UiO : Department of Technology Systems
University of Oslo

Lecture 3.3
Robust estimation with RANSAC

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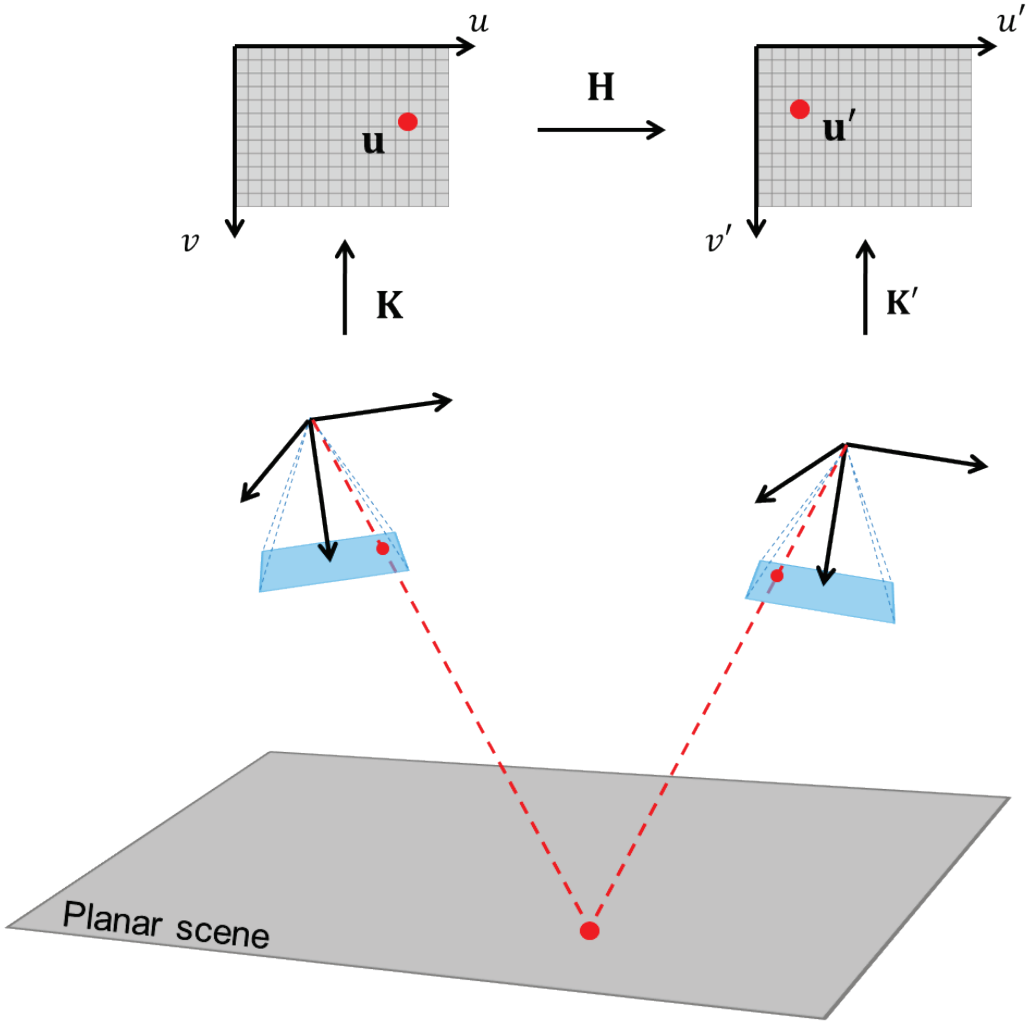
Motivation

- If two perspective cameras captures images of a planar scene, their images are related by a homography $\mathbf{H}$



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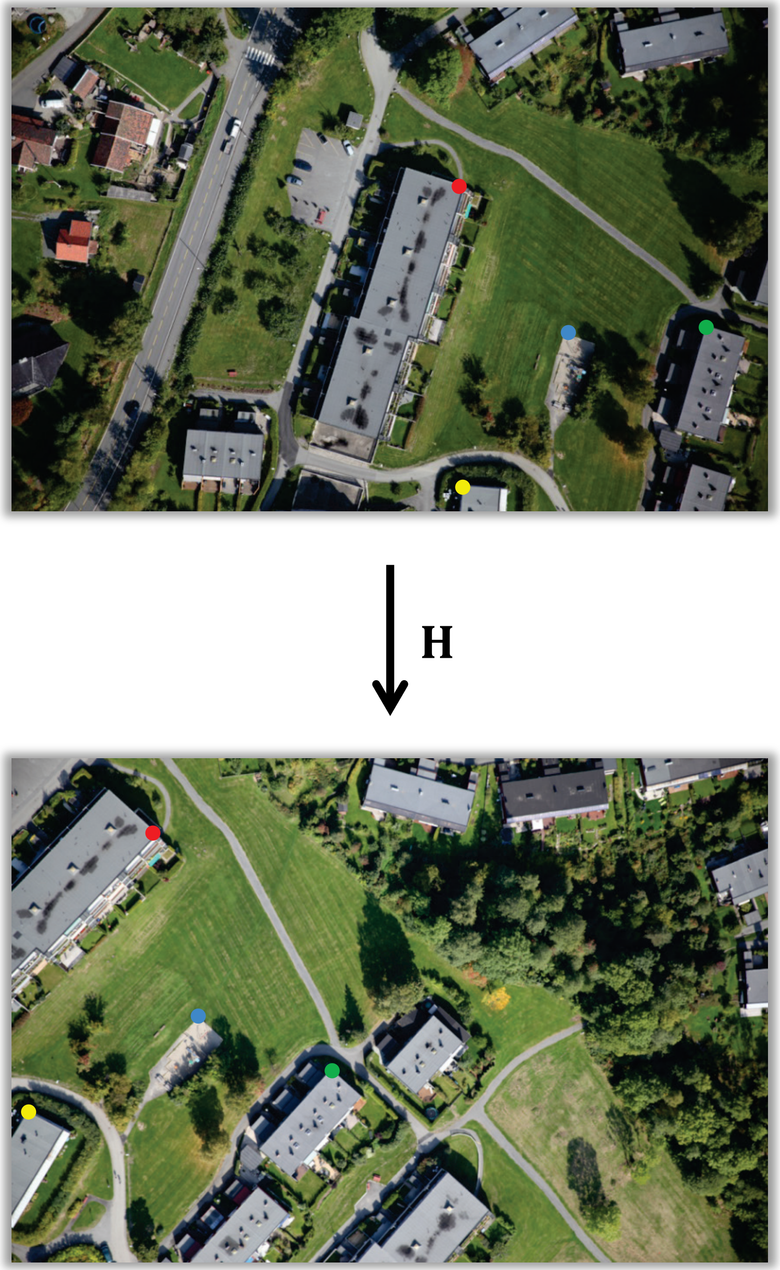
- It can be estimated if we know at least 4 point-correspondences $\mathbf{u}_i \leftrightarrow \mathbf{u}'_i$



