UiO : Department of Technology Systems
University of Oslo

Lecture 6.4
Nonlinear pose estimation

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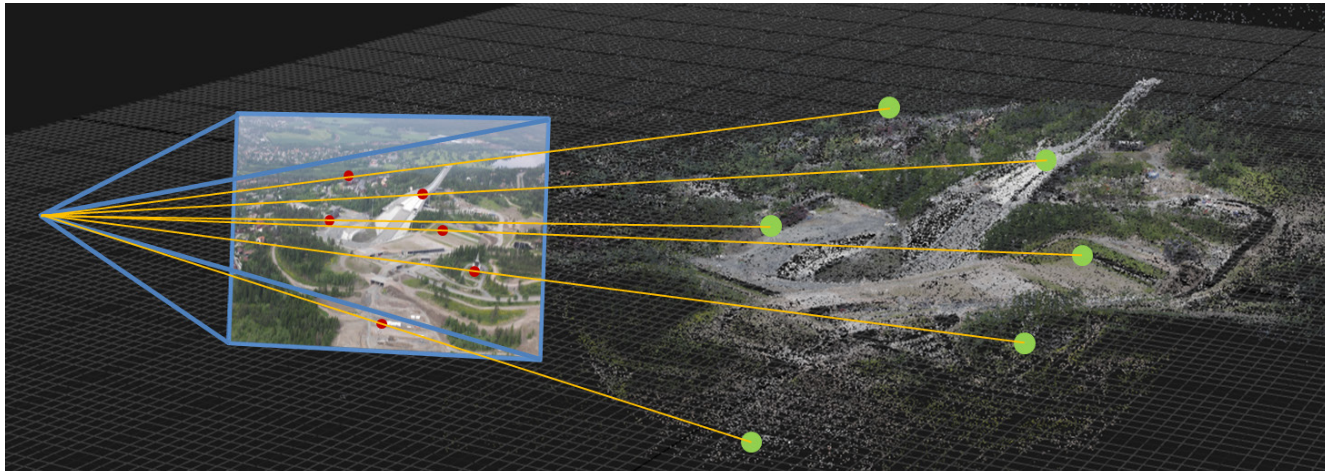
The indirect tracking method

Minimize **geometric error** over the **camera pose**

This is also sometimes called **Motion-Only Bundle Adjustment**

$$\mathbf{T}_{cw}^{*} = \underset{\mathbf{T}_{cw}}{\operatorname{argmin}} \sum_{i} \left\| \pi(\mathbf{T}_{cw}\tilde{\mathbf{x}}_{i}^{w}) - \mathbf{u}_{i} \right\|^{2}$$



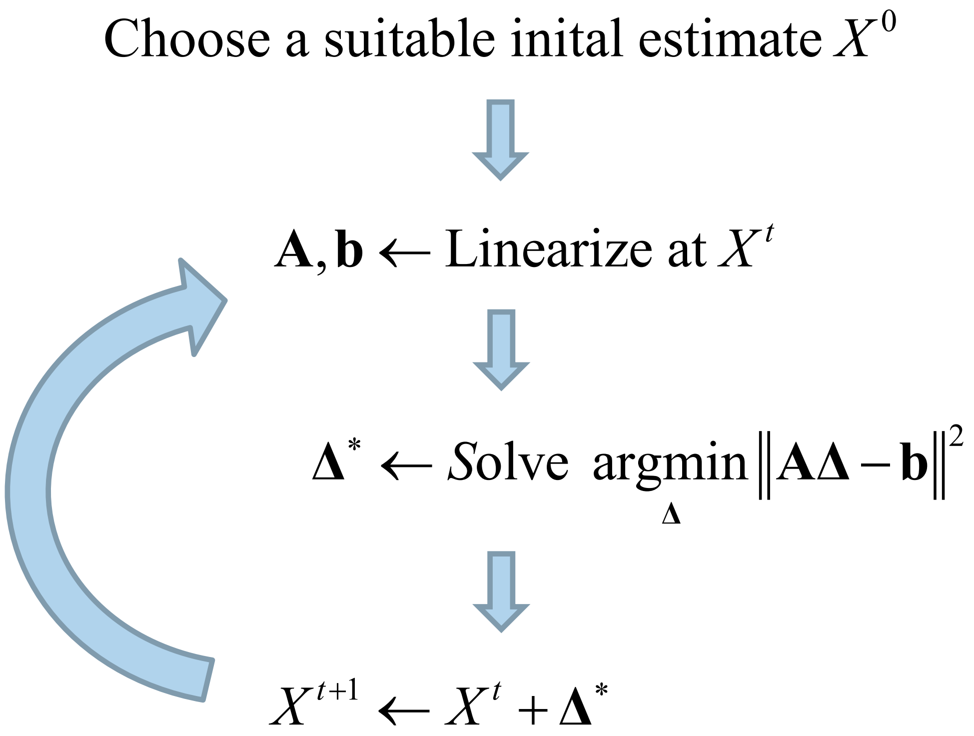
Nonlinear state estimation

We have seen how we can find the MAP estimate of our unknown states given measurements

