UiO : Department of Technology Systems
University of Oslo

Lecture 2.1
Image filtering

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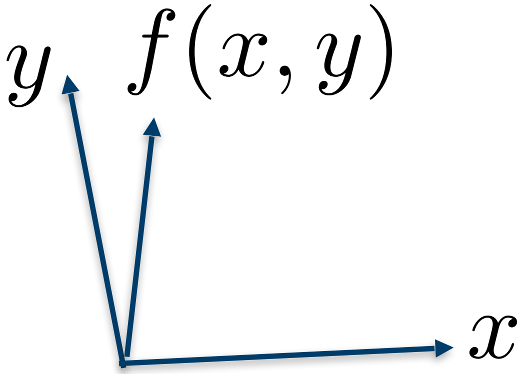
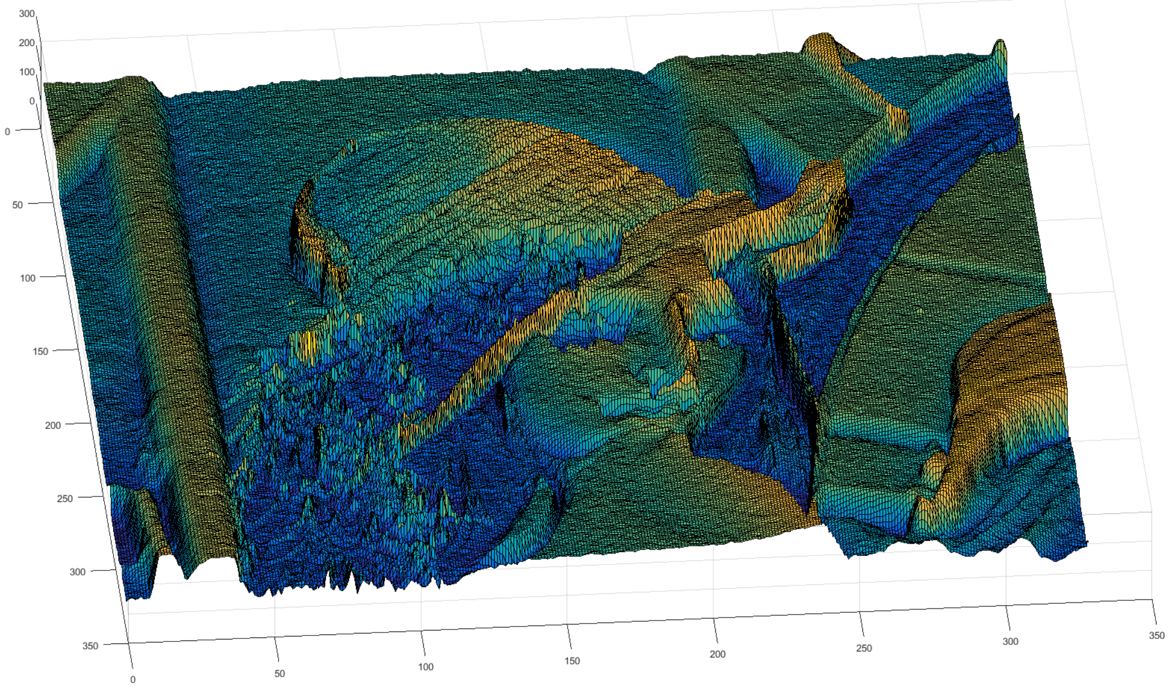
Image function

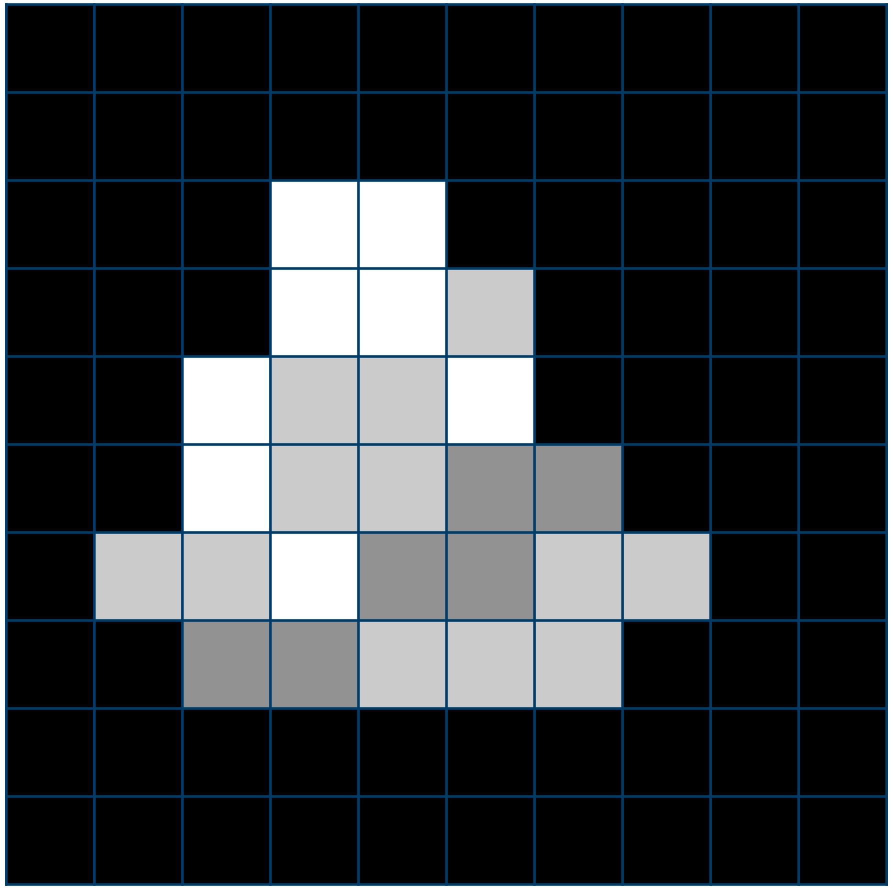


$$f : \mathrm{R}^2 \rightarrow \mathrm{R}$$



2D signal where $f(x, y)$ gives the **intensity** at position (x, y)

Digital Image



=

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	255	255	0	0	0	0	0
0	0	0	255	255	170	0	0	0	0
0	0	255	170	170	255	0	0	0	0
0	0	255	170	170	85	85	0	0	0
0	170	170	255	85	85	170	170	0	0
0	0	85	85	170	170	170	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

Discrete (**sampled** and **quantized**) version of the (continuous) image function $f(x, y)$

Image Processing

- Point operators
- Image filtering in spatial domain
  - Linear filters
  - Non-linear filters
- Image filtering in frequency domain
  - Fourier transform

$$f[i,j] \rightarrow g[i,j]$$

Point Operators

- Pixel transforms
  - Brightness adjustment
  - Contrast adjustment
  - …
- Colour transforms
- Histogram equalization
- …

$$g[i,j] = h(f[i,j])$$

(Pixel-by-pixel transformation)

$f[i,j]$

10	5	3
4	5	1
1	1	7

$\xrightarrow{h}$

$g[i,j]$

20	10	6
8	10	2
2	2	14

$$g[i,j] = 2f[i,j]$$

(Each pixel multiplied by 2)

Pixel transforms - example

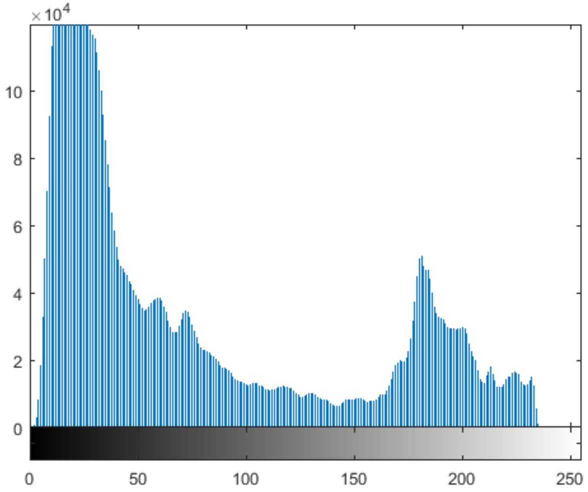
Original image

$$f[i,j]$$

Processed image

$$g[i,j] = \sqrt{f[i,j]}$$

Histogram equalization



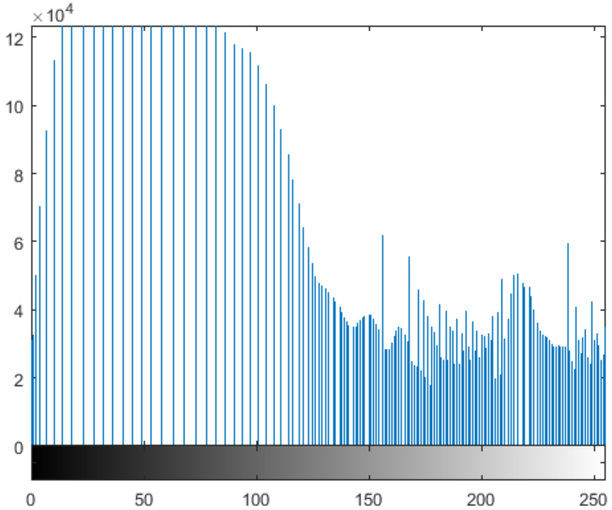


Image filtering

- **Image filters in the spatial domain:**
  - Filtering is a mathematical operation on a grid of numbers
  - Smoothing, sharpening (enhancing the image)
  - Feature extraction (measuring texture, finding edges, distinctive points and patterns).

- **Image filters in the frequency domain:**
  - Filtering is a way to modify the (spatial) frequencies of images
  - Noise removal, (re)sampling, image compression.