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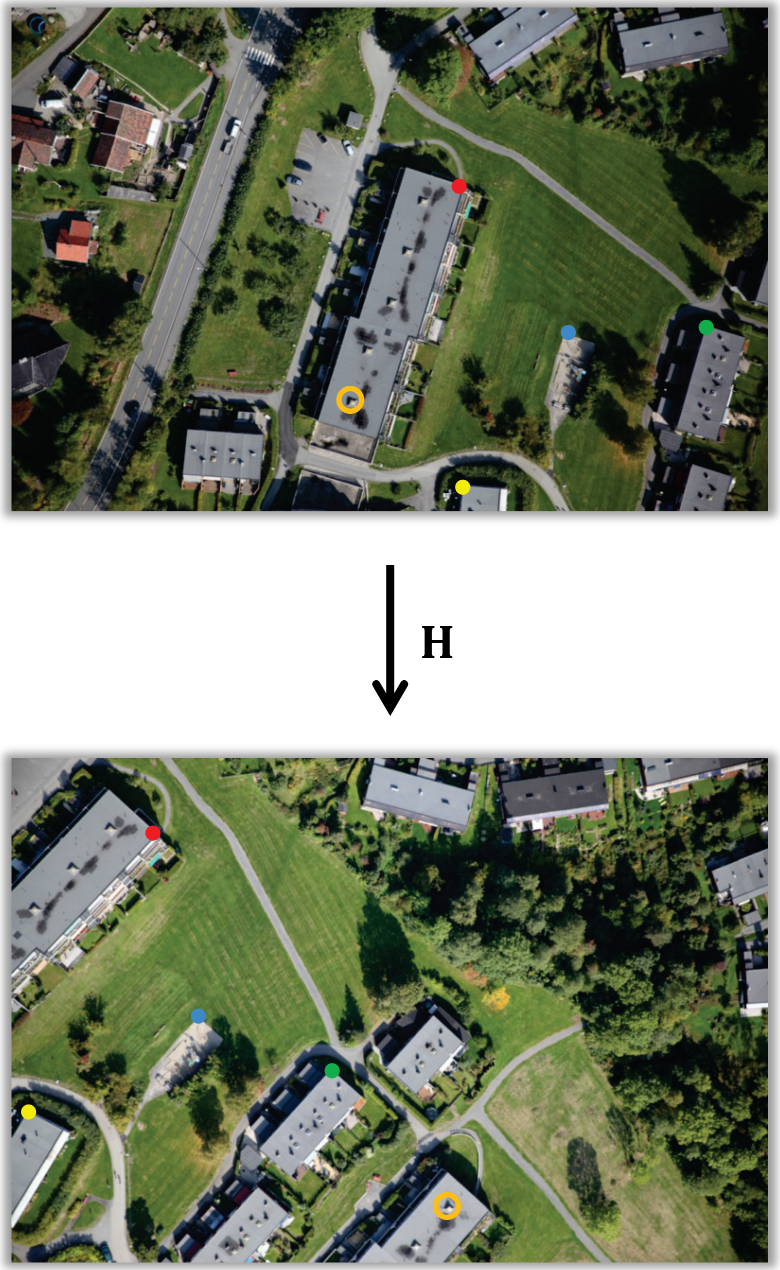
- Correspondences can be found automatically, but typically some of them will be wrong

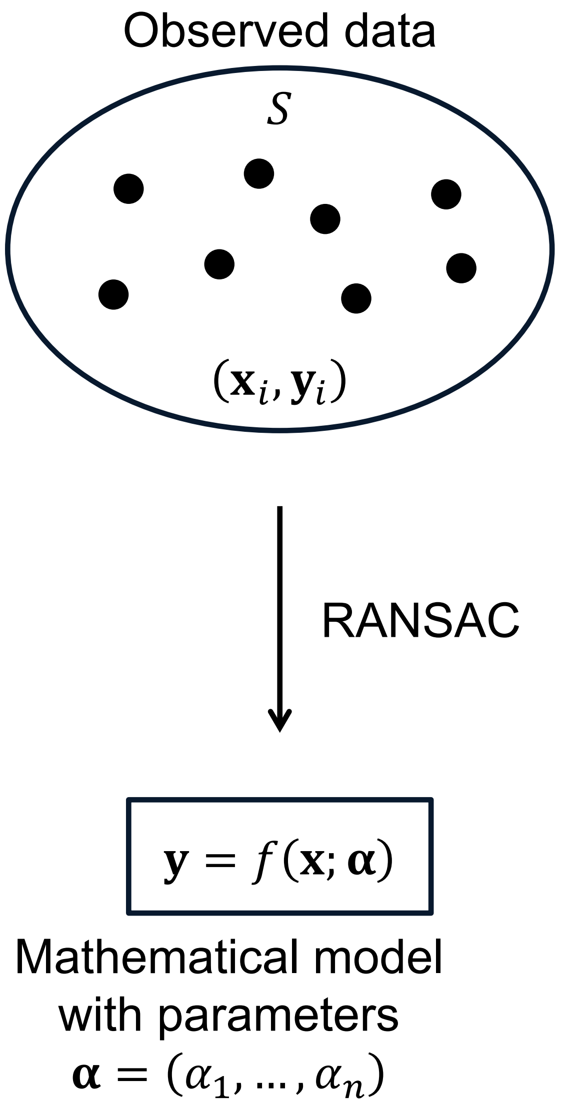


Motivation

- If two perspective cameras captures images of a planar scene, their images are related by a homography $\mathbf{H}$
- It can be estimated if we know at least 4 point-correspondences $\mathbf{u}_i \leftrightarrow \mathbf{u}'_i$
- Correspondences can be found automatically, but typically some of them will be wrong
- Despite outliers in the set of point-correspondences, we are still able to estimate $\mathbf{H}$

