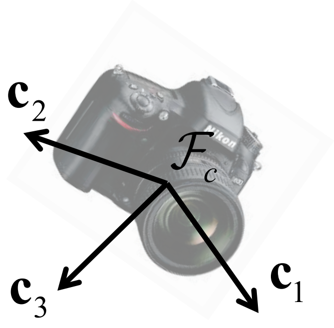
Lecture 5.1
Orientation in 3D

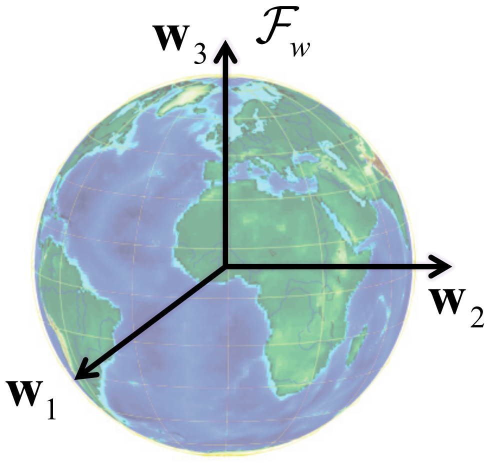
Thomas Opsahl

What is orientation?

- A term describing the relationship between coordinate frames

- Orientation ↔ Rotation





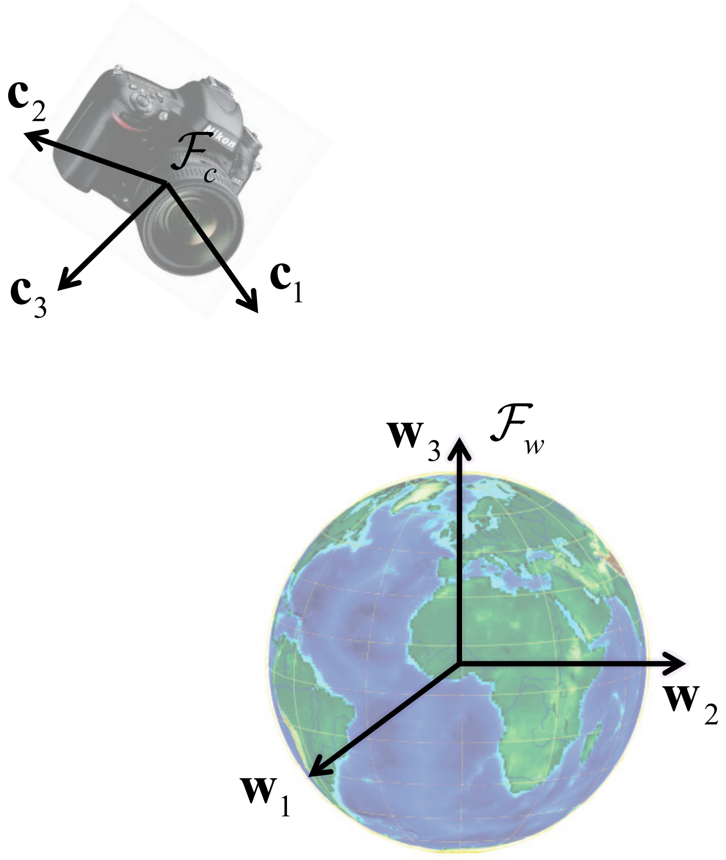
What is orientation?

- A term describing the relationship between coordinate frames
- Orientation ↔ Rotation

The orientation of $\mathcal{F}_c$ relative to $\mathcal{F}_w$

⇕

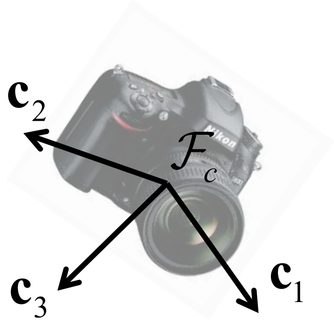
How $\mathcal{F}_w$ should rotate in order to align with $\mathcal{F}_c$

