangle$ is possible?

Weighted voting games

How many permutations of $C = \langle 100; 99,99,1 \rangle$ is possible?

$(99_1, 99_2, 1)$
$(99_1, 1, 99_2)$
$(99_2, 99_1, 1)$
$(99_2, 1, 99_1)$
$(1,99_1, 99_2)$
$(1, 99_2, 99_2)$

Weighted voting games

What is the marginal contribution of voter in each permutation?

{1,2,3}

$(99_1, 99_2, 1) \quad \rightarrow (0,1,0)$

$(99_1, 1, 99_2)$

$(99_2, 99_1, 1)$

$(99_2, 1, 99_1)$

$(1,99_1, 99_2)$

$(1, 99_2, 99_2)$

Weighted voting games

Permutations of {1,2,3} Marginal contribution of {1,2,3}

$(99_1, 99_2, 1) \rightarrow (0,1,0)$

$(99_1, 1, 99_2) \rightarrow (0,0,1)$

$(99_2, 99_1, 1) \rightarrow (1,0,0)$

$(99_2, 1, 99_1) \rightarrow (0,0,1)$

$(1, 99_1, 99_2) \rightarrow (1,0,0)$

$(1, 99_2, 99_1) \rightarrow (0,1,0)$

Weighted voting games

Permutations of {1,2,3}	Marginal contribution of {1,2,3}
$(99_1, 99_2, 1)$	$\rightarrow (0,1,0)$
$(99_1, 1, 99_2)$	$\rightarrow (0,0,1)$
$(99_2, 99_1, 1)$	$\rightarrow (1,0,0)$
$(99_2, 1, 99_1)$	$\rightarrow (0,0,1)$
$(1, 99_1, 99_2)$	$\rightarrow (1,0,0)$
$(1, 99_2, 99_1)$	$\rightarrow (0,1,0)$

Shapley value $\Rightarrow \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

Weighted voting games

The Shapley value is NP-hard to compute,
[Deng and Papadimitriou, 1994]

Weighted voting games

The core is much easier to compute, i.e. to check if there are non-empty coalitions. This can be done in polynomial time.

The core is non-empty, iff there is an agent present in every winning coalition. Formally, for a coalition C

$$\sum_{j \in C} w_j < q \text{ and } \sum_{j \in C \cap \{i\}} w_j \geq q$$

where q is quota, w_j is weight and i is voter i.

k-weighted voting games

Weighted voting games are not complete (representation of simple games).

k-weighted voting games are complete.

We define a number k of different weighted voting games with the same set of players the overall winner is the coalition that wins in all k coalition games.

k-weighted voting games

European Union three-weight voting game:

Germany, the UK, France, Italy, Spain, Poland, Romania, the Netherlands, Greece, Czech Republic, Belgium, Hungary, Portugal, Sweden, Bulgaria, Austria, Slovak Republic, Denmark, Finland, Ireland, Lithuania, Latvia, Slovenia, Estonia, Cyprus, Luxembourg, Malta

$$g_1 = \langle 255; 29,29,29,29,27,27,14,13,12,12,12,12,12,10,10,10,7,7,7,7,7,4,4,4,4,4,3 \rangle$$
$$g_2 = \langle 14; 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1 \rangle$$
$$g_3 = \langle 620; 170,123,122,120,82,80,47,33,22,21,21,21,21,18,17,17,11,11,11,8,8,5,4,3,2,1,1 \rangle$$

(where g_1 is number of commissioners, g_2 simple majority, g_3 is population size)

Image: Figure 13.4, Wooldridge 2009

k-weighted voting games

How many k-dimensions are needed in order to be a complete representation of a simple game?

Upper bound is 2^N, but a check if games could be represented by a smaller number of components is NP-complete.

A game of non-reducible k is called ‘minimal’.

Network flow games

Directed graph with edges representing capacities between nodes as routers.