$$X^{MAP} = \underset{X}{\operatorname{argmax}}\; p(X \mid Z)$$

by representing it as
a nonlinear least squares problem

$$X^{*} = \underset{X}{\operatorname{argmin}} \sum_{i=1}^{m} \left\| h_i(X_i) - \mathbf{z}_i \right\|_{\mathbf{\Sigma}_i}^{2}$$



Objective function

Minimize error over the **state variable** $X = \mathbf{T}_{wc}$

The optimization problem is

$$X^* = \underset{X}{\operatorname{argmin}} \sum_j \left\| \pi(g(\mathbf{T}_{wc}, \mathbf{x}_j^w)) - \mathbf{x}_{n,j} \right\|_{\boldsymbol{\Sigma}_j}^2$$

For simpler notation,
we assume that the measurements are pre-calibrated to normalized image coordinates

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For simpler notation,
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$$\mathbf{x}_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{K}^{-1} \begin{bmatrix} \mathbf{u} \\ 1 \end{bmatrix} = \begin{bmatrix} \dfrac{u - c_u}{f_u} \\ \dfrac{v - c_v}{f_v} \end{bmatrix}$$

(and distortion…)

Measurement prediction

This gives us the **measurement prediction function**

$$\hat{\mathbf{x}}_n = h(\mathbf{T}_{wc}; \mathbf{x}^w) = \pi(g(\mathbf{T}_{wc}, \mathbf{x}^w))$$

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where

$$g(\mathbf{T}_{wc},\mathbf{x}^w) = \mathbf{R}_{wc}^T(\mathbf{x}^w - \mathbf{t}_{wc}^w) = \begin{bmatrix} x^c \\ y^c \\ z^c \end{bmatrix} = \mathbf{x}^c$$

(Coordinate transformation)

$$\pi(\mathbf{x}^c) = \frac{1}{z^c}\begin{bmatrix} x^c \\ y^c \end{bmatrix} = \begin{bmatrix} \hat{x}_n \\ \hat{y}_n \end{bmatrix} = \hat{\mathbf{x}}_n$$

(Camera model)

Linearization

We can **linearize** the measurement prediction function with a local Taylor expansion

$$h(\mathbf{T}_{wc}\exp(\boldsymbol{\xi}_{\Delta}^{\wedge});\mathbf{x}^{w}) \approx h(\mathbf{T}_{wc};\mathbf{x}^{w}) + \mathbf{F}\boldsymbol{\xi}_{\Delta}$$

where $\exp(\boldsymbol{\xi}_{\Delta}^{\wedge})$ is a small perturbation in the camera frame.
The **measurement Jacobian** is given by

$$\mathbf{F} = \left.\frac{\partial h(\mathbf{T}_{wc}\exp(\boldsymbol{\xi}^{\wedge});\mathbf{x}^{w})}{\partial \boldsymbol{\xi}}\right|_{\boldsymbol{\xi}=0} = \left.\frac{\partial \pi(\mathbf{x}^{c})}{\partial \mathbf{x}^{c}}\right|_{\mathbf{x}^{c}=g(\mathbf{T}_{wc},\mathbf{x}^{w})} \left.\frac{\partial g(\mathbf{T}_{wc}\exp(\boldsymbol{\xi}^{\wedge}),\mathbf{x}^{w})}{\partial \boldsymbol{\xi}}\right|_{\boldsymbol{\xi}=0}$$

Jacobians

$$\left.\frac{\partial g(\mathbf{T}_{wc}\exp(\boldsymbol{\xi}^\wedge), \mathbf{x}^w)}{\partial \boldsymbol{\xi}}\right|_{\boldsymbol{\xi}=0}$$

$$g(\mathbf{T}_{wc}, \mathbf{x}^w) = \mathbf{R}_{wc}^T(\mathbf{x}^w - \mathbf{t}_{wc}^w) = \mathbf{x}^c$$

Jacobians

$$\left.\frac{\partial g(\mathbf{T}_{wc}\exp(\boldsymbol{\xi}^{\wedge}),\mathbf{x}^{w})}{\partial \boldsymbol{\xi}}\right|_{\xi=0} = \left.\frac{\partial (\mathbf{T}_{wc}\exp(\boldsymbol{\xi}^{\wedge}))^{-1}\oplus \mathbf{x}^{w}}{\partial \boldsymbol{\xi}}\right|_{\xi=0}$$