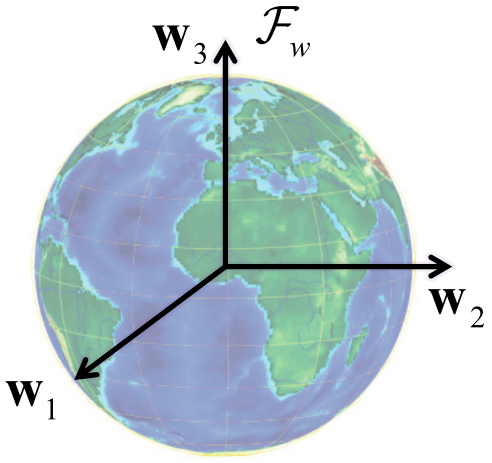


Orientation

- The orientation of the camera frame $\mathcal{F}_c$ with respect to the world frame $\mathcal{F}_w$ can be represented by an orthonormal rotation matrix

$$\mathbf{R}_{wc} \in SO(3)$$





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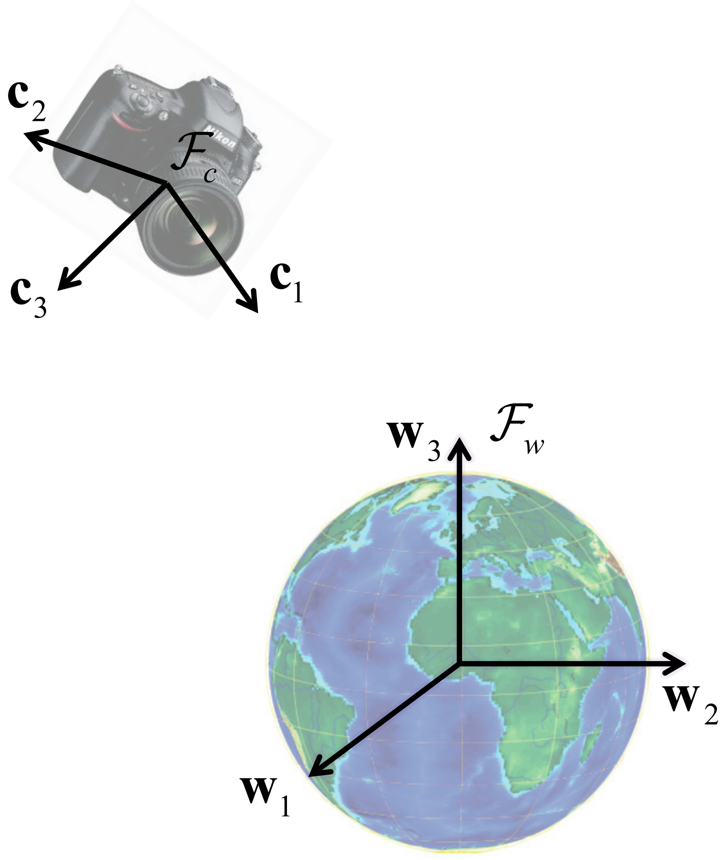
orthonormal matrix

$\Updownarrow$

1. All rows and columns have norm 1
2. Any two rows are orthogonal
3. Any two columns are orthogonal



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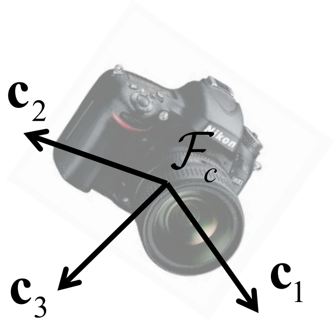
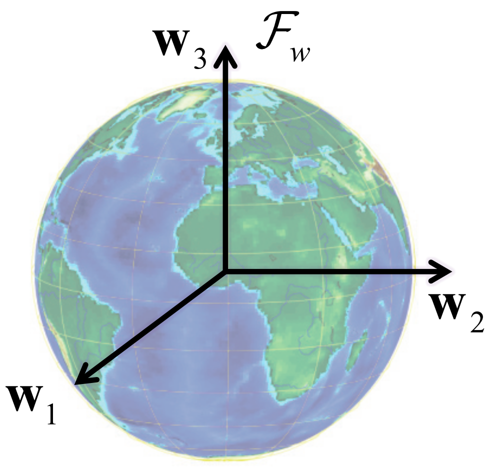
- Special orthogonal group

$$SO(3) = \left\{ \mathbf{R} \in \mathbb{R}^{3\times3} \mid \mathbf{R}\mathbf{R}^T = \mathbf{1}, \det \mathbf{R} = 1 \right\}$$