RANdom SAmple Consensus - RANSAC

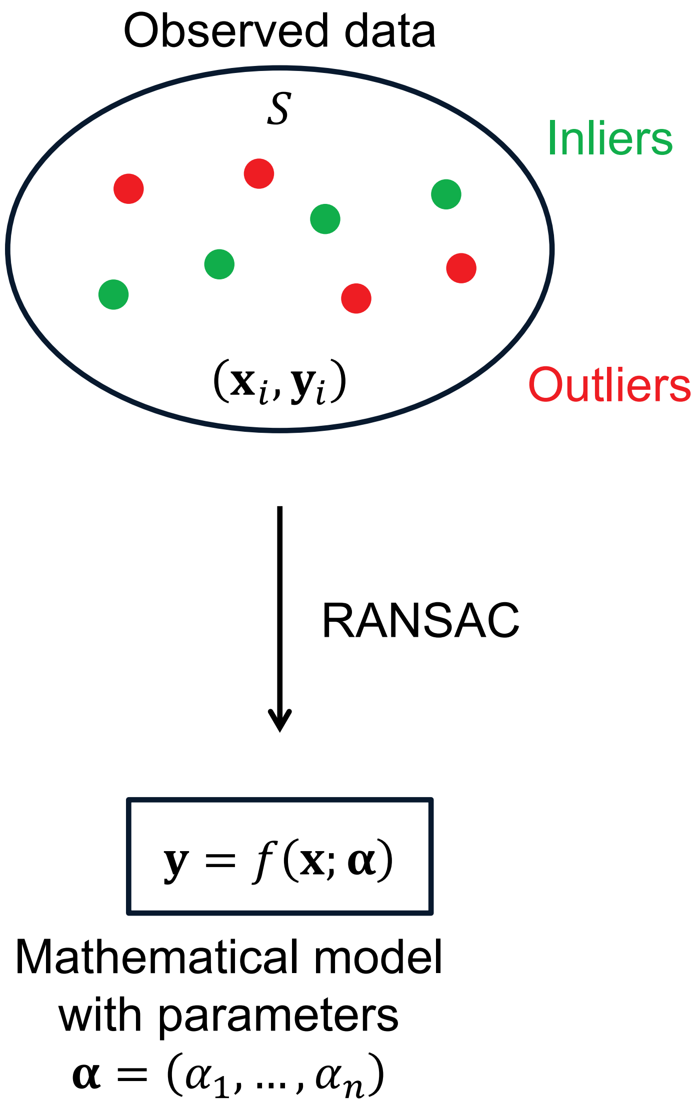
- RANSAC is an iterative method for estimating the parameters of a mathematical model from a set of observed data that contains outliers



RANdom SAmple Consensus - RANSAC

- RANSAC is an iterative method for estimating the parameters of a mathematical model from a set of observed data that contains outliers

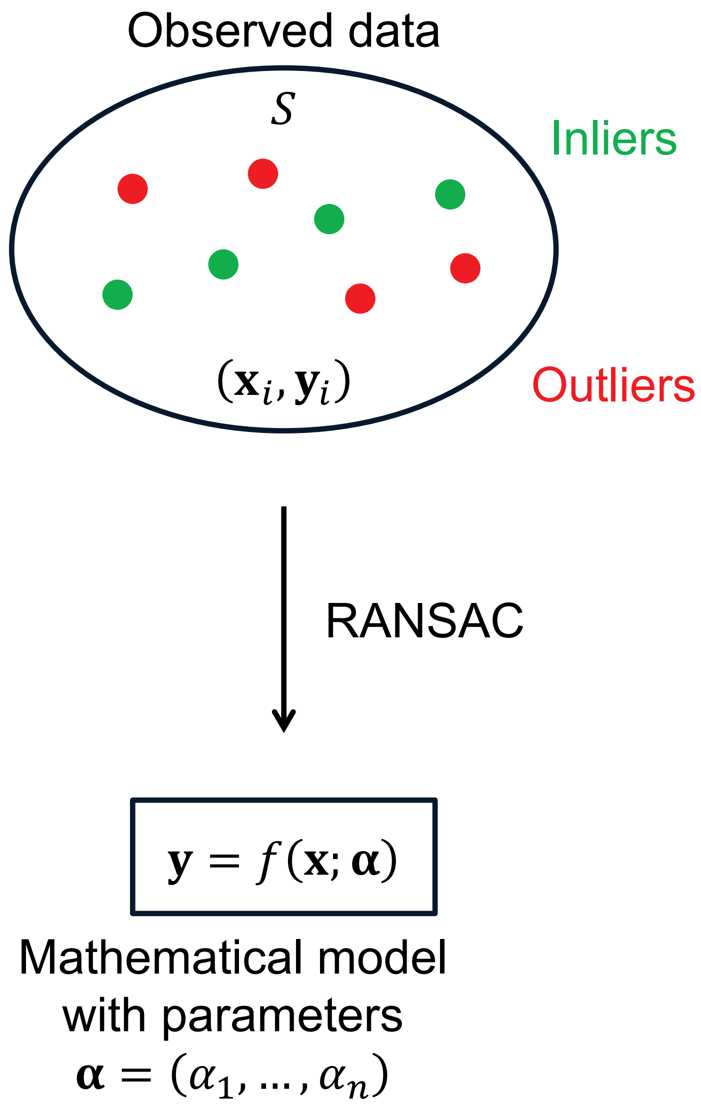
- The RANSAC estimation process divides the observed data into inliers and outliers, so it can also be regarded as an outlier detection method