Image filtering in spatial domain

Modify the pixels in an image based on some function of a local neighborhood of each pixel:

$f[i,j]$

10	5	3
4	5	1
1	1	7

Local image data

$\xrightarrow{h}$

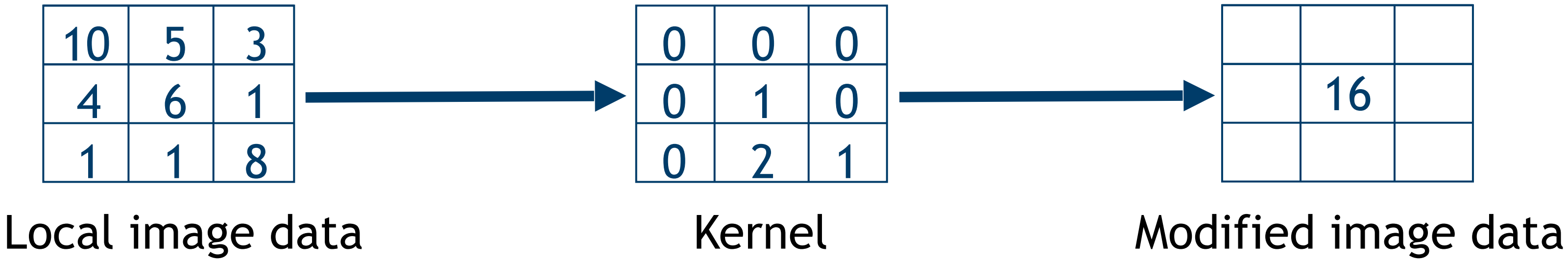
$g[i,j]$

	10	

Modified image data

Linear filtering

Convolution or **cross-correlation** where each pixel in the filtered image is a linear combination of the pixels in a local neighborhood in the original image:



The coefficients of the linear combination is contained in the “kernel” (filter mask).

Cross-correlation

Let ***f*** be the image, ***h*** be the kernel (of size 2k+1 x 2k+1), and ***g*** be the output image:

$$g[i,j] = \sum_{u=-k}^{k} \sum_{v=-k}^{k} h[u,v] f[i+u, j+v]$$

This is called a **cross-correlation** operation:

$$g = h \otimes f$$

Linear filtering

$$
f = \begin{bmatrix}
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&90&90&0&0&0&0&0\\
0&0&0&90&90&90&0&0&0&0\\
0&0&90&90&90&90&0&0&0&0\\
0&0&90&90&90&90&90&0&0&0\\
0&90&90&90&90&90&90&90&0&0\\
0&0&90&90&90&90&90&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0
\end{bmatrix}
\otimes \frac{1}{9}\underbrace{\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}}_{h}
=
\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0\\
0&0&10&20&20&10&0&0&0&0\\
0&0&20&&&&&&&\\
&&&&&&&&&\\
&&&&&&&&&\\
&&&&&&&&&\\
&&&&&&&&&\\
&&&&&&&&&\\
&&&&&&&&&\\
&&&&&&&&&
\end{bmatrix} = g
$$

$$
g[i,j] = \sum_{u,v} h[u,v] f[i+u, j+v]
$$

Linear filtering

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	0	0	0	0	0
0	0	0	90	90	90	0	0	0	0
0	0	90	90	90	90	0	0	0	0
0	0	90	90	90	90	90	0	0	0
0	90	90	90	90	90	90	90	0	0
0	0	90	90	90	90	90	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$$\otimes \frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} =$$

0	0	0	0	0	0	0	0	0	0
0	0	10	20	20	10	0	0	0	0
0	0	20	40	50	30	10	0	0	0
0	10	40	70	80	50	20	0	0	0
0	20	50	80	90	70	40	10		

Linear filtering

$$
\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&90&90&0&0&0&0&0\\
0&0&0&90&90&90&0&0&0&0\\
0&0&90&90&90&90&0&0&0&0\\
0&0&90&90&90&90&90&0&0&0\\
0&90&90&90&90&90&90&90&0&0\\
0&0&90&90&90&90&90&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0
\end{bmatrix}
\otimes \frac{1}{9}
\begin{bmatrix}
1&1&1\\
1&1&1\\
1&1&1
\end{bmatrix}
=
\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0\\
0&0&10&20&20&10&0&0&0&0\\
0&0&20&40&50&30&10&0&0&0\\
0&10&40&70&80&50&20&0&0&0\\
0&20&50&80&90&70&40&10&0&0\\
0&40&70&90&90&80&60&30&10&0\\
10&40&70&90&90&90&70&40&10&0\\
0&30&50&60&60&60&50&30&10&0\\
0&10&20&30&30&30&20&10&0&0\\
0&0&0&0&0&0&0&0&0&0
\end{bmatrix}
$$

Moving average filter (box filter)

3 x 3 kernel

$$\frac{1}{9}\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$$

Replaces each pixel with an average of its neighborhood (smoothing effect)

$$g[i,j] = \sum_{u,v} h[u,v] f[i+u, j+v]$$

9 x 9 kernel

Sharpening filter

Enhances differences with local average

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Convolution

- Same as cross-correlation, except that the kernel is “flipped” (horizontally and vertically):

$$g[i,j] = \sum_{u=-k}^{k} \sum_{v=-k}^{k} h[u,v] f[i-u, j-v]$$

- This is called a **convolution** operation:

$$g = h * f$$

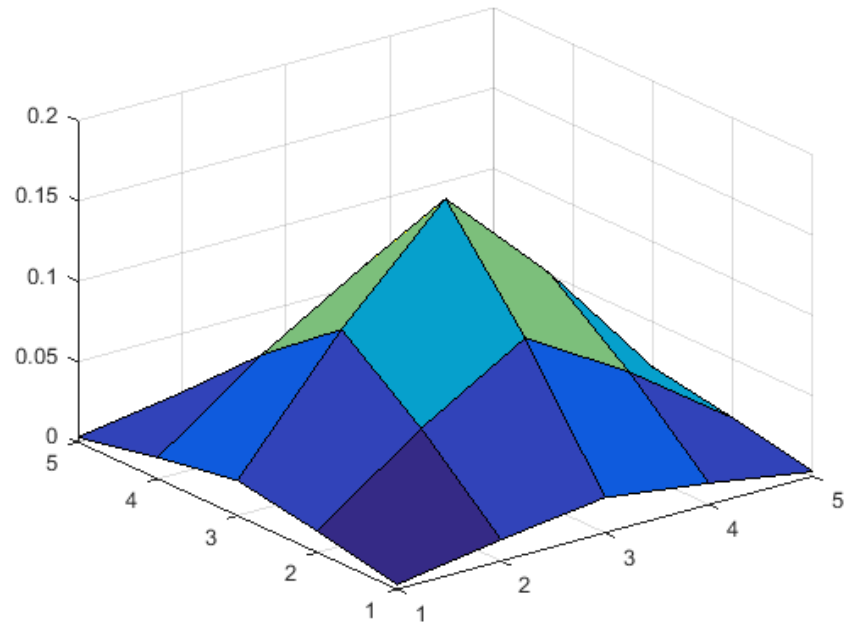
- Convolution is **commutative** and **associative** *(no difference between filter and image):*

$$a * b = b * a \qquad a * (b * c) = (a * b) * c$$

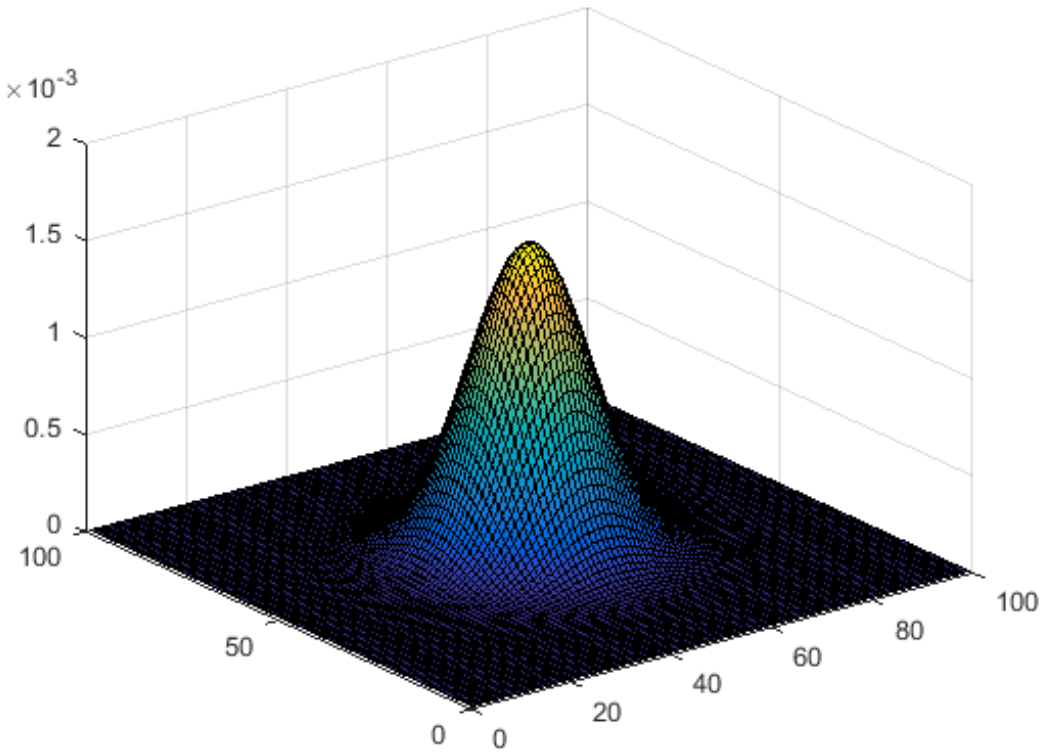
- Apply several filters, one after the other:

$$(((a * b_1) * b_2) * b_3) = a * (b_1 * b_2 * b_3)$$

Gaussian filter (smoothing)



$$5 \times 5, \sigma = 1$$



$$100 \times 100, \sigma = 10$$

$$G_\sigma = \frac{1}{2\pi\sigma^2} e^{-\left(\frac{x^2+y^2}{2\sigma^2}\right)}$$

Gaussian filtering

Original image

$\sigma = 2$ pixels

$\sigma = 4$ pixels

$\sigma = 8$ pixels

Separable filters - example

The 2D Gaussian kernel can be expressed as a product of two 1D kernels:

$$G_\sigma(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) \times \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y^2}{2\sigma^2}\right)$$

Discrete 3 x 3 approximation:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

More efficient to perform two 1D convolutions compared to a single 2D convolution!