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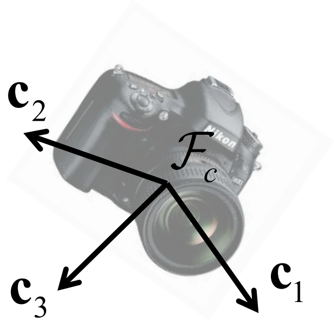

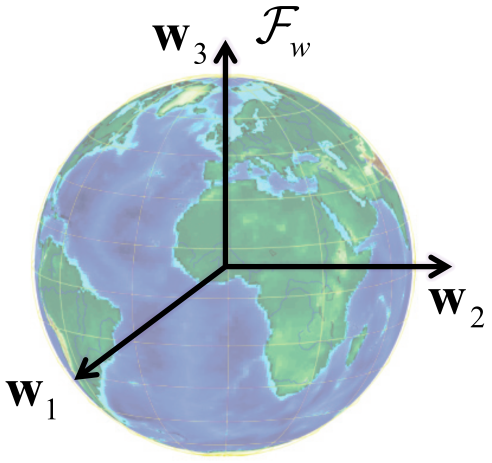

Group

1. Closed under matrix multiplication $\mathbf{R}_1\mathbf{R}_2 \in SO(3)$
2. Neutral element $\mathbf{1} \in SO(3)$
3. Inverse $\mathbf{R}^{-1} \in SO(3)$
4. Associativity $\mathbf{R}_1(\mathbf{R}_2\mathbf{R}_3) = (\mathbf{R}_1\mathbf{R}_2)\mathbf{R}_3$

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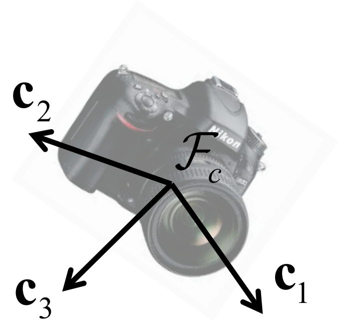
!

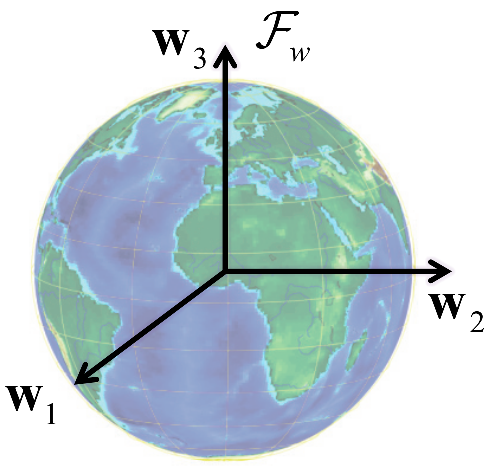
?

$$\mathbf{R}^{-1} = \mathbf{R}^T$$

$$\mathbf{R}\mathbf{R}^\mathbf{T} = \mathbf{1} \Rightarrow \det \mathbf{R} = \pm 1$$

What about $\det \mathbf{R} = -1$?





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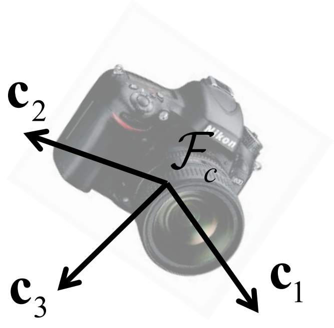
- Construction from orthonormal basis vectors