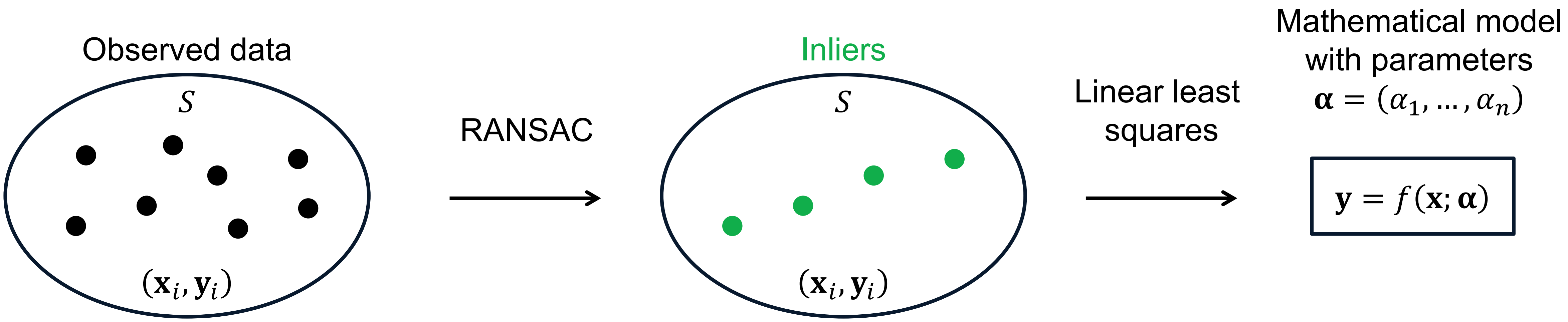
RANdom SAmple Consensus - RANSAC

- RANSAC is an iterative method for estimating the parameters of a mathematical model from a set of observed data that contains outliers
  - Secondary application!

- The RANSAC estimation process divides the observed data into inliers and outliers, so it can also be regarded as an outlier detection method
  - Main application!



RANdom SAmple Consensus - RANSAC



- In order to estimate a model from data containing outliers, it is common to use RANSAC in combination with another estimation method e.g. linear least squares

- The model estimated in RANSAC is usually ignored
  - It is based on a small subset of the observed data
  - Different RANSAC estimations will typically return different models (RANSAC is non-deterministic)

Basic RANSAC

Objective

Robustly fit a model $\mathbf{y} = f(\mathbf{x}; \boldsymbol{\alpha})$ to a data set $S = \{\mathbf{x}_i\}$

Algorithm

Repeat steps 1-3 until N models have been tested

1. Determine a test model $\mathbf{y} = f(\mathbf{x}; \boldsymbol{\alpha}_{tst})$ from n random data points $\{(\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2), \dots, (\mathbf{x}_n, \mathbf{y}_n)\}$

2. Check how well the data points in S fit with the test model
   - Data points within a distance t of the model constitute a set of inliers $S_{tst} \subseteq S$
   - The remaining data points are outliers

3. If S_{tst} is larger than all previous set of inliers, we update the RANSAC model $f(\mathbf{x}; \boldsymbol{\alpha}) = f(\mathbf{x}; \boldsymbol{\alpha}_{tst})$ and the set of inliers $S_{IN} = S_{tst}$

Comments

- The number of tests, N, is directly related to the probability of sampling at least one random n-tuple $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ with no outliers in it

  The test model and inlier set corresponding to such an n-tuple should be an acceptable result of the RANSAC estimation

- If ω is the probability of a random data point being an inlier, then the number of test N and the probability p to sample at least one random n-tuple with no outliers is related by

$$N = \frac{log(1-p)}{log(1-\omega^n)}$$

- By keeping n as small as possible, we also minimize the number of required tests N for a desired level of confidence p

Comments

- Standard value $p = 0.99$
- We rarely know the ratio of inliers in our dataset, so in most situations, ω is unknown
- Instead of choosing a small ω just to be on the safe side, leading to a larger than necessary N, we can modify RANAC to adaptively estimate N as we perform the iterations

$\omega = P(inlier)$

n	N	0.9	0.8	0.7	0.6	0.5
	2	3	5	7	11	17
	3	4	7	11	19	35
	4	5	9	17	34	72
	5	6	12	26	57	146
	6	7	16	37	97	293
	7	8	20	54	163	588
	8	9	26	78	272	1177

$$N = \frac{log(1-p)}{log(1-\omega^n)}$$

$$p = 0.99$$

Adaptive RANSAC

Objective

Robustly fit a model $\mathbf{y} = f(\mathbf{x}; \boldsymbol{\alpha})$ to a data set $S = \{\mathbf{x}_i\}$

Algorithm

Let $N = \infty$, $S_{IN} = \emptyset$

While ($num\_iterations$ < N) repeat steps 1-4

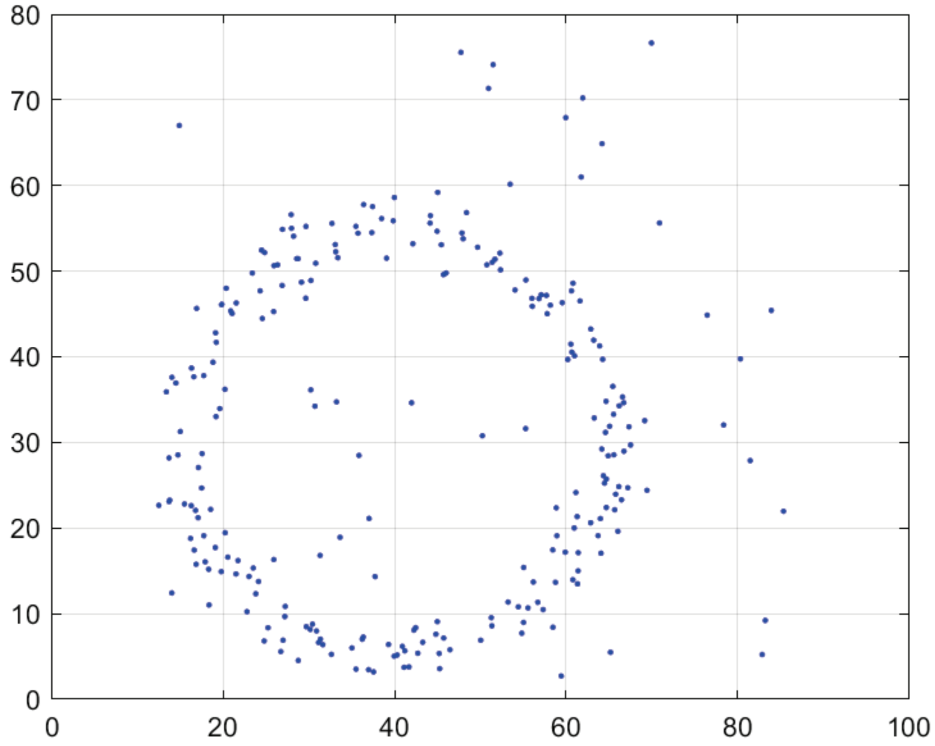
1. Determine a test model $\mathbf{y} = f(\mathbf{x}; \boldsymbol{\alpha}_{tst})$ from n random data points $\{(\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2), \dots, (\mathbf{x}_n, \mathbf{y}_n)\}$