2D convolution

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 1 \\ \hline 2 & 4 & 2 \\ \hline 1 & 2 & 1 \\ \hline \end{array} \quad * \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 1 & 4 & 4 \\ \hline 3 & 3 & 5 \\ \hline \end{array} \quad = \quad \begin{array}{|c|c|c|}\hline & & \\ \hline & 47 & \\ \hline & & \\ \hline \end{array}$$

Filter kernel

3 x 3 image window

Result (center pixel only)

1D convolution along rows and columns

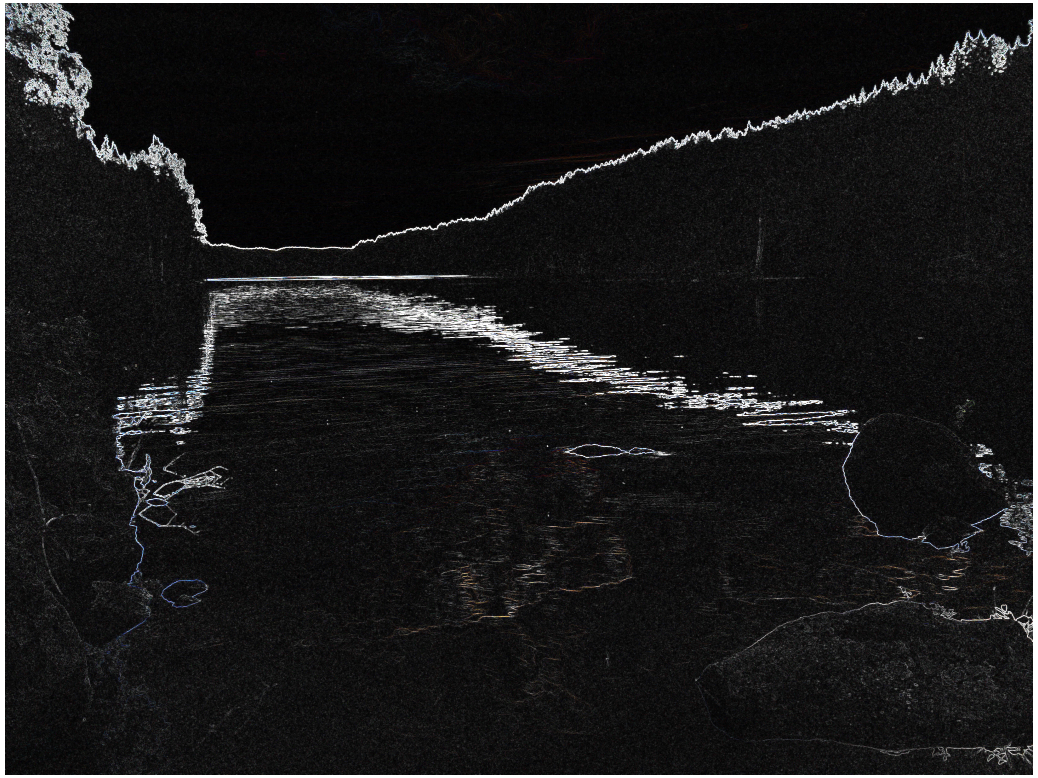
Convolution along rows:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 1 \\ \hline\end{array} * \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 1 & 4 & 4 \\ \hline 3 & 3 & 5 \\ \hline\end{array} = \begin{array}{|c|c|c|}\hline & 7 & \\ \hline & 13 & \\ \hline & 14 & \\ \hline\end{array}$$

Convolution along remaining column:

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 1 \\ \hline\end{array} * \begin{array}{|c|c|c|}\hline & 7 & \\ \hline & 13 & \\ \hline & 14 & \\ \hline\end{array} = \begin{array}{|c|c|c|}\hline & & \\ \hline & 47 & \\ \hline & & \\ \hline\end{array}$$

Edge detection



Edges and image derivatives

- An edge is a place of rapid change of the image intensity function
- Corresponds to extrema of the first derivative of the image intensity function
- Discrete approximation to the image derivatives:

$$\frac{\partial f}{\partial x}[i,j] \approx f[i+1,j] - f[i,j]$$

$$\frac{\partial f}{\partial y}[i,j] \approx f[i,j+1] - f[i,j]$$

Image gradient:

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]$$

Gradient magnitude:

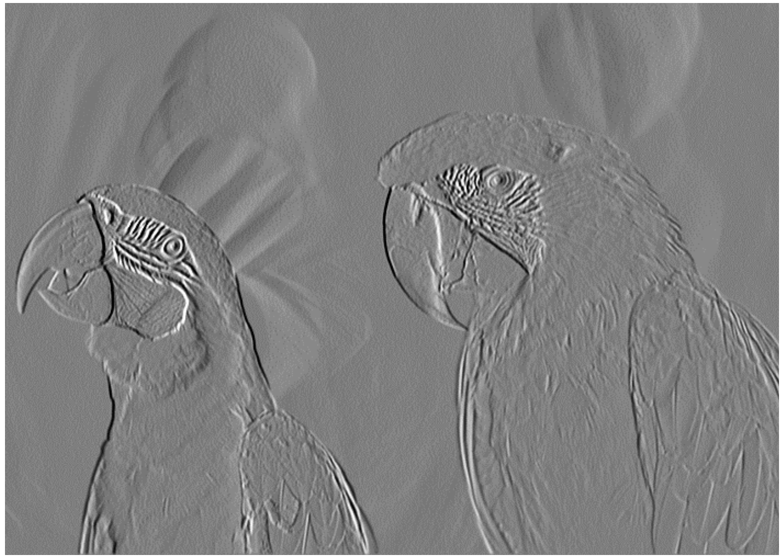
$$||\nabla f|| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

Prewitt operator:

$$G_x = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \qquad G_y = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