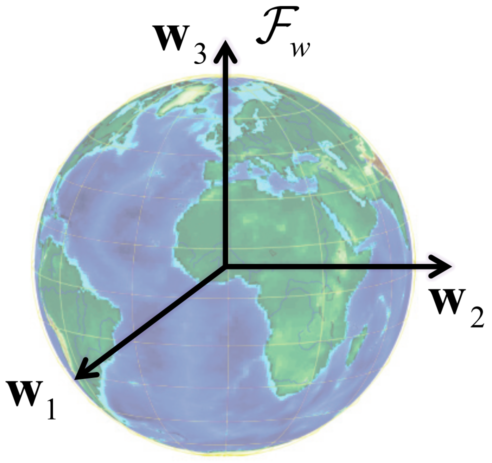
$$\mathbf{R}_{wc} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{matrix} \mathbf{w}_1^{cT} \\ \mathbf{w}_2^{cT} \\ \mathbf{w}_3^{cT} \end{matrix}$$

$$\begin{matrix} \mathbf{c}_1^w & \mathbf{c}_2^w & \mathbf{c}_3^w \end{matrix}$$

$$\mathcal{F}_w = span\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$$

$$\mathcal{F}_c = span\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$$

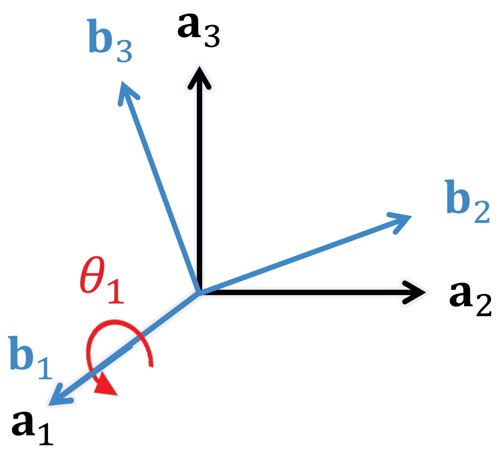




Principal rotations

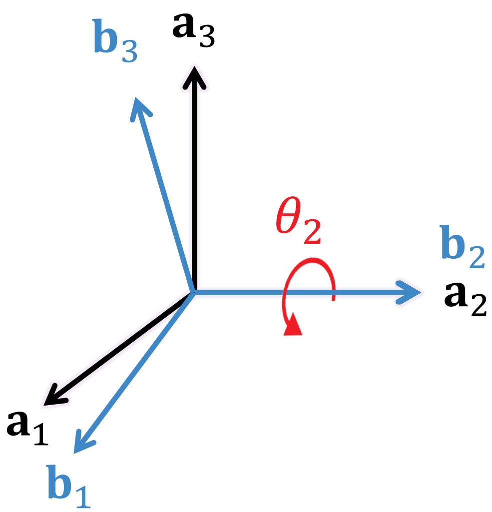
$\mathbf{R}_x(\theta_1)$

$$\mathbf{R}_{ab} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$



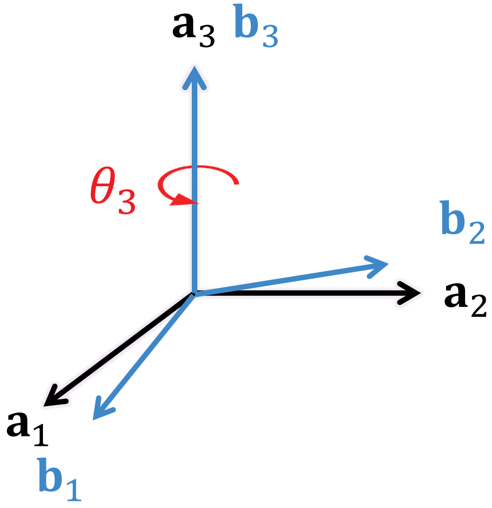
$\mathbf{R}_y(\theta_2)$

$$\mathbf{R}_{ab} = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}$$



$\mathbf{R}_z(\theta_3)$

$$\mathbf{R}_{ab} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

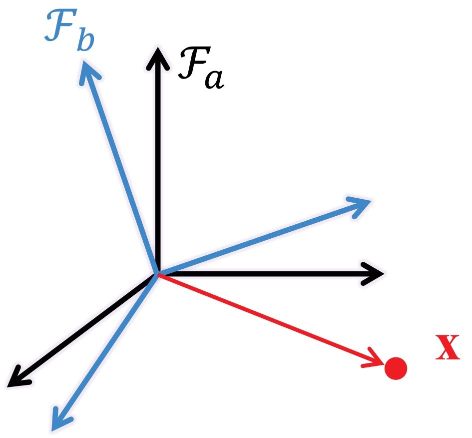


Action on points

- The matrix $\mathbf{R}_{ab}$ represents the orientation of $\mathcal{F}_b$ relative to $\mathcal{F}_a$, but it is also a point transformation from $\mathcal{F}_b$ to $\mathcal{F}_a$ given that the frames have the same origin