2. Check how well the data points in S fit with the test model
   - Data points within a distance t of the model constitute a set of inliers $S_{tst} \subseteq S$

3. If S_{tst} is larger than S_{IN}, we update the RANSAC model $f(\mathbf{x}; \boldsymbol{\alpha}) = f(\mathbf{x}; \boldsymbol{\alpha}_{tst})$ and the set of inliers $S_{IN} = S_{tst}$

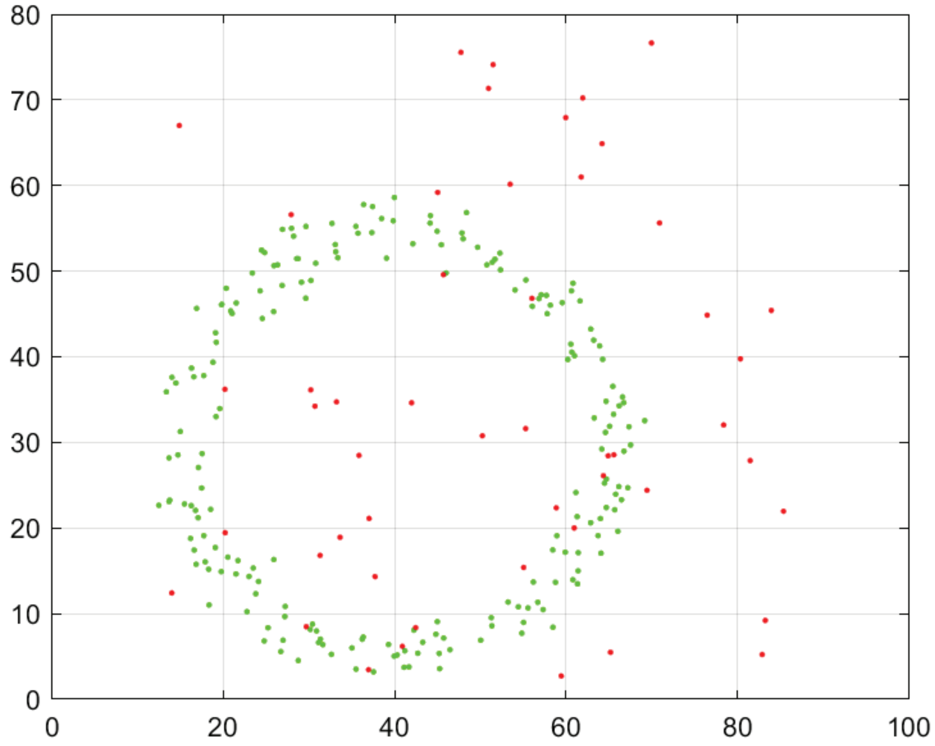
4. Compute $N = \frac{log(1-p)}{log(1-\omega^n)}$ using that $\omega = \frac{|S_{IN}|}{|S|}$ and $p = 0.99$

Example



- Fit a circle $(x - x_0)^2 + (y - y_0)^2 = r^2$ to these data points by estimating the 3 parameters x_0, y_0 and r

Example



Random points on a circle + Gaussian noise

Random points

- Fit a circle $(x - x_0)^2 + (y - y_0)^2 = r^2$ to these data points by estimating the 3 parameters x_0, y_0 and r

- The data set consists of random points on a circle with some Gaussian noise added to them and some additional random points

Example

Linear least squares approach
Separate observables from parameters:

$$(x-x_0)^2+(y-y_0)^2=r^2$$

$$x^2-2xx_0+x_0^2+y^2-2yy_0+y_0^2=r^2$$

$$2xx_0+2yy_0+r^2-x_0^2-y_0^2=x^2+y^2$$

$$\begin{bmatrix} x & y & 1 \end{bmatrix}\begin{bmatrix} 2x_0 \\ 2y_0 \\ r^2-x_0^2-y_0^2 \end{bmatrix}=\begin{bmatrix} x^2+y^2 \end{bmatrix}$$

$$\begin{bmatrix} x & y & 1 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}=\begin{bmatrix} x^2+y^2 \end{bmatrix}$$

So for each observation (x_i, y_i) we get one equation

$$\begin{bmatrix} x_i & y_i & 1 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}=\begin{bmatrix} x_i{}^2+y_i{}^2 \end{bmatrix}$$

From all our n observations we get a system of linear equations

$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ & \vdots & \\ x_n & y_n & 1 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}=\begin{bmatrix} x_1^2+y_1^2 \\ x_2^2+y_2^2 \\ \vdots \\ x_n^2+y_n^2 \end{bmatrix}$$

$$\mathbf{Ap}=\mathbf{b}$$

Example

- Here we have $n > 3$ data points and only 3 parameters
  - Overdetermined set of equations
  - Typically no exact solution

- The linear least squares solution to the problem is the parameter $\mathbf{p}^*$ that minimizes the sum of squares of residuals

$$\mathbf{p}^* = \underset{\mathbf{p}}{\operatorname{argmin}} \|\mathbf{A}\mathbf{p} - \mathbf{b}\|^2$$

- This can be found by solving the following equation

$$\frac{\partial}{\partial \mathbf{p}} \left(\|\mathbf{A}\mathbf{p} - \mathbf{b}\|^2 \right) = \mathbf{0}$$