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$$= \left.\frac{\partial(\exp(-\boldsymbol{\xi}^{\wedge})\mathbf{T}_{wc}^{-1})\oplus\mathbf{x}^{w}}{\partial\boldsymbol{\xi}}\right|_{\xi=0}$$

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$$= -\left.\frac{\partial (\exp(\boldsymbol{\xi}^\wedge)\mathbf{T}_{cw})\oplus \mathbf{x}^w}{\partial \boldsymbol{\xi}}\right|_{\xi=0}$$

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$$= -\begin{bmatrix}\mathbf{I}_{3\times3} & -[\mathbf{T}_{cw}\oplus\mathbf{x}^w]^\wedge\end{bmatrix}$$

Jacobians

$$
\left.\frac{\partial g(\mathbf{T}_{wc}\exp(\boldsymbol{\xi}^{\wedge}),\mathbf{x}^{w})}{\partial\boldsymbol{\xi}}\right|_{\boldsymbol{\xi}=0} = \left.\frac{\partial(\mathbf{T}_{wc}\exp(\boldsymbol{\xi}^{\wedge}))^{-1}\oplus\mathbf{x}^{w}}{\partial\boldsymbol{\xi}}\right|_{\boldsymbol{\xi}=0}
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$$

$$
= \begin{bmatrix}-\mathbf{I}_{3\times3} & [\mathbf{x}^{c}]^{\wedge}\end{bmatrix}
$$

Jacobians

$$\left.\frac{\partial \pi(\mathbf{x}^c)}{\partial \mathbf{x}^c}\right|_{\mathbf{x}^c = g(\mathbf{T}_{wc}, \mathbf{x}^w)}$$

$$\pi(\mathbf{x}^c) = \frac{1}{z^c}\begin{bmatrix} x^c \\ y^c \end{bmatrix}$$

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$$\left.\frac{\partial \pi(\mathbf{x}^c)}{\partial \mathbf{x}^c}\right|_{\mathbf{x}^c = g(\mathbf{T}_{wc}, \mathbf{x}^w)} = \frac{1}{z^c}\begin{bmatrix} 1 & 0 & -x^c/z^c \\ 0 & 1 & -y^c/z^c \end{bmatrix}$$

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$$= d\begin{bmatrix} 1 & 0 & -x_n \\ 0 & 1 & -y_n \end{bmatrix}$$

$$d = \frac{1}{z^c}$$

Jacobians

$$\mathbf{F} = \left.\frac{\partial h(\mathbf{T}_{wc}\exp(\boldsymbol{\xi}^\wedge);\mathbf{x}^w)}{\partial \boldsymbol{\xi}}\right|_{\boldsymbol{\xi}=0} = \left.\frac{\partial \pi(\mathbf{x}^c)}{\partial \mathbf{x}^c}\right|_{\mathbf{x}^c = g(\mathbf{T}_{wc},\mathbf{x}^w)} \left.\frac{\partial g(\mathbf{T}_{wc}\exp(\boldsymbol{\xi}^\wedge),\mathbf{x}^w)}{\partial \boldsymbol{\xi}}\right|_{\boldsymbol{\xi}=0}$$

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$$= d\begin{bmatrix} 1 & 0 & -x_n \\ 0 & 1 & -y_n \end{bmatrix}\begin{bmatrix} -\mathbf{I}_{3\times 3} & [\mathbf{x}^c]^\wedge \end{bmatrix}$$

Jacobians

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$$= d\begin{bmatrix} 1 & 0 & -x_n \\ 0 & 1 & -y_n \end{bmatrix}\begin{bmatrix} -\mathbf{I}_{3\times 3} & [\mathbf{x}^c]^\wedge \end{bmatrix}$$

$$= \begin{bmatrix} -d & 0 & dx_n & x_n y_n & -1-x_n^2 & y_n \\ 0 & -d & dy_n & 1+y_n^2 & -x_n y_n & -x_n \end{bmatrix}$$

Linear least-squares

We can then obtain a **linear least-squares problem**

$$\boldsymbol{\xi}_{\Delta}^{*} = \underset{\boldsymbol{\xi}_{\Delta}}{\operatorname{argmin}} \sum_{j} \left\| h(\mathbf{T}_{wc}; \mathbf{x}_{j}^{w}) + \mathbf{F}_{j}\boldsymbol{\xi}_{\Delta} - \mathbf{x}_{n,j} \right\|_{\boldsymbol{\Sigma}_{j}}^{2}$$