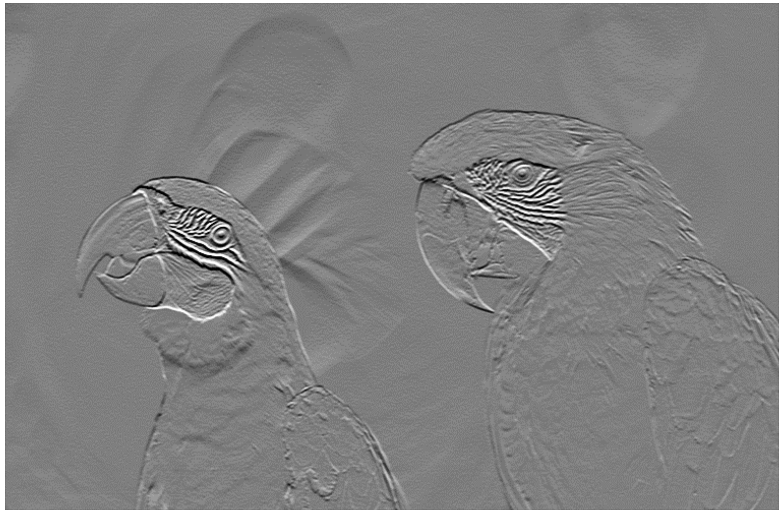
Image gradient

$$f$$



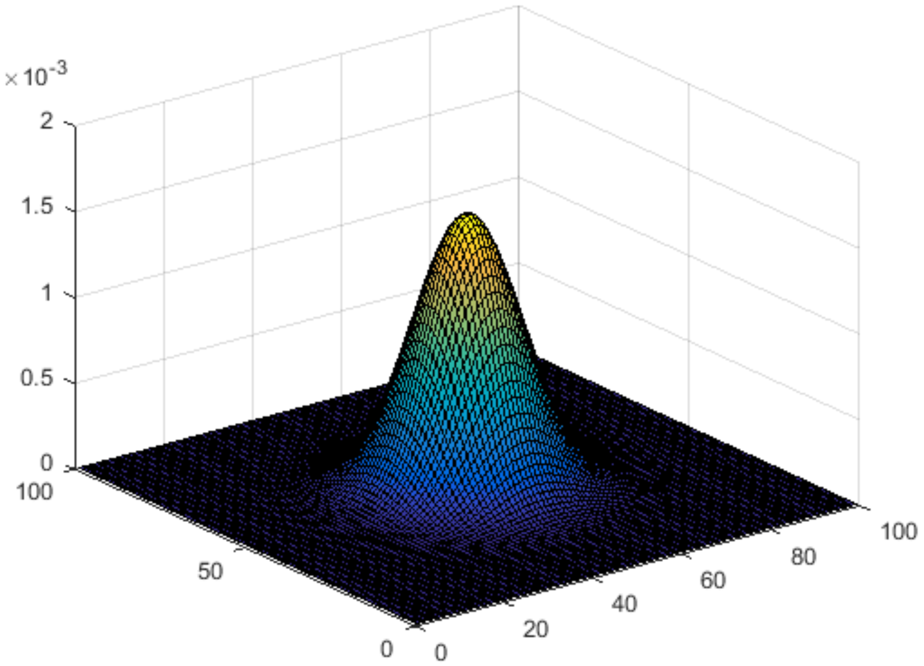
$$\frac{\partial f}{\partial x}$$

$$||\nabla f||$$

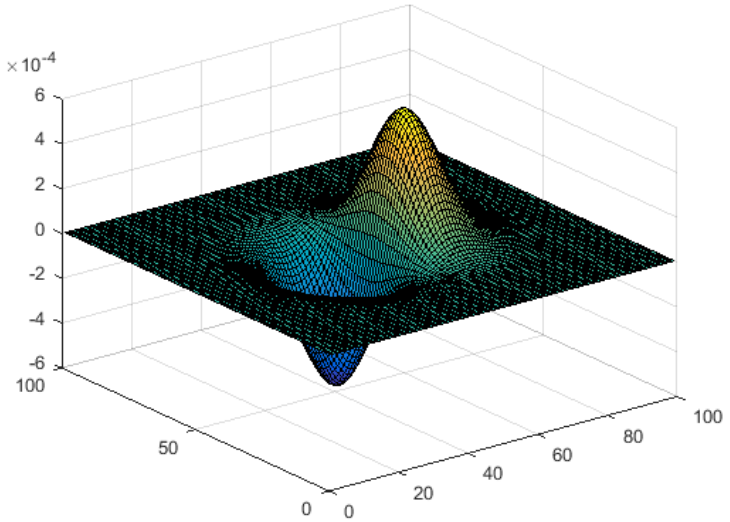


$$\frac{\partial f}{\partial y}$$

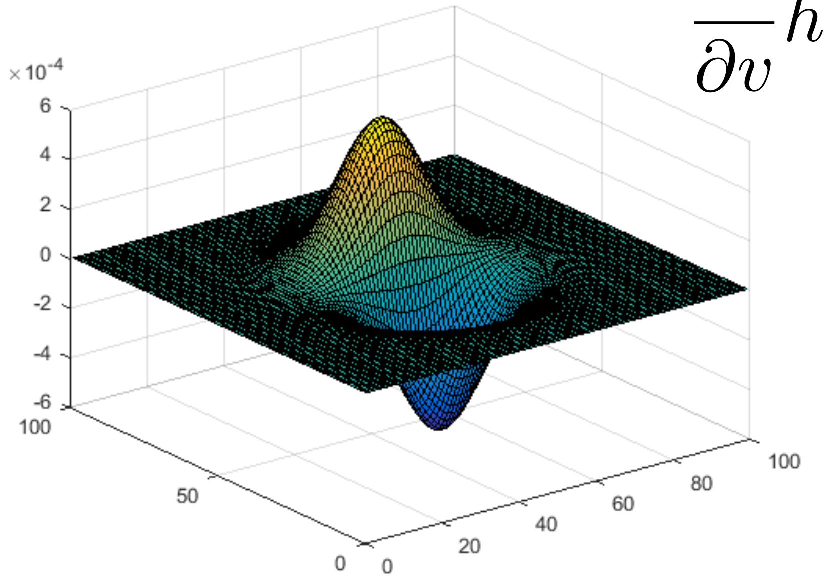
Derivative of Gaussians



$$h_\sigma(u,v) = \frac{1}{2\pi\sigma^2} e^{-\left(\frac{u^2+v^2}{2\sigma^2}\right)}$$



$$\frac{\partial}{\partial u} h_\sigma(u,v)$$



$$\frac{\partial}{\partial v} h_\sigma(u,v)$$

Sobel operator

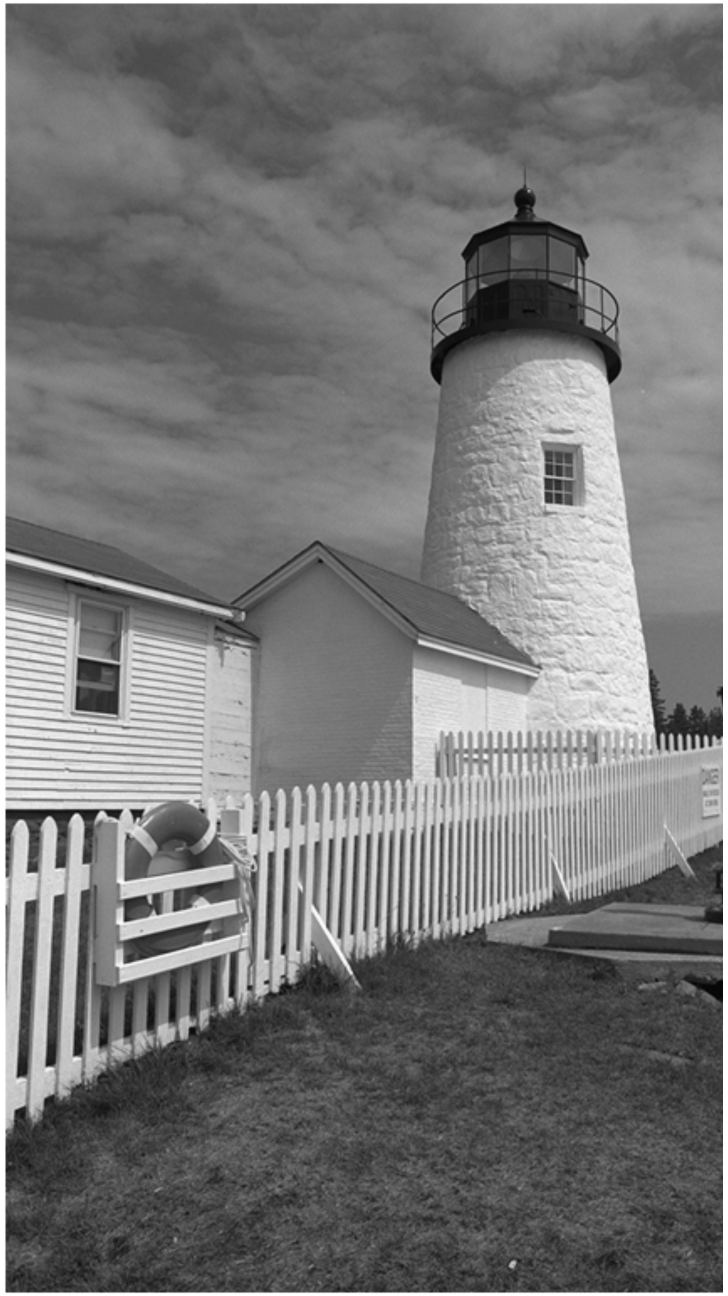
Common approximation of the derivative of a Gaussian:

-1	0	1
-2	0	2
-1	0	1

x-direction

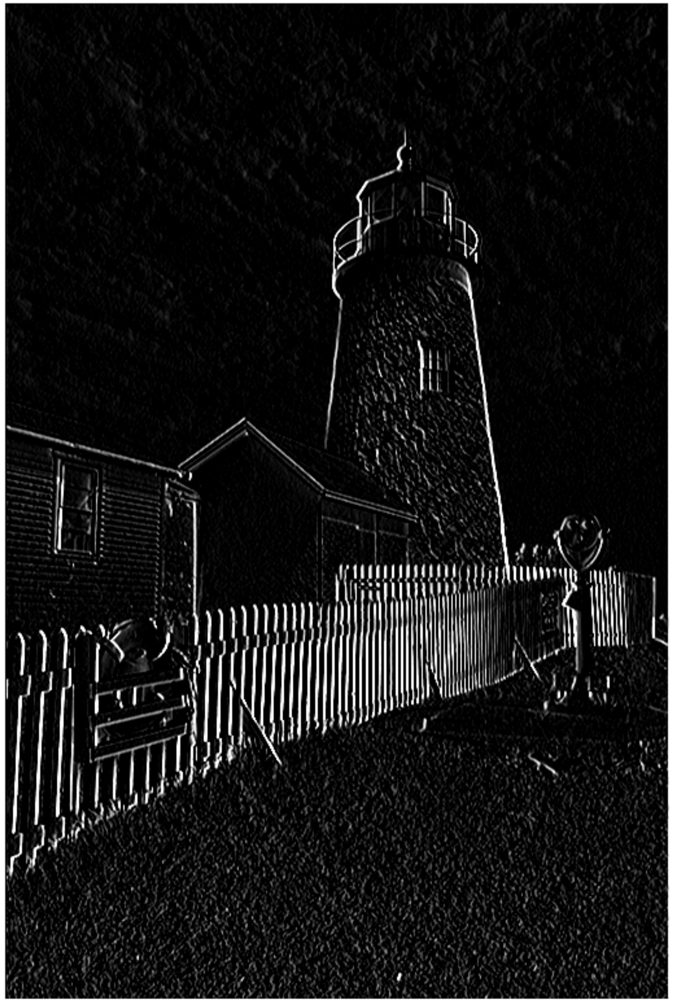
-1	-2	-1
0	0	0
1	2	1

y-direction

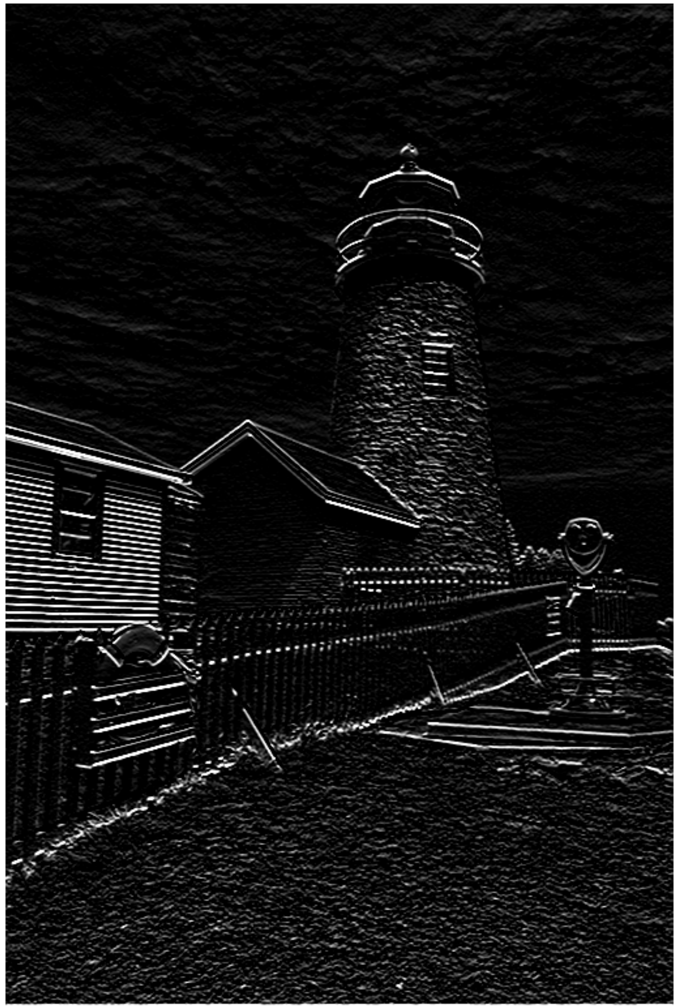


Sobel operator - example

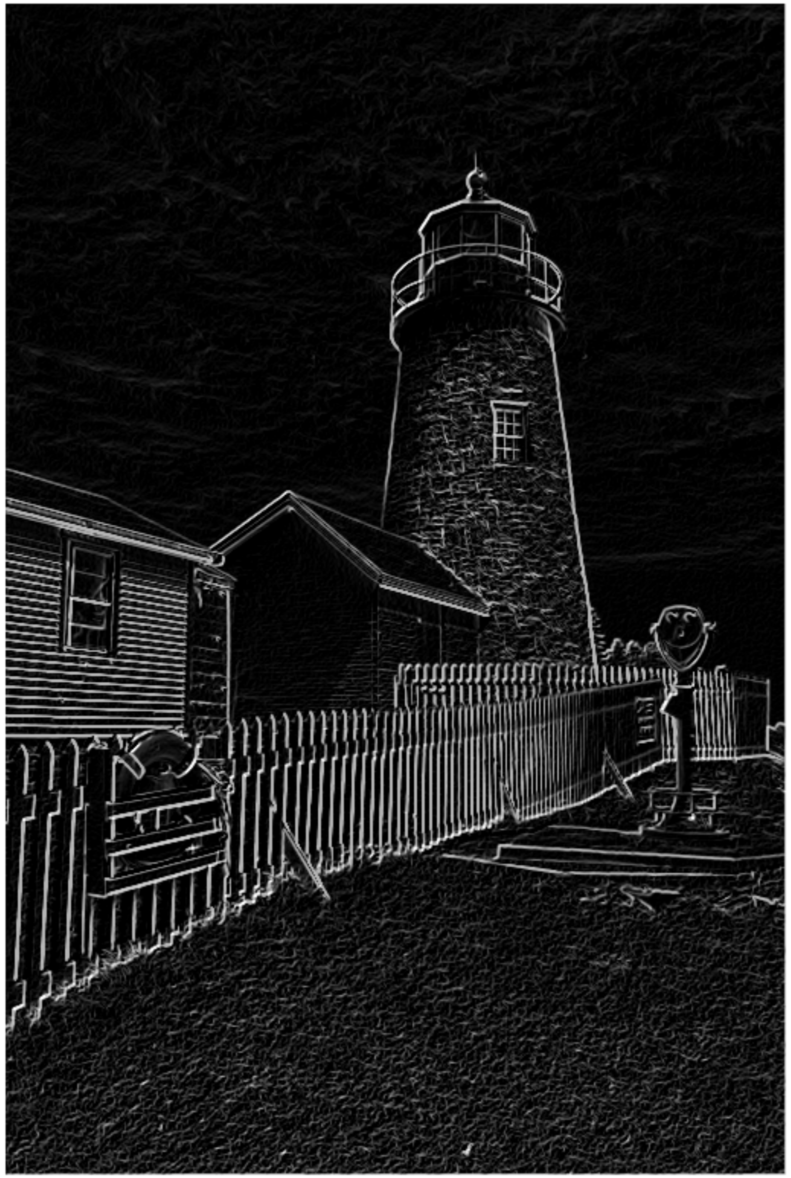
x-direction



y-direction

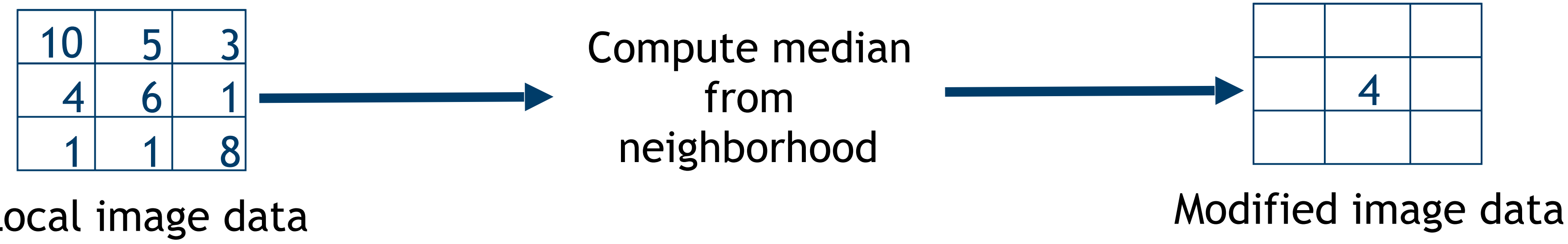


Gradient magnitude



Non-linear filtering - Median filter

A **median filter** operates over a neighborhood in the input image by selecting the median intensity:



Other non-linear filters:

- Bilateral filters (outlier rejection)
- Anisotropic diffusion
- Morphological operations (on binary images)
- …

Median filtering - example

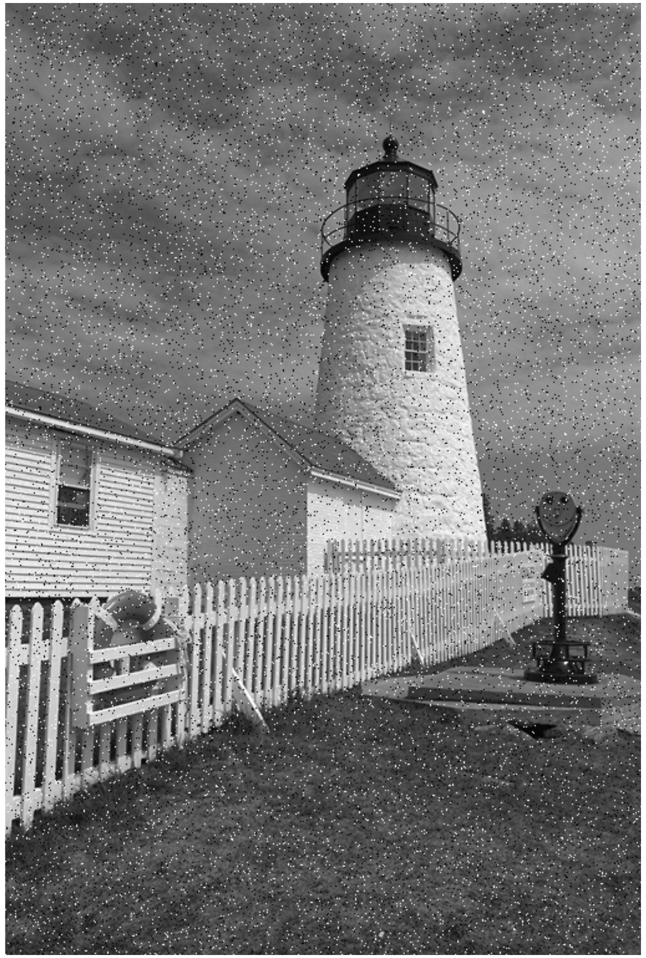


Image with Salt & Pepper noise

Image after median filtering

Morphological operations

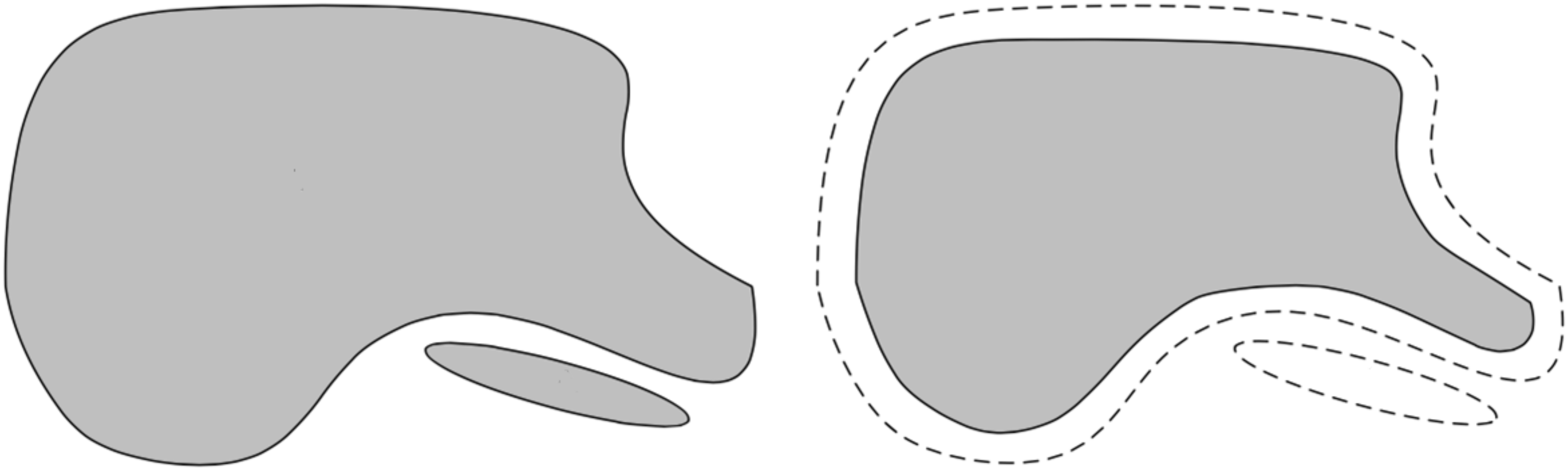
- Non-linear filtering
- Typically used to clean up binary images
- Erosion: replace pixel value with minimum in local neighborhood
- Dilation: replace pixel value with maximum in local neighborhood
- Structuring element used to define the local neighborhood:

0	1	0
1	1	1
0	1	0

A shape (in blue) and its morphological dilation (in green) and erosion (in yellow) by a diamond-shape structuring element.