Example

- This leads to the so called normal equations

$$\frac{\partial}{\partial \mathbf{p}}(\|\mathbf{Ap}-\mathbf{b}\|^2) = \mathbf{0}$$

$$2\mathbf{A}^T(\mathbf{Ap}-\mathbf{b}) = \mathbf{0}$$

$$\mathbf{A}^T\mathbf{Ap} = \mathbf{A}^{\mathrm{T}}\mathbf{b}$$

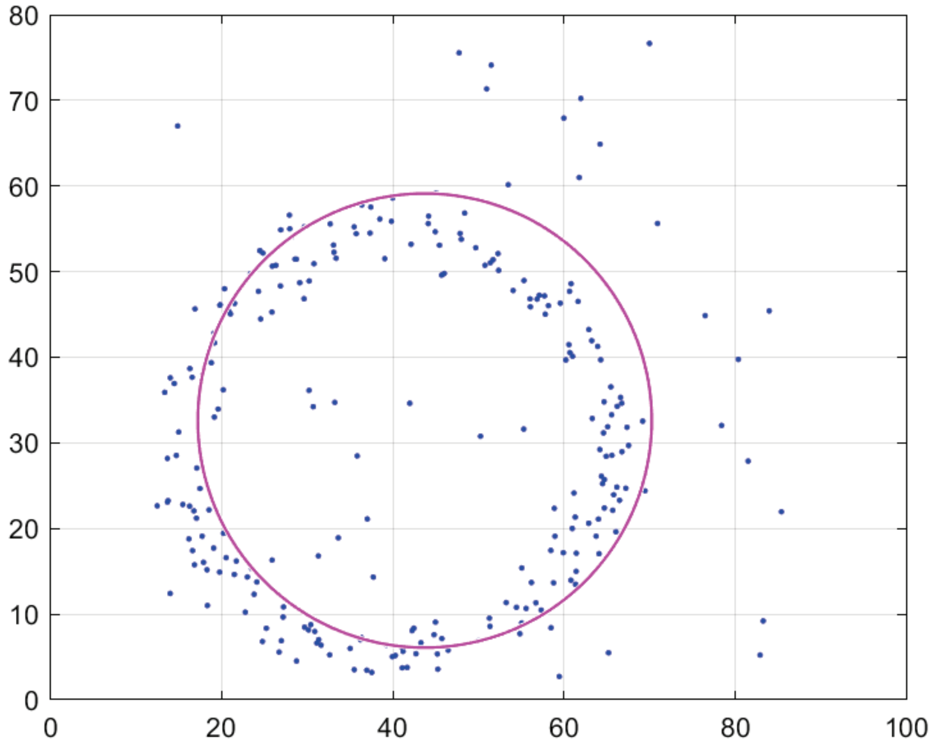
- Hence the linear least squares solution is

$$\mathbf{p}^* = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{b}$$

Example

- The linear least squares solution for our problem looks like this...

$$\underbrace{\begin{bmatrix} 2x_0 \\ 2y_0 \\ r^2 - x_0^2 - y_0^2 \end{bmatrix}}_{\mathbf{p}^*} = \underbrace{\left(\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ & \vdots & \\ x_n & y_n & 1 \end{bmatrix}^T \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ & \vdots & \\ x_n & y_n & 1 \end{bmatrix} \right)^{-1}}_{(\mathbf{A}^T\mathbf{A})^{-1}} \underbrace{\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ & \vdots & \\ x_n & y_n & 1 \end{bmatrix}^T}_{\mathbf{A}^\mathrm{T}} \underbrace{\begin{bmatrix} x_1^2 + y_1^2 \\ x_2^2 + y_2^2 \\ \vdots \\ x_n^2 + y_n^2 \end{bmatrix}}_{\mathbf{b}}$$



NOT GOOD!

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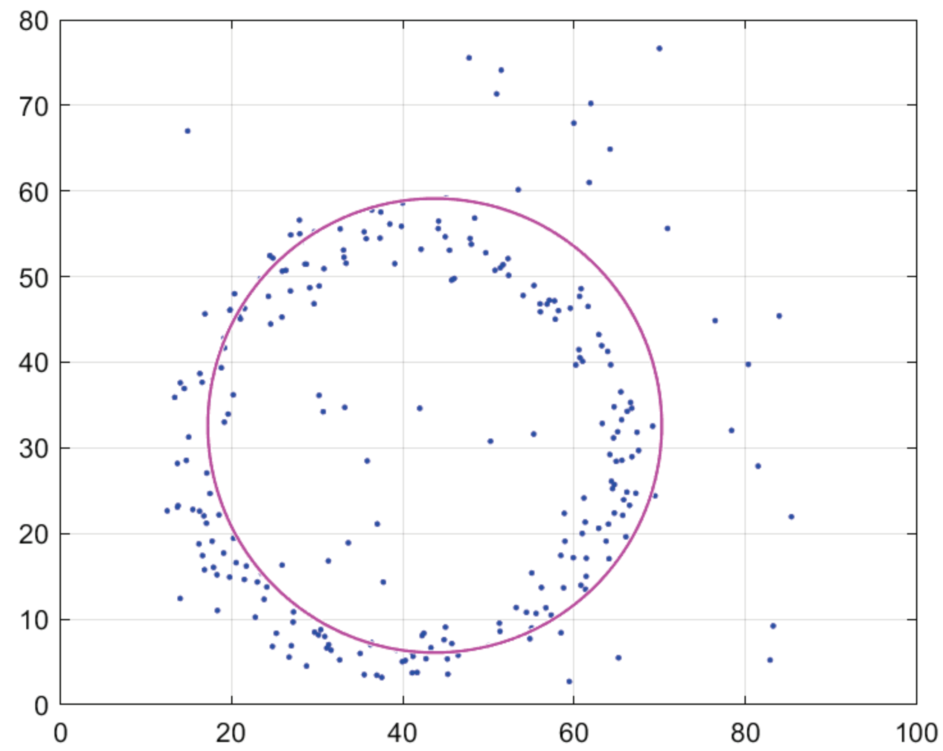
- Since all points are treated equal, the random points shift the estimated circle away from the desired solution

- Now let us try RANSAC



NOT GOOD!