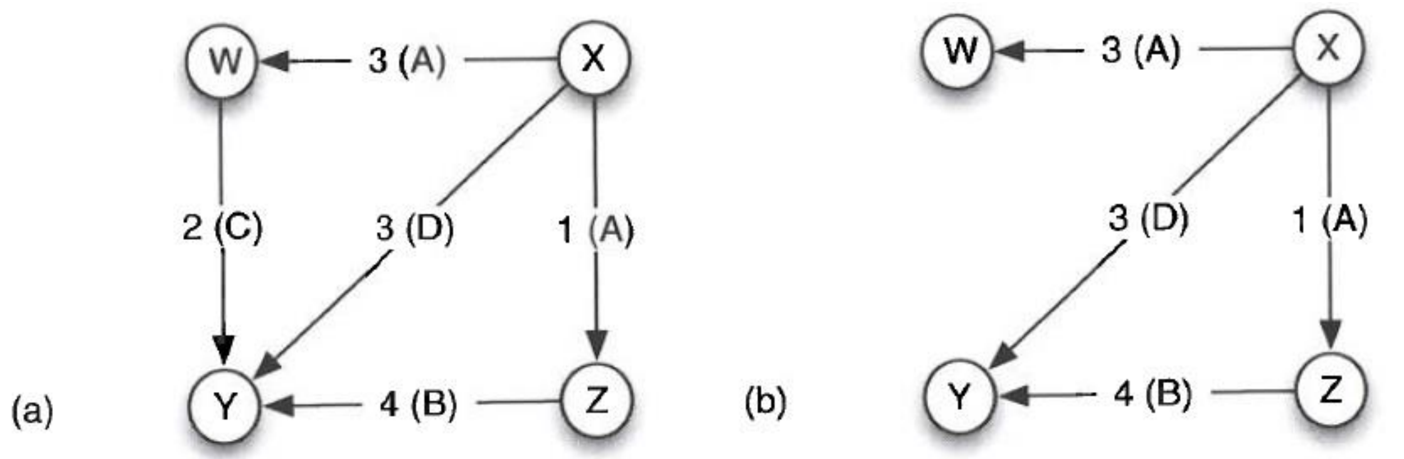


Image: Figure 13.5, Wooldridge 2009

Network flow games

Now, what coalitions of nodes allow for a capacity b between l_1 and l_2 in the graph? We get

$$v(C) = \begin{cases} 1 \text{ if } N \downarrow C \text{ allows a flow of } b \text{ from } l_1 \text{ to } l_2 \\ 0 \text{ otherwise} \end{cases}$$

where $N \downarrow C$ is the network flow of coalition C.

Coalitional games with goals

Qualitative Coalitional Games (QCG)

In QCG the numerical valued payoffs are replaced with goals.

The player has a set of goals it wants to achieve and they do not prioritize among them, they only want one of them to be achieved.

The coalition is successful if it can cooperate in such a way that all of its members are satisfied.

Coalitional games with goals

Coalitional Resouce Games (CRG)

To achieve a goal requires some resources that agents are endowed with. The coalition will pool resources in order to achieve some mutual set of goals.

Coalition structure formation

Central forming of coalition

1. All nodes owned by a single designer
2. Maximizing social welfare

The problem is to find the coalition structure that maximizes the aggregated outcome.

This is a search through a partition of the overall set of agents Ag into mutually disjoint coalitions.

Coalition structure formation

Example using $Ag = \{1,2,3\}$

Coalitions

$\{1\}, \{2\}, \{3\}, \{1,2\}, \{1, 3\}, \{2,3\}, \{1,2,3\}$

Coalition structures

$\{1,2,3\}, \{1\}\{2,3\}, \{2\}\{1,3\}, \{3\}\{1,2\}, \{1\}\{2\}\{3\}$

Coalition structure formation

The social choice is:

$$CS^* = \arg \max_{CS \in partition\ of\ Ag} V(CS)$$

where $V(CS) = \sum_{C \in CS} v(C)$ is the outcome of CS

The problem is that the number of possible coalition structures is exponentially more than coalitions over Ag.