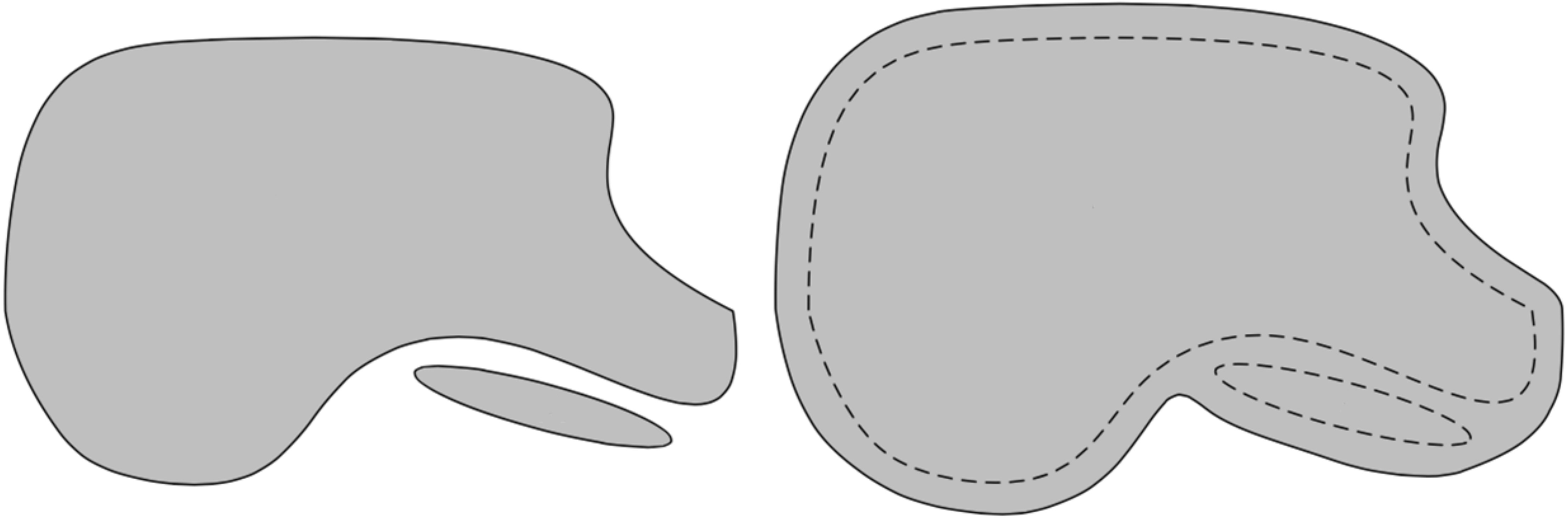
Morphological operations - Erosion

Structuring element (disk shaped) 

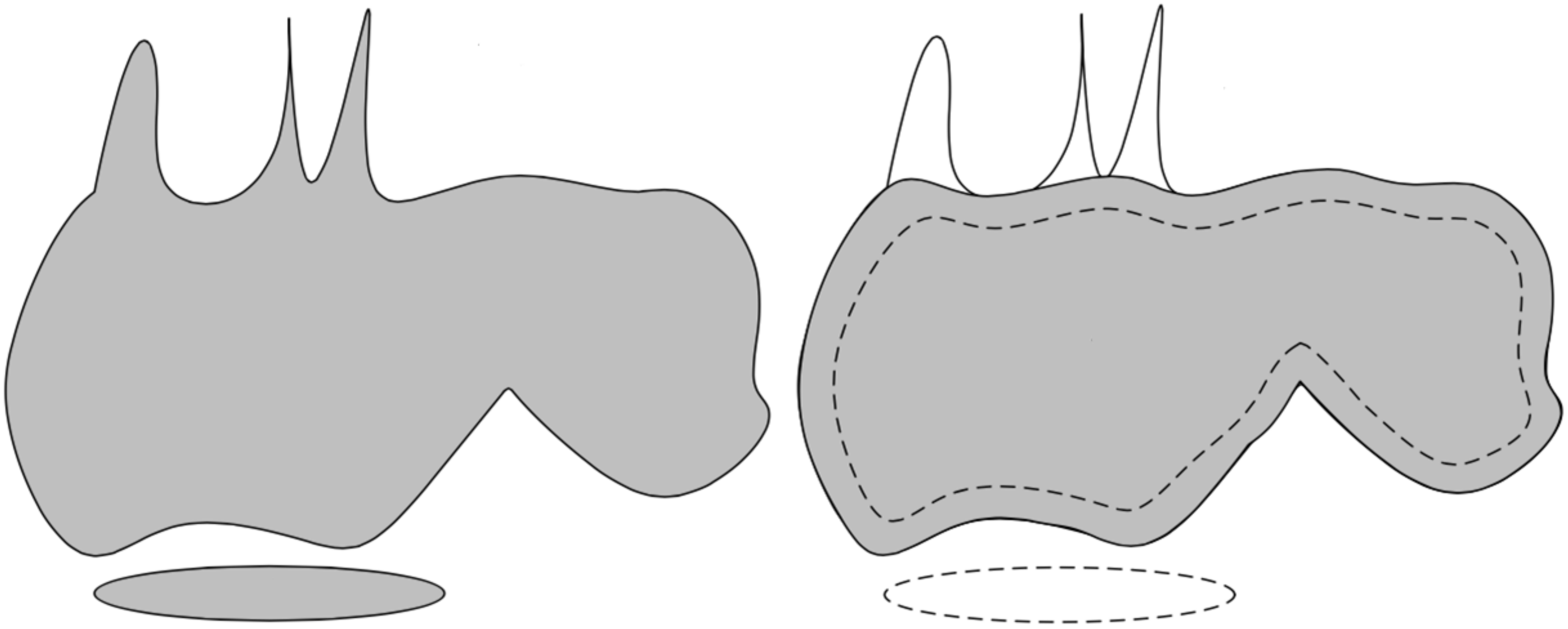


Morphological operations - Dilation

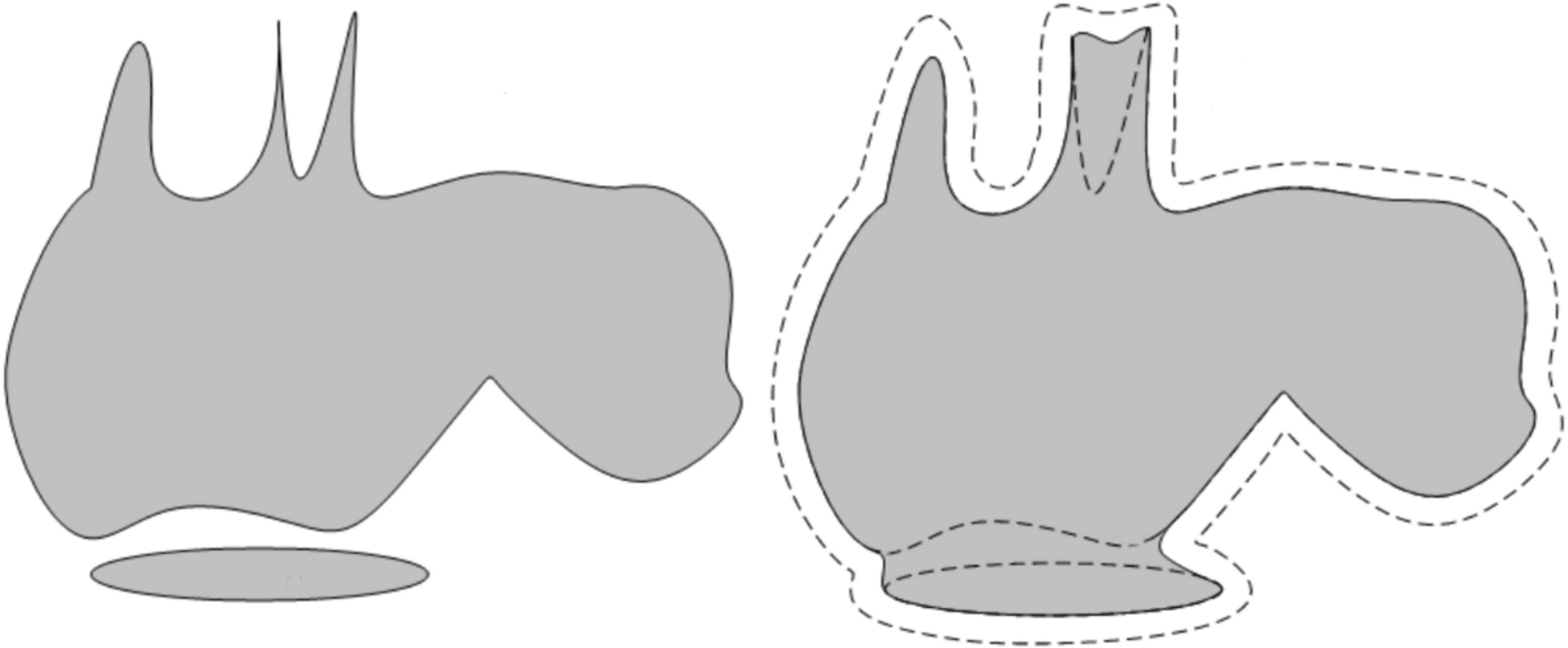
Structuring element (disk shaped) 



Opening = Erosion + Dilation



Closing = Dilation + Erosion



Filtering in frequency domain

Fourier (1807):

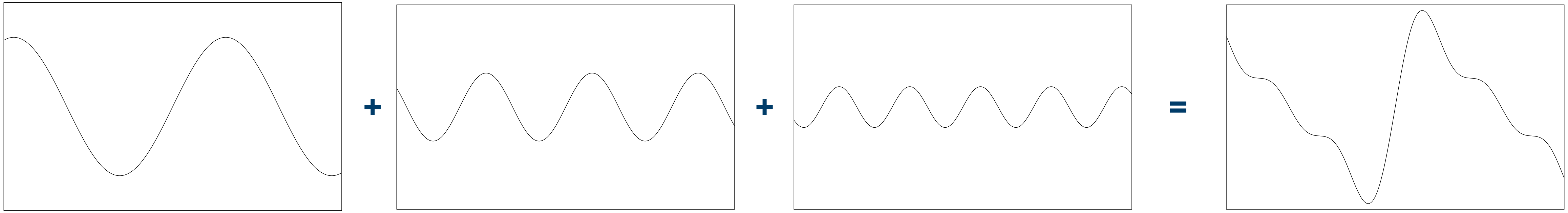
> ***Any*** *univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies (true with some subtle restrictions).*

This leads to:

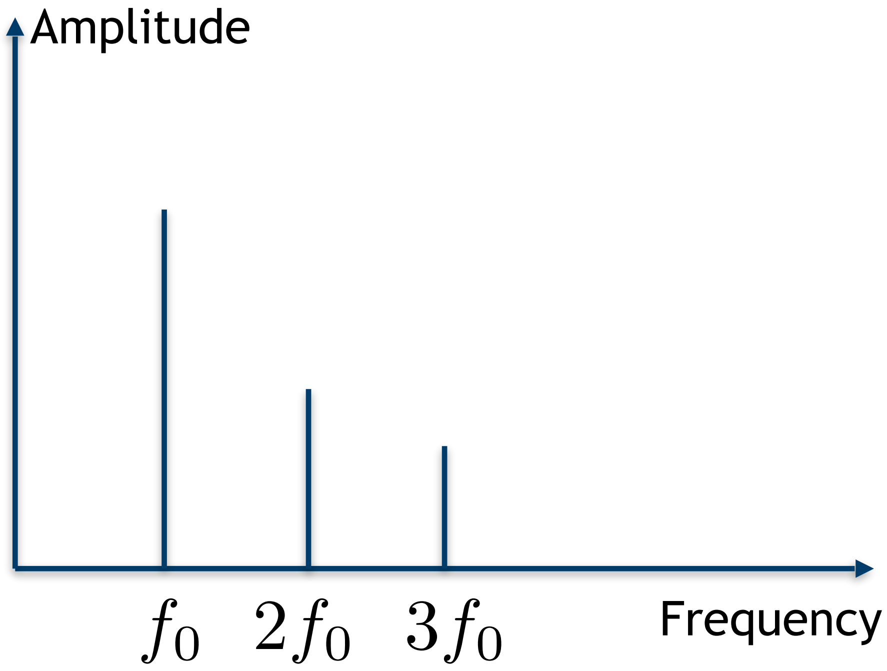
- Fourier Series
- Fourier Transform (continuous and discrete)
- Fast Fourier Transform (FFT)

Jean Baptiste Joseph Fourier (1768-1830)

Sum of sines



$$A \sin(\omega x + \phi)$$



The Fourier transform stores the magnitude and phase at each frequency

Amplitude:

$$A = \pm\sqrt{R(\omega)^2 + I(\omega)^2}$$

Phase:

$$\phi = \tan^{-1}\frac{I(\omega)}{R(\omega)}$$

Two-dimensional Fourier transform

Continous transform:

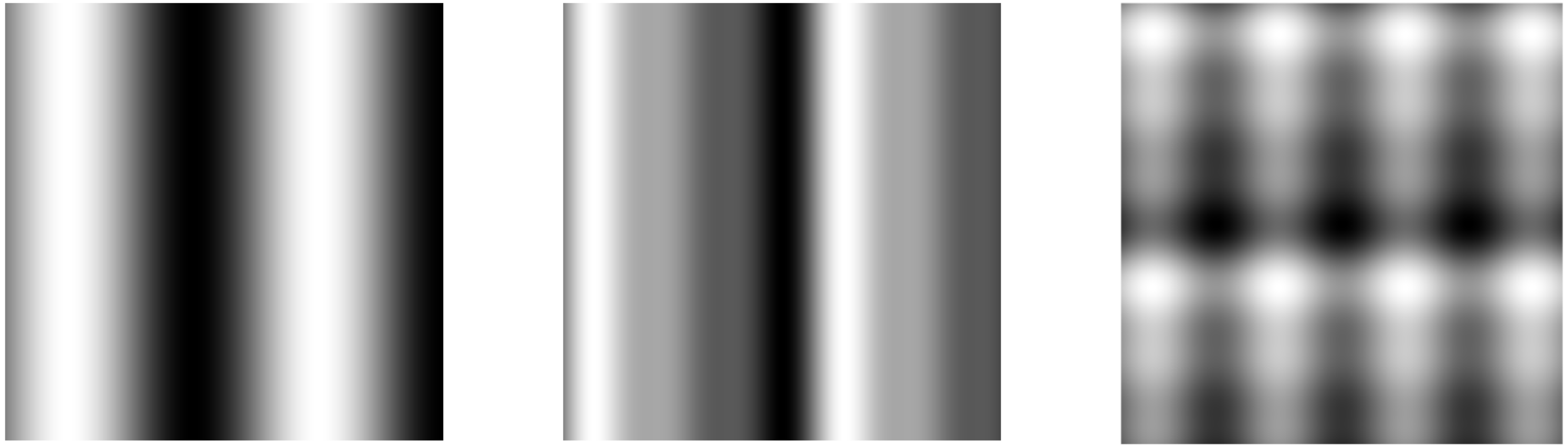
$$F(\omega_x, \omega_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j(\omega_x x + \omega_y y)} \mathrm{d}x\mathrm{d}y$$

Discrete transform:

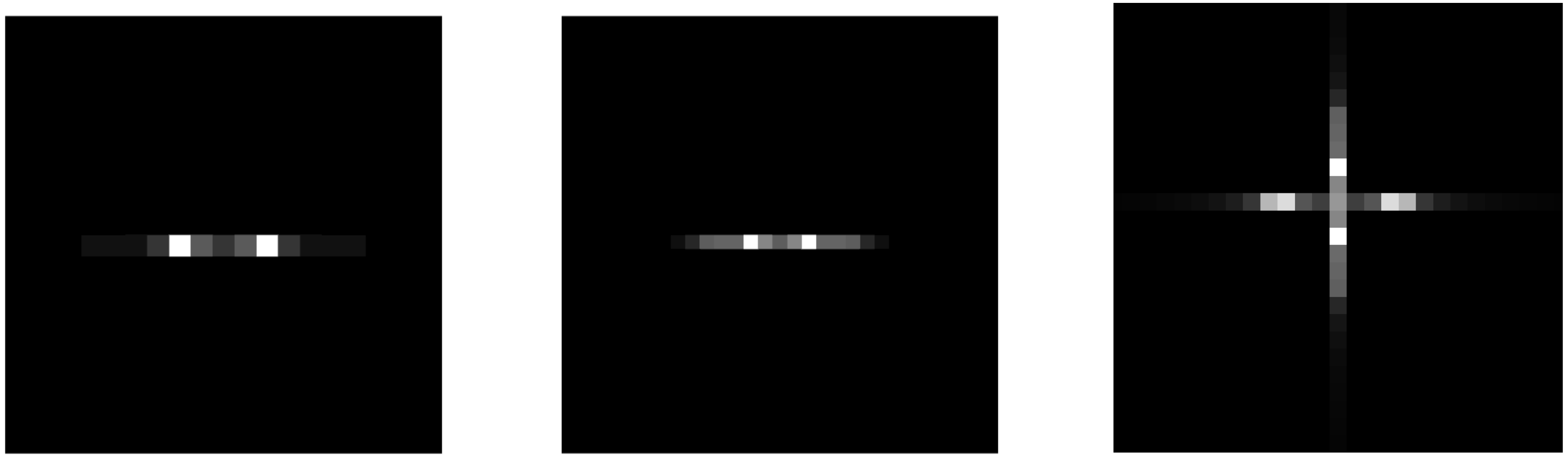
$$F[k_m, k_n] = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m, n] e^{-2\pi j \frac{(k_m m + k_n n)}{MN}}$$

Fourier analysis in images

Intensity images



Fourier images



The Convolution Theorem

The Fourier transform of the convolution of two functions is the product of their Fourier transforms:

$$F[g * h] = F[g]F[h]$$

Convolution in spatial domain is equivalent to multiplication in frequency domain:

$$g * h = F^{-1}[F[g]F[h]]$$

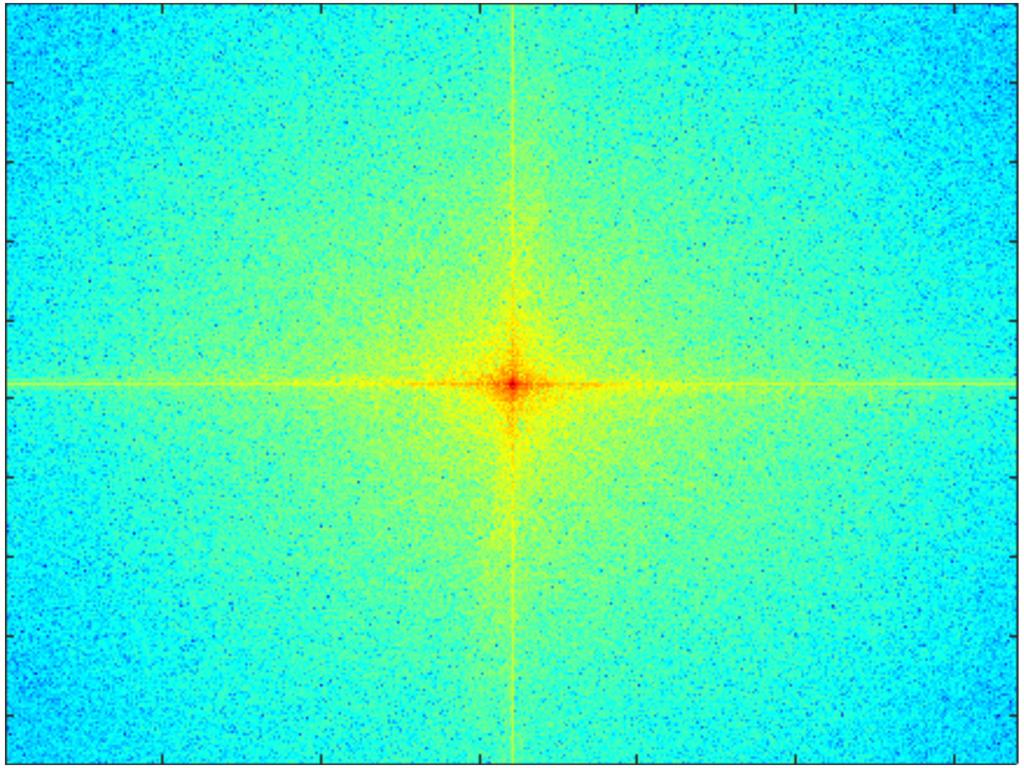
Example – Gaussian (low pass) filtering

Original image

g

FFT

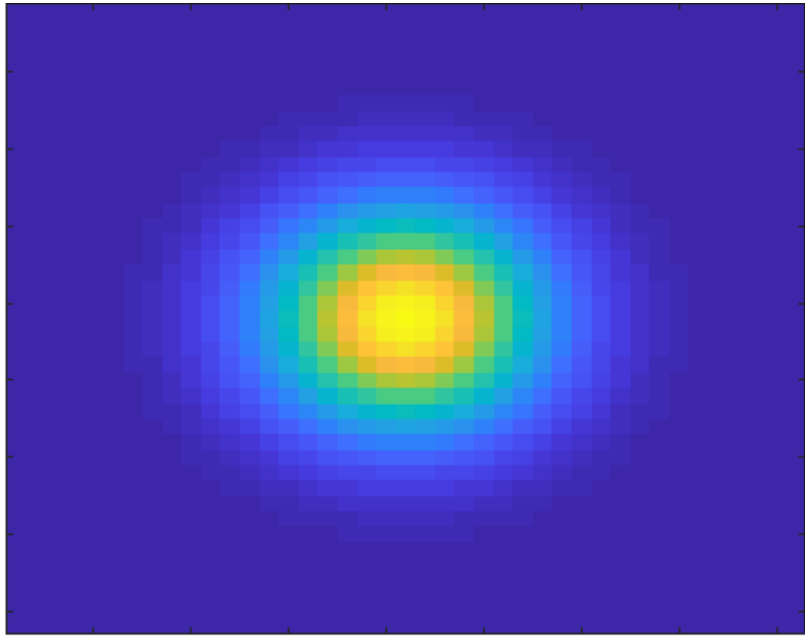
Fourier transform (absolute value)