Linear least-squares

We can then obtain a **linear least-squares problem**

$$\boldsymbol{\xi}_\Delta^* = \underset{\boldsymbol{\xi}_\Delta}{\operatorname{argmin}} \sum_j \left\| h(\mathbf{T}_{wc}; \mathbf{x}_j^w) + \mathbf{F}_j \boldsymbol{\xi}_\Delta - \mathbf{x}_{n,j} \right\|_{\boldsymbol{\Sigma}_j}^2$$

$$= \underset{\boldsymbol{\xi}_\Delta}{\operatorname{argmin}} \sum_j \left\| \mathbf{F}_j \boldsymbol{\xi}_\Delta - \left\{ \mathbf{x}_{n,j} - h(\mathbf{T}_{wc}; \mathbf{x}_j^w) \right\} \right\|_{\boldsymbol{\Sigma}_j}^2$$

Linear least-squares

We can then obtain a **linear least-squares problem**

$$\begin{aligned}
\boldsymbol{\xi}_\Delta^* &= \underset{\boldsymbol{\xi}_\Delta}{\operatorname{argmin}} \sum_j \left\| h(\mathbf{T}_{wc};\mathbf{x}_j^w) + \mathbf{F}_j \boldsymbol{\xi}_\Delta - \mathbf{x}_{n,j} \right\|_{\boldsymbol{\Sigma}_j}^2 \\
&= \underset{\boldsymbol{\xi}_\Delta}{\operatorname{argmin}} \sum_j \left\| \mathbf{F}_j \boldsymbol{\xi}_\Delta - \left\{ \mathbf{x}_{n,j} - h(\mathbf{T}_{wc};\mathbf{x}_j^w) \right\} \right\|_{\boldsymbol{\Sigma}_j}^2 \\
&= \underset{\boldsymbol{\xi}_\Delta}{\operatorname{argmin}} \sum_j \left\| \mathbf{A}_j \boldsymbol{\xi}_\Delta - \mathbf{b}_j \right\|^2
\end{aligned}$$

Linear least-squares

We can then obtain a **linear least-squares problem**

$$\begin{aligned}
\boldsymbol{\xi}_\Delta^* &= \underset{\boldsymbol{\xi}_\Delta}{\operatorname{argmin}} \sum_j \left\| h(\mathbf{T}_{wc};\mathbf{x}_j^w) + \mathbf{F}_j \boldsymbol{\xi}_\Delta - \mathbf{x}_{n,j} \right\|_{\boldsymbol{\Sigma}_j}^2 \\
&= \underset{\boldsymbol{\xi}_\Delta}{\operatorname{argmin}} \sum_j \left\| \mathbf{F}_j \boldsymbol{\xi}_\Delta - \left\{ \mathbf{x}_{n,j} - h(\mathbf{T}_{wc};\mathbf{x}_j^w) \right\} \right\|_{\boldsymbol{\Sigma}_j}^2 \\
&= \underset{\boldsymbol{\xi}_\Delta}{\operatorname{argmin}} \sum_j \left\| \mathbf{A}_j \boldsymbol{\xi}_\Delta - \mathbf{b}_j \right\|^2 \\
&= \underset{\boldsymbol{\xi}_\Delta}{\operatorname{argmin}} \left\| \mathbf{A} \boldsymbol{\xi}_\Delta - \mathbf{b} \right\|^2
\end{aligned}$$

Linear least-squares

With the **state update vector**

$$\boldsymbol{\xi}_{\Delta}$$

For an example with *three points*,
the **measurement Jacobian** $\mathbf{A}$ and the **prediction error** $\mathbf{b}$ are

$$\mathbf{A} = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \mathbf{F}_3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix}$$

Solution to the linearized problem

The solution can be found by solving **the normal equations**

$$\left(\mathbf{A}^T\mathbf{A}\right)\xi_{\Delta}^{*} = \mathbf{A}^T\mathbf{b}$$

Choose a suitable inital estimate X^0

$\mathbf{A},\mathbf{b} \leftarrow \text{Linearize at } X^t$

$\boldsymbol{\Delta}^* \leftarrow \text{Solve } \underset{\boldsymbol{\Delta}}{\operatorname{argmin}} \left\|\mathbf{A}\boldsymbol{\Delta} - \mathbf{b}\right\|^2$

$X^{t+1} \leftarrow X^t + \boldsymbol{\Delta}^*$

Gauss-Newton optimization

Given a good initial estimate $\mathbf{T}_{wc}^{0}$.