Coalition structure formation

In the coalition graph the highest $CS^*_{1,2}$ of the two lowest levels are no worse than $1/N \cdot CS^*$.

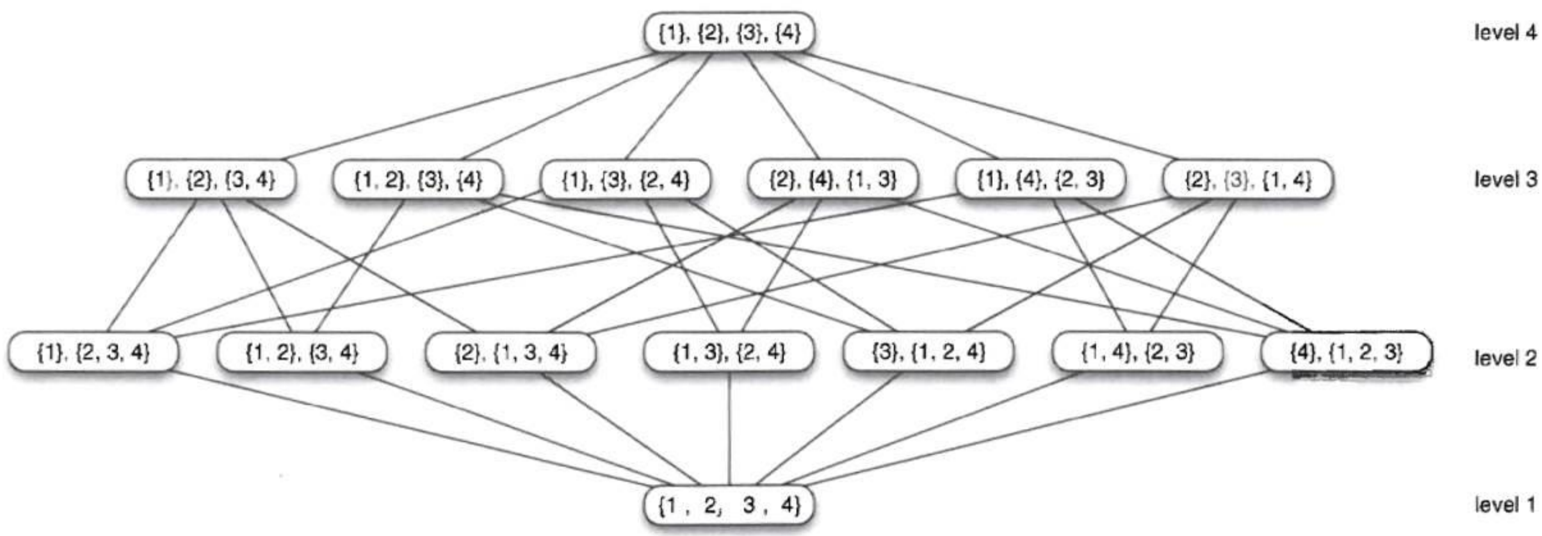


Image: Figure 13.6, Wooldridge 2009

Coalition structure formation

[Sandholm et *al*., 1999] propose a search algorithm:

1. Search the 2 lowest levels for $CS^*_{1,2}$.
2. If time, search rest of coalition structure graph, using breadth-first search from top until exhausted or time is up.
3. Return coalition structure with highest value seen.

Summary lecture 10 – Cooperative game theory*

- Cooperative games – forming coalitions
  - Characteristic function (value of coalition)
  - The Core (stability, non-empty)
  - The Shapley value (‘fairness’ axioms, induced subgraphs, marginal contribution nets)
- Simple games – ‘yes/no’ games
  - Weighted voting games, k-weighted voting games, network flow games
- Coalition structure formation – ‘central planner’

*Wooldridge, 2009: chapter 13