$F[g]$

Example – Gaussian filtering

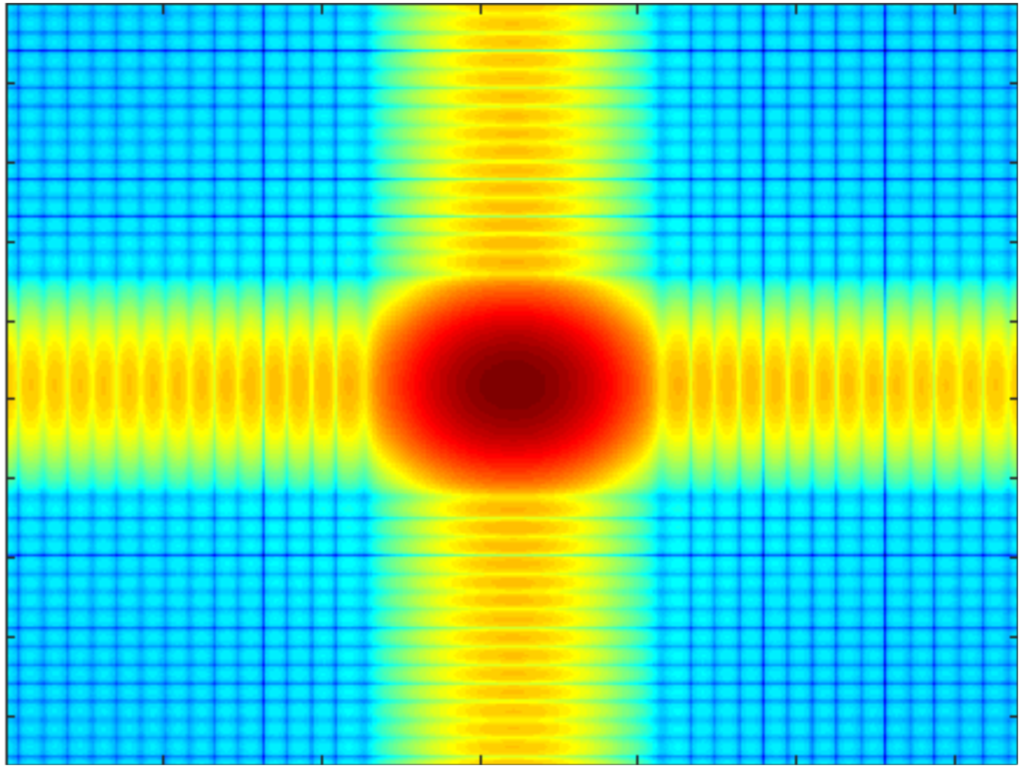
Gaussian kernel (41 x 41), $\sigma = 5$



h

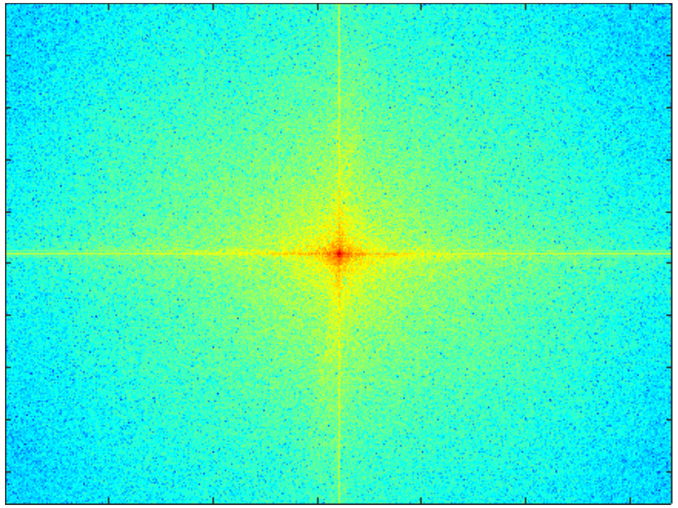
FFT

Fourier transform (absolute value)

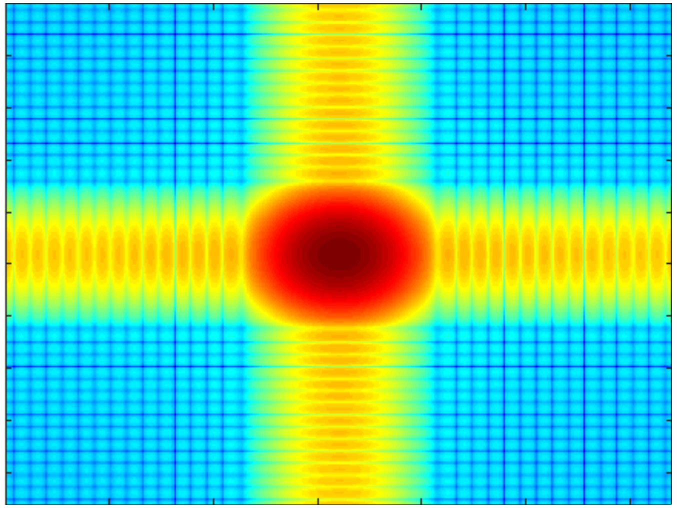


$F[h]$

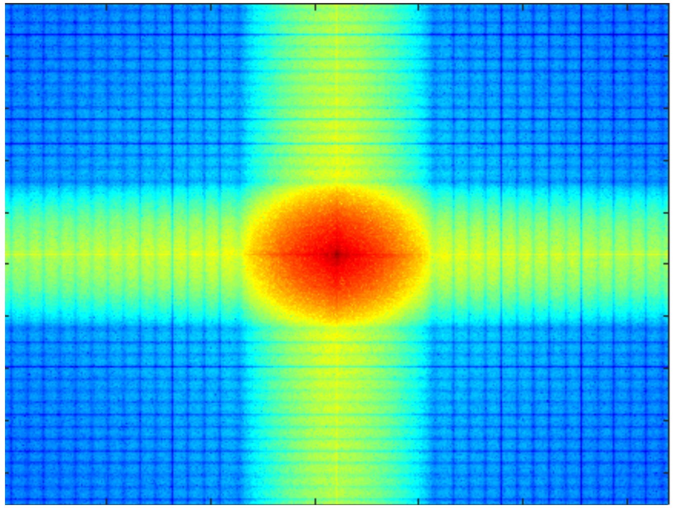
Example – Gaussian filtering



$F[g]$



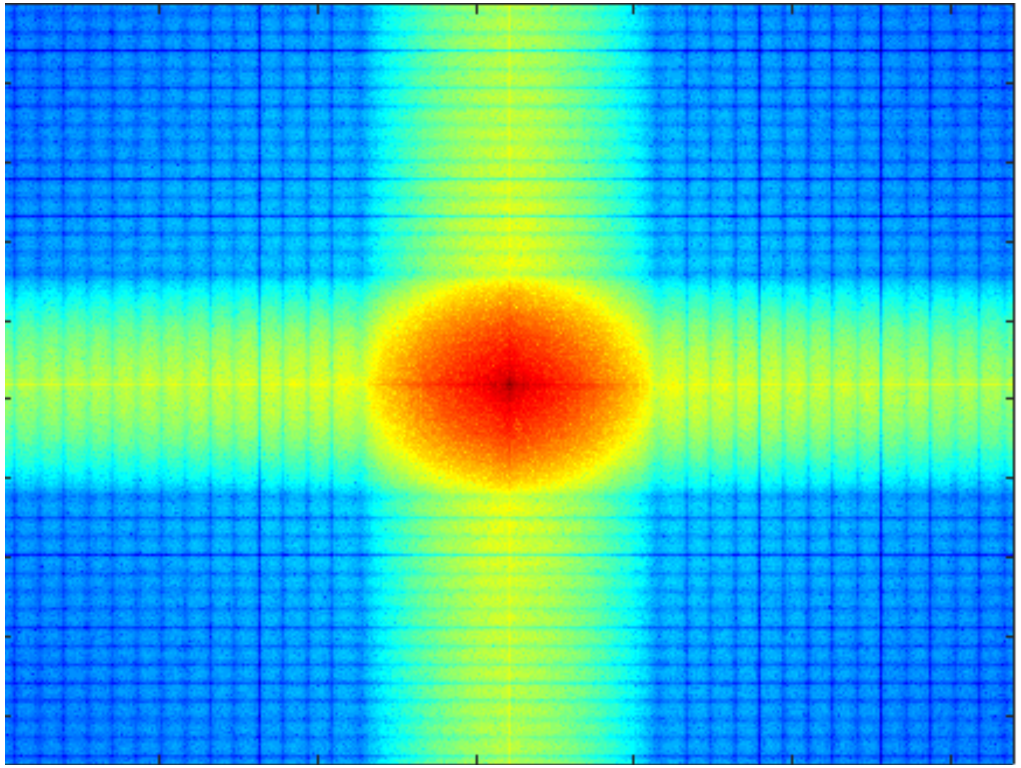
$F[h]$



$F[g]F[h]$

Example – Gaussian filtering

Fourier transform of filtered image



$$F[g]F[h]$$

Inverse FFT

Filtered image

$$g * h = F^{-1}[F[g]F[h]]$$

Summary

Image Processing

- Point operators

- Image filtering in spatial domain
  – Linear filters
  – Non-linear filters

- Image filtering in frequency domain
  – Fourier transforms
  – Gaussian (low pass) filtering

More information: Szeliski 3.1 – 3.4