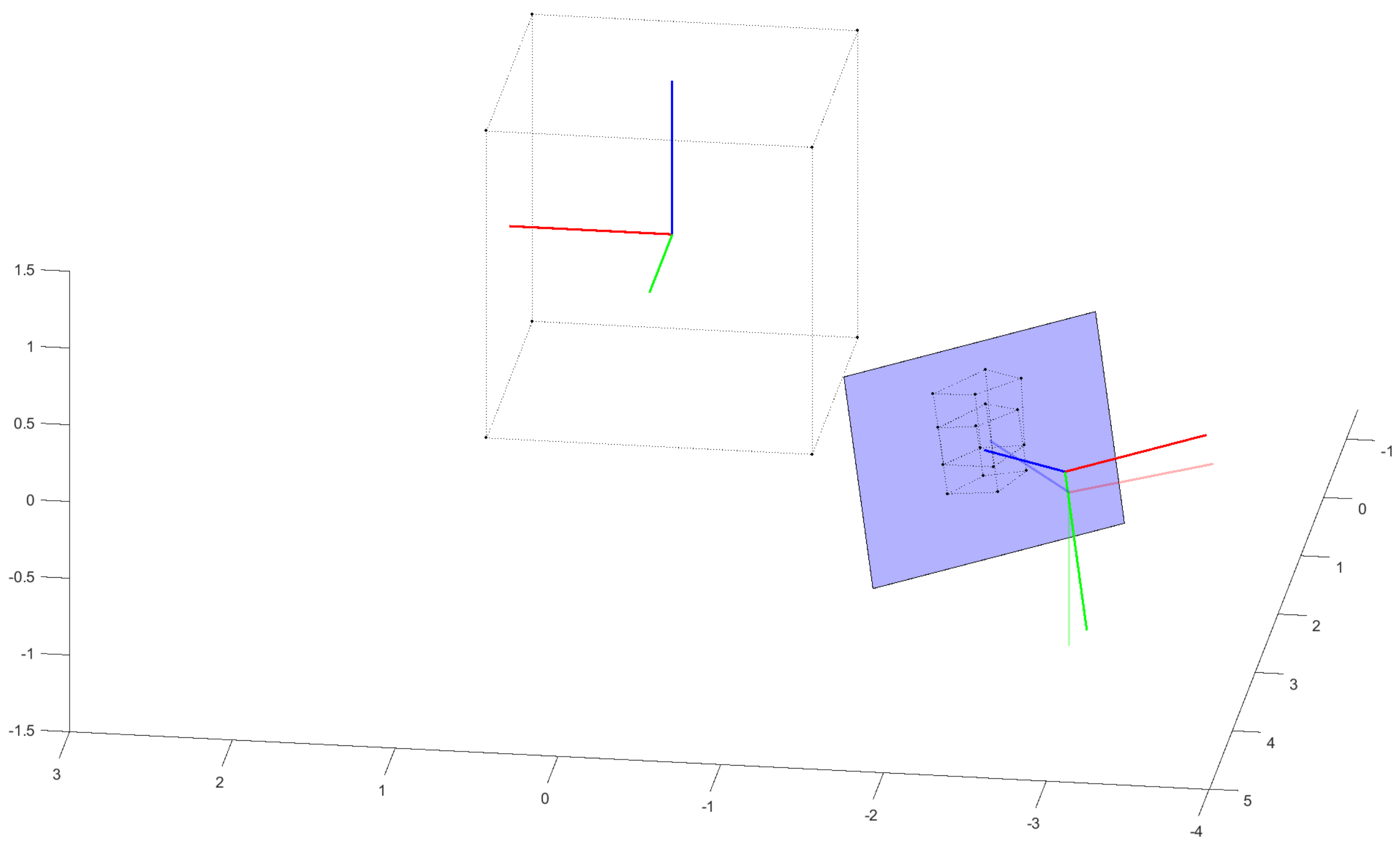
For $t = 0, 1, \ldots, t^{max}$

$\mathbf{A}, \mathbf{b} \leftarrow$ Linearize at $\mathbf{T}_{wc}^{t}$

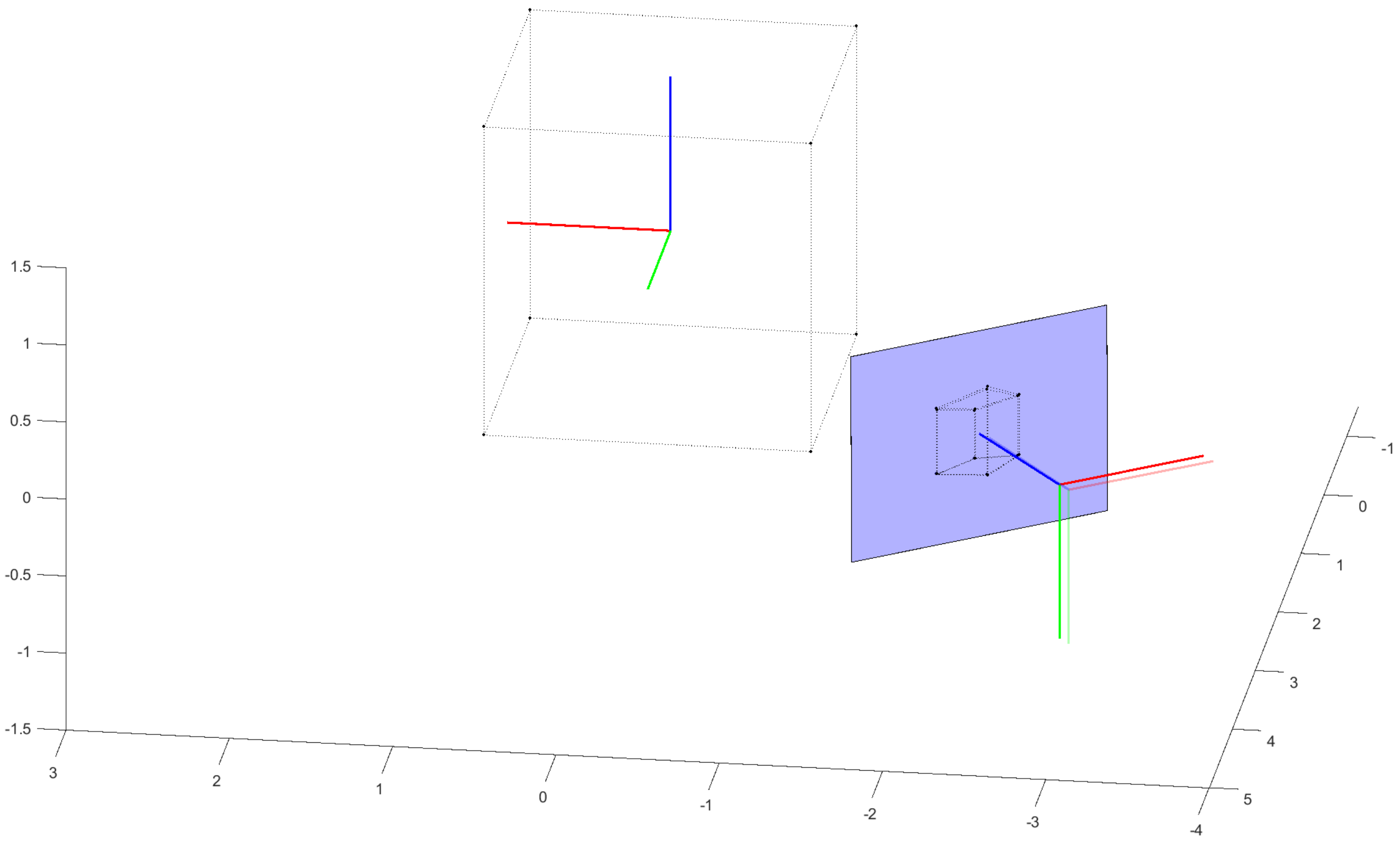
$\boldsymbol{\Delta} \leftarrow$ Solve the linearized problem with $\mathbf{A}^T \mathbf{A} \boldsymbol{\Delta} = \mathbf{A}^T \mathbf{b}$

$\mathbf{T}_{wc}^{t+1} \leftarrow \mathbf{T}_{wc}^{t} \exp(\boldsymbol{\xi}_{\Delta}^{*\wedge})$