- A point $\mathbf{x}$ can be transformed from $\mathcal{F}_b$ to $\mathcal{F}_a$ by

$$\mathbf{x}^a = \mathbf{R}_{ab}\mathbf{x}^b$$

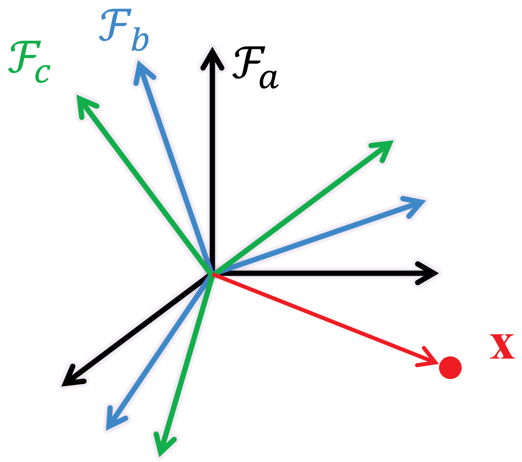


Composition

- We can chain together consecutive orientations

- If $\mathbf{R}_{ab}$ is the orientation of $\mathcal{F}_b$ relative to $\mathcal{F}_a$ and $\mathbf{R}_{bc}$ is the orientation of $\mathcal{F}_c$ relative to $\mathcal{F}_b$, then the orientation of $\mathcal{F}_c$ relative to $\mathcal{F}_a$ is given by

$$\mathbf{R}_{ac} = \mathbf{R}_{ab}\mathbf{R}_{bc}$$



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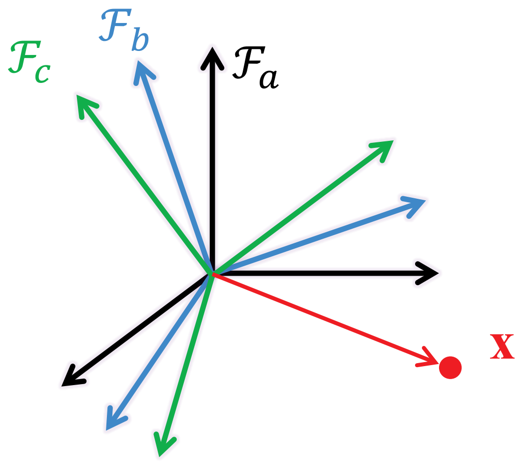

Note
The indexes are always pairwise equal

$$\mathbf{x}^a = \mathbf{R}_{ab}\mathbf{R}_{bc}\mathbf{x}^c$$

destination frame

intermediate frame

source frame

Composition

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- If $\mathbf{R}_{ab}$ is the orientation of $\mathcal{F}_b$ relative to $\mathcal{F}_a$ and $\mathbf{R}_{bc}$ is the orientation of $\mathcal{F}_c$ relative to $\mathcal{F}_b$, then the orientation of $\mathcal{F}_c$ relative to $\mathcal{F}_a$ is given by

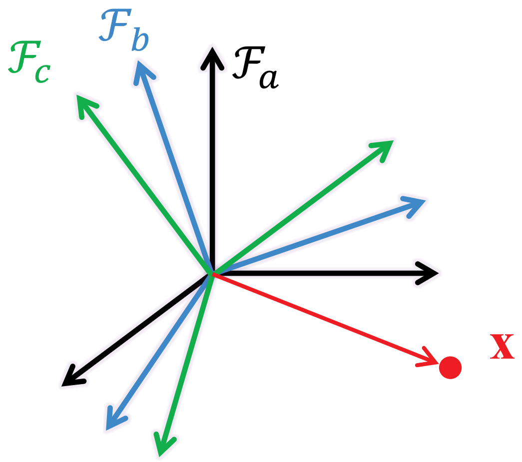
$$\mathbf{R}_{ac} = \mathbf{R}_{ab}\mathbf{R}_{bc}$$