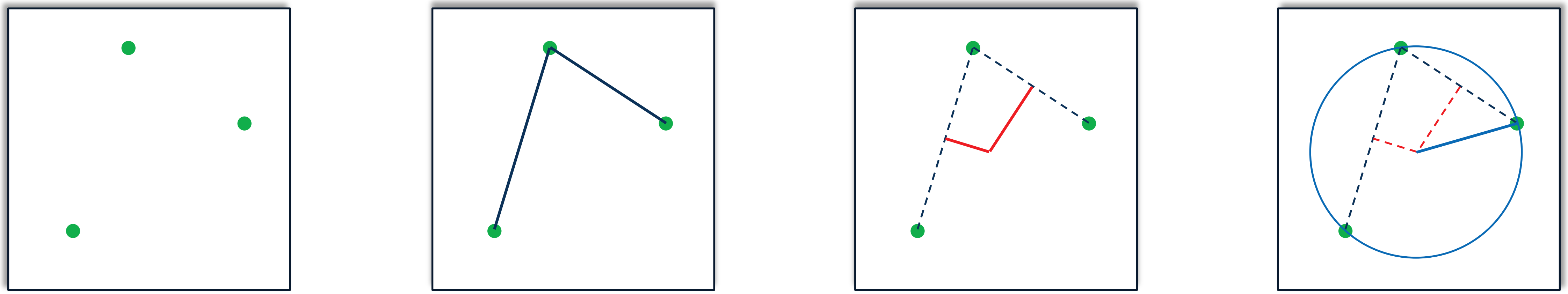
Example

- RANSAC requires two things
  1. A way to estimate a circle from n points, where n is as small as possible
  2. A way to determine which of the points are inliers for an estimated circle

Example

- RANSAC requires two things
  1. **A way to estimate a circle from n points, where n is as small as possible**
  2. A way to determine which of the points are inliers for an estimated circle

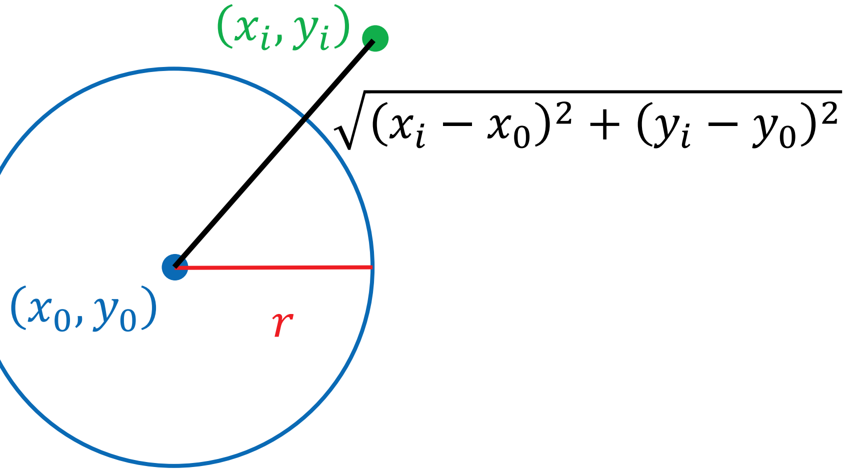
- The smallest number of points required to determine a circle is 3, i.e. $n = 3$, and the algorithm for computing the circle is quite simple



Example

- RANSAC requires two things
  1. A way to estimate a circle from n points, where n is as small as possible
  2. **A way to determine which of the points are inliers for an estimated circle**

- The distance from a point (x_i, y_i) to a circle $(x - x_0)^2 + (y - y_0)^2 = r^2$ is given by $\left|\sqrt{(x_i - x_0)^2 + (y_i - y_0)^2} - r\right|$



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- So for a threshold value t, we say that (x_i, y_i) is an inlier if $\left|\sqrt{(x_i - x_0)^2 + (y_i - y_0)^2} - r\right| < t$

- The value of t should be chosen according to the noise/uncertainty we expect in the data points (x_i, y_i)
  – In the case of Gaussian noise with standard deviation $\sigma = \sigma_x = \sigma_y$, $t = 3\sigma$ should enable us to find a large set of inliers

Example

Objective

Robustly fit the model $(x - x_0)^2 + (y - y_0)^2 = r^2$ to our data set $S = \{(x_i, y_i)\}$

Algorithm

Let $N = \infty$, $S_{IN} = \emptyset$, $p = 0.99$, $t = 3 \cdot expected\ noise$

While ($num\_iterations$ < N) repeat steps 1-4

1. Determine test circle $(x_{tst}, y_{tst}, r_{tst})$ from three random points $\{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$ from S

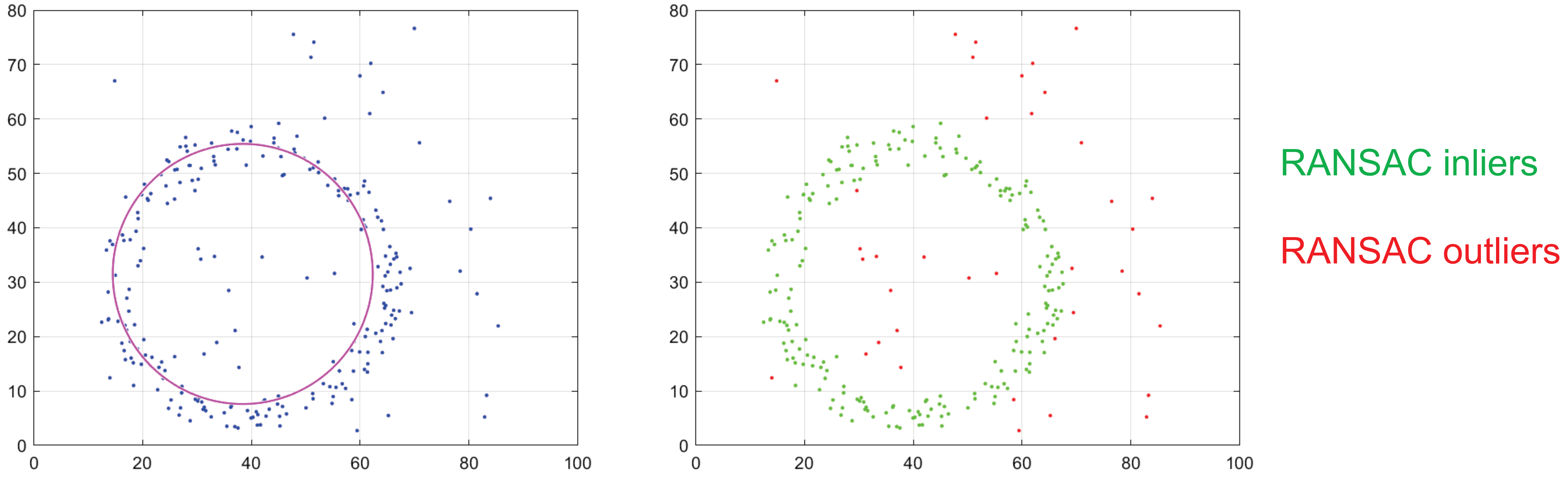
2. Check how well each individual data point in S fits with the test model

$$S_{tst} = \left\{(x_i, y_i) \in S \text{ such that } \left|\sqrt{(x_i - x_{tst})^2 + (y_i - y_{tst})^2} - r_{tst}\right| < t \right\}$$