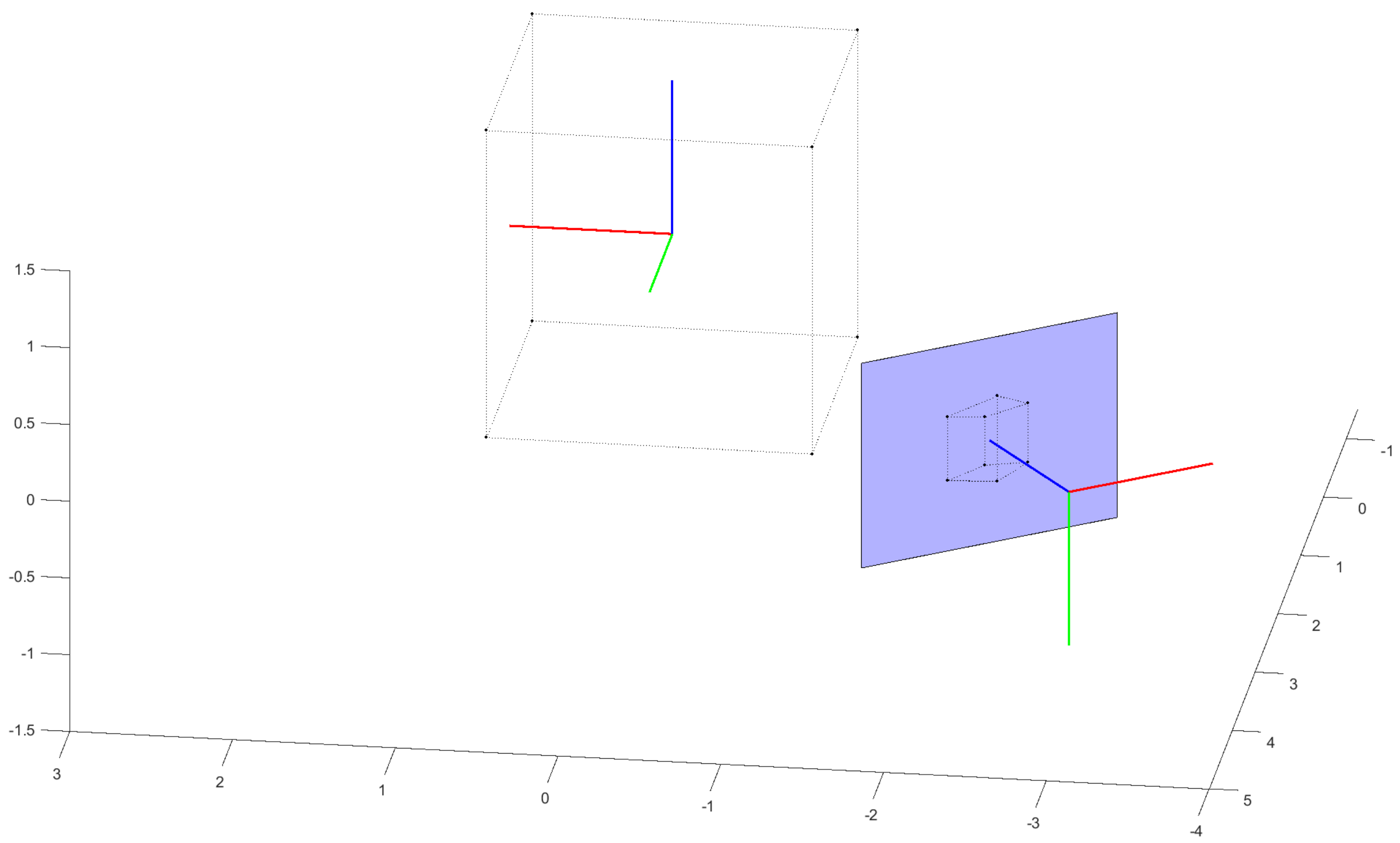
Example



Example



Example



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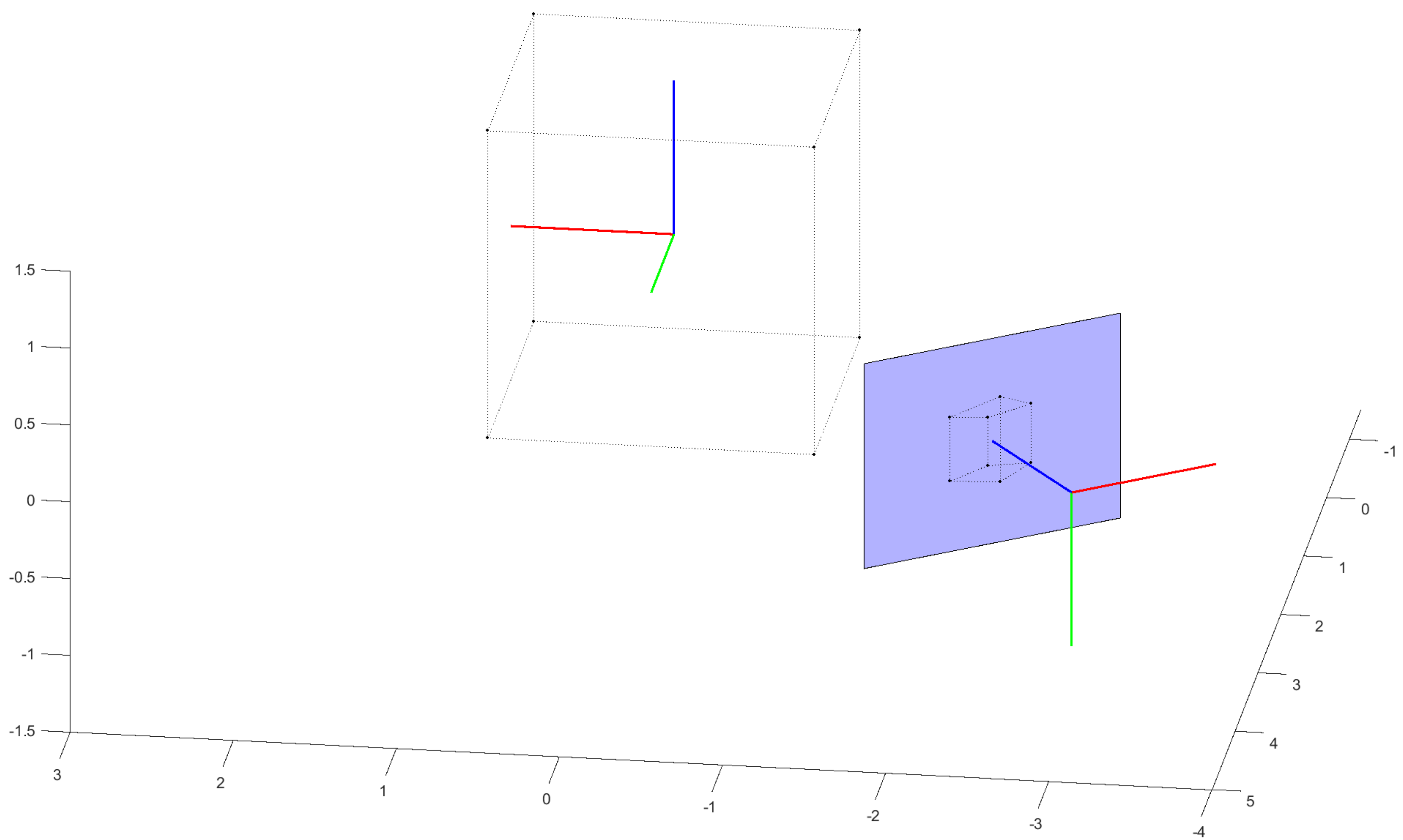
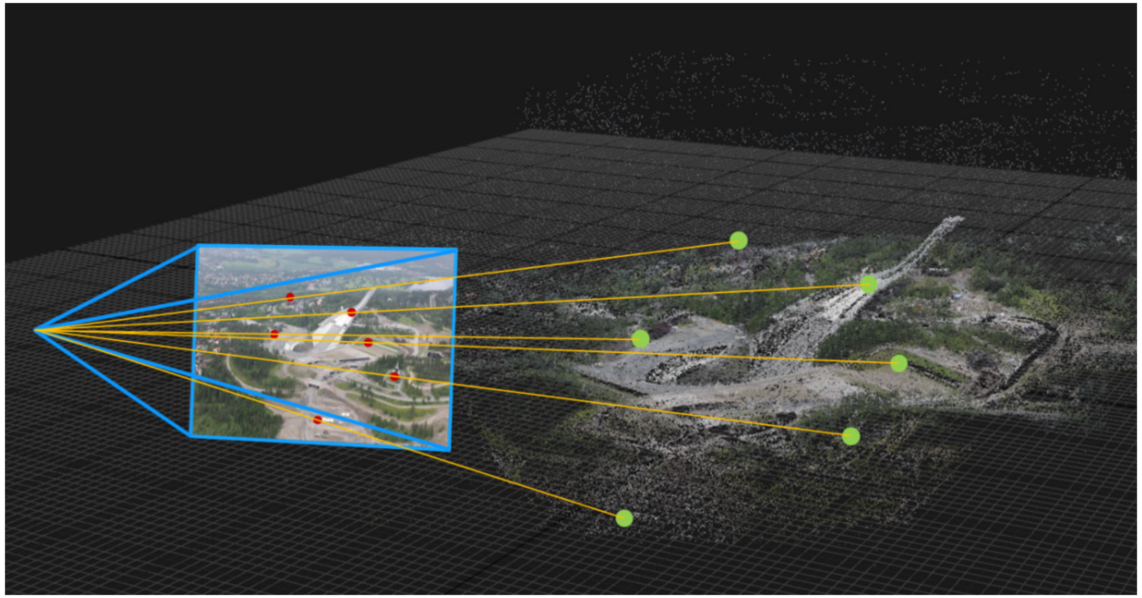


Pose estimation relative to known 3D points

- *n*-Point Pose Problem (P*n*P)
  - Typically fast non-iterative methods
  - Minimal in number of points
  - Accuracy comparable to iterative methods
  - Good initial estimates

- Examples
  - P3P, EPnP
  - P4Pf
    - Estimate pose and focal length
  - P6P
    - Estimates **P** with DLT
  - R6P
    - Estimate pose with rolling shutter



Summary

- Pose estimation relative to a world plane
  - Pose from homography
- Nonlinear optimization over poses
- Pose estimation relative to known 3D points
  - Iterative methods
  - P*n*P