- Issue with numerical precision

$$\mathbf{R}_{az} = \mathbf{R}_{ab}\mathbf{R}_{bc}\cdots\mathbf{R}_{yz} \notin SO(3)$$

- Normalization

$$\mathbf{R}_{az} \overset{SVD}{=} \mathbf{USV}^T \notin SO(3) \quad \Rightarrow \quad \mathbf{R}_{az} = \mathbf{UV}^T \in SO(3)$$



Other representations – Euler angles

- Any orientation can be decomposed into a sequence of three principal rotations

$$\mathbf{R} = \mathbf{R}_z(\theta_3)\mathbf{R}_y(\theta_2)\mathbf{R}_x(\theta_1)$$

- The orientation can be represented by the three angles $(\theta_1, \theta_2, \theta_3)$ known as **Euler angles**

$$\mathbf{R} \rightarrow (\theta_1, \theta_2, \theta_3)$$

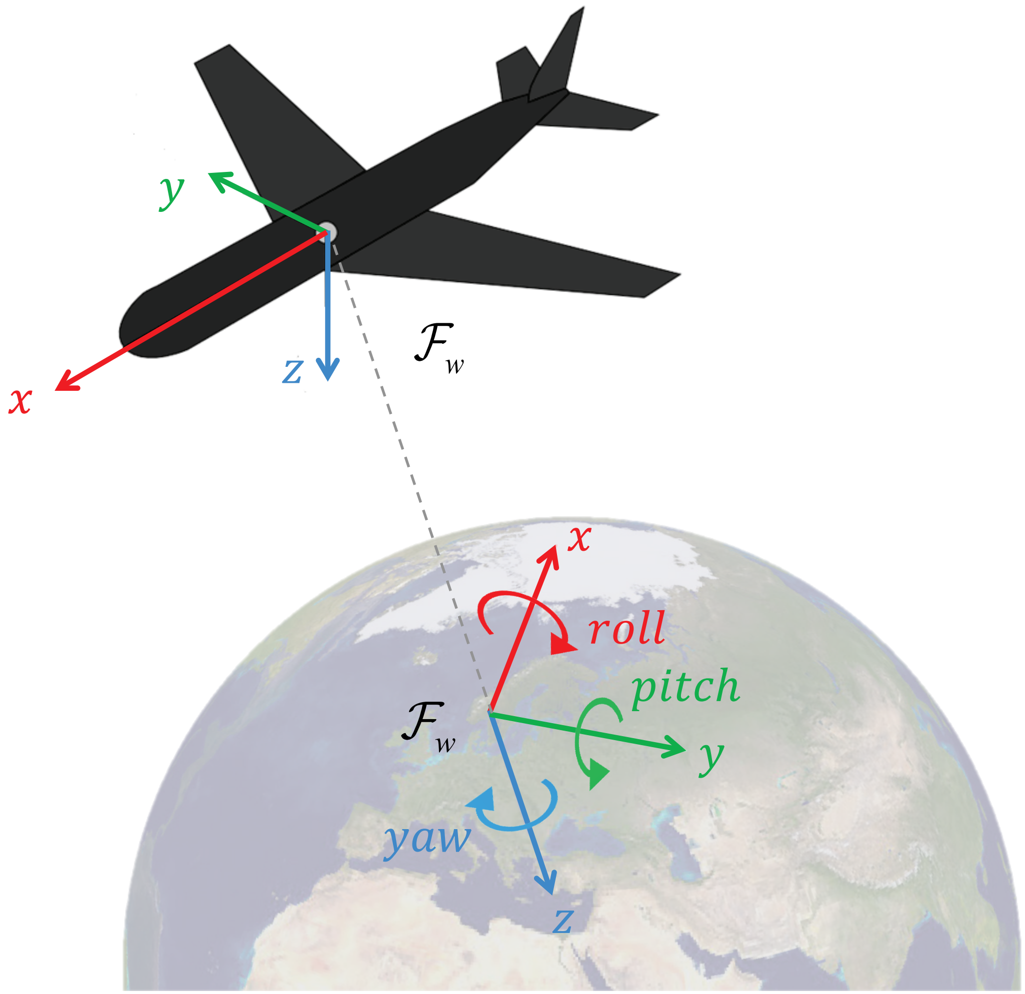
- Several sequences can be used
  - $\mathbf{R}_z\mathbf{R}_y\mathbf{R}_x$, $\mathbf{R}_x\mathbf{R}_y\mathbf{R}_z$, $\mathbf{R}_z\mathbf{R}_x\mathbf{R}_z$,...
  - To understand Euler angles, we must know the sequence they came from!
  - All sequences have singularities, i.e. orientations where the angles of the sequence are not unique
  - Problematic if we want to recover Euler angles from a rotation matrix