3. If S_{tst} is the largest set of inliers encountered so far, we keep this model
   – Set $S_{IN} = S_{tst}$ and $(x_0, y_0, r) = (x_{tst}, y_{tst}, r_{tst})$

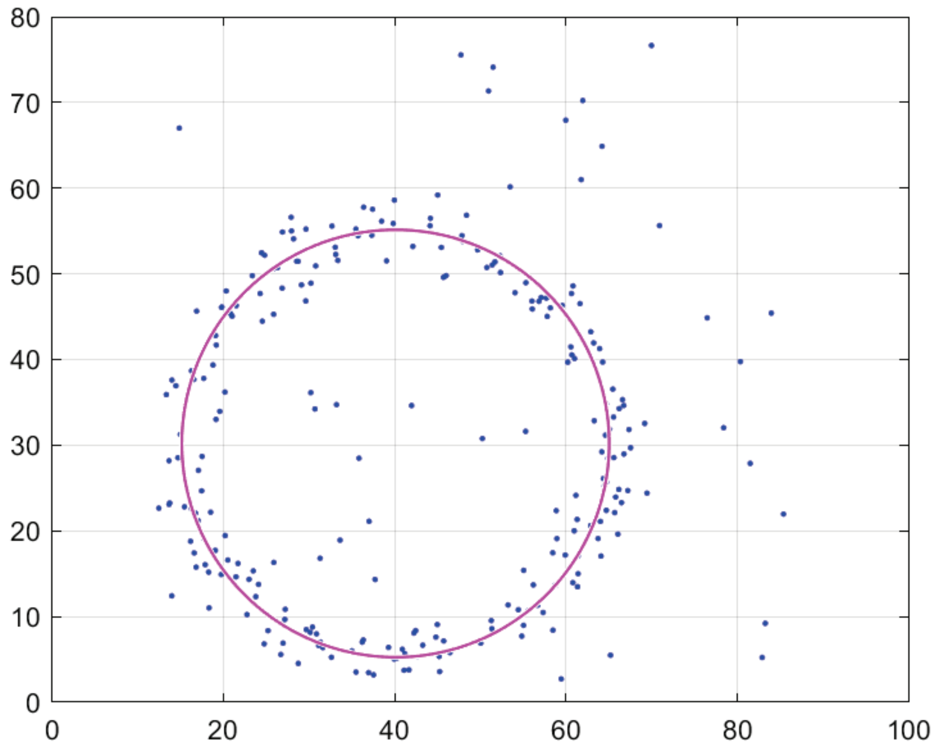
4. Recompute $N = \frac{log(1-p)}{log(1-\omega^3)}$ using that $\omega = \frac{|S_{IN}|}{|S|}$ and $p = 0.99$

Example



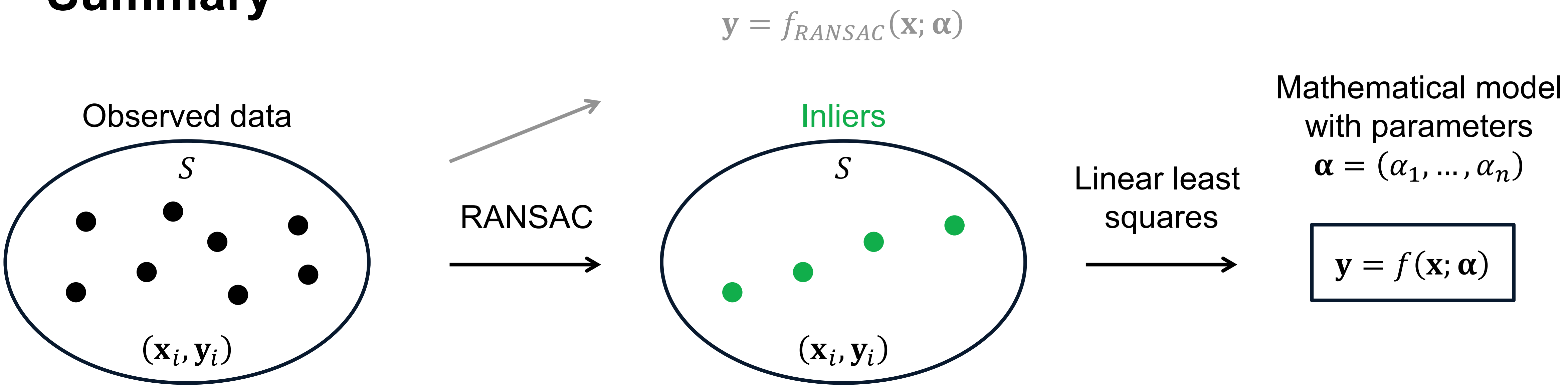
- RANSAC output (an example)
  - The RANSAC estimated circle typically changes from one estimation to another
  - The RANSAC estimated inliers are more consistent

Example



- Linear least squares solution based on RANSAC inliers

Summary



- RANSAC is an inlier detection method commonly used in combination with an estimation method like linear least squares to estimate a mathematical model from a dataset containing outliers

- RANSAC also provides an estimate for the mathematical model,
  - Typically estimated from only a small subset of the inliers
  - Typically different from one estimation to another

Further reading

- Online book by Richard Szeliski, Computer Vision: Algorithms and Applications
  http://szeliski.org/Book/drafts/SzeliskiBook_20100903_draft.pdf
  - Chapter 6.1.4

- Online book by Timothy D. Barfoot, State Estimation for Robotics
  http://asrl.utias.utoronto.ca/~tdb/bib/barfoot_ser17.pdf
  - Chapter 5.3