Other representations – Euler angles

- $(roll, pitch, yaw)$ is often used in navigation to represent the orientation of a vehicle

- The orientation is often described relative to a local North-East-Down (NED) coordinate frame $\mathcal{F}_w$ in the world situated directly below the body frame $\mathcal{F}_b$

- Then the yaw angle is commonly referred to as «heading» since it corresponds to the compass direction
  - North corresponds to $0°$, east $90°$ and so on



Other representations – Euler angles

- The **roll-pitch-yaw** sequence $\mathbf{R}_z(\theta_3)\mathbf{R}_y(\theta_2)\mathbf{R}_x(\theta_1)$ is singular when $\theta_2 = \frac{\pi}{2}$

$$\mathbf{R}(\theta_1,\theta_2,\theta_3) = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

$\theta_3 = \text{yaw}$ $\theta_2 = \text{pitch}$ $\theta_1 = \text{roll}$

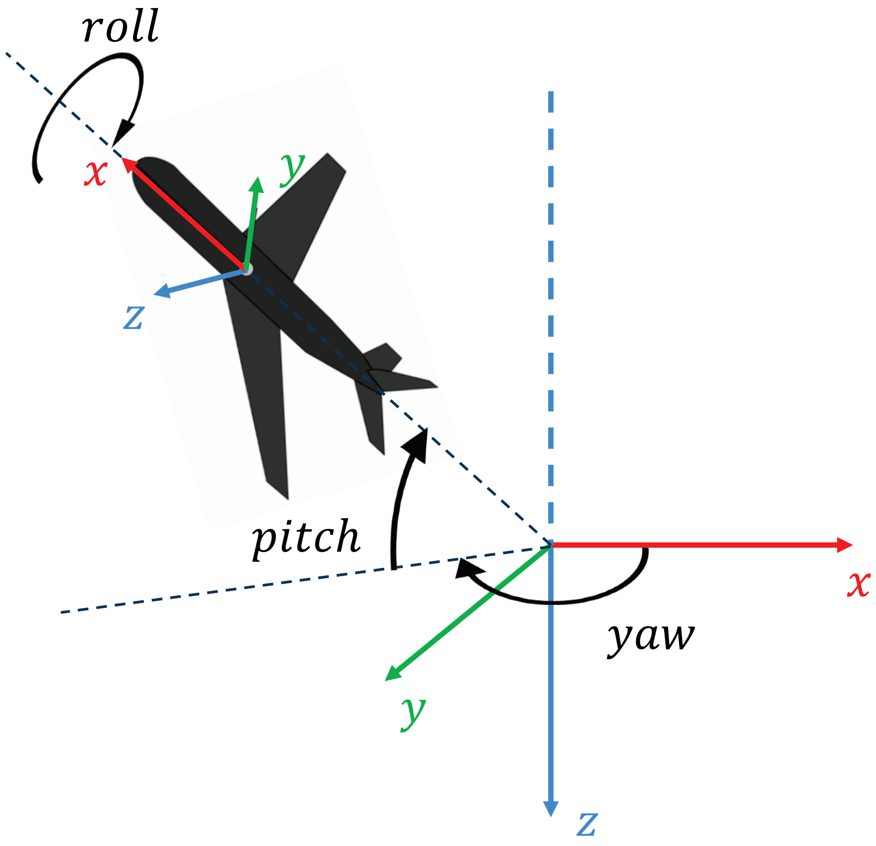
- If we use the notation $c_i = cos(\theta_i)$ and $s_i = sin(\theta_i)$ then we can write

$$\mathbf{R}(\theta_1,\theta_2,\theta_3) = \begin{bmatrix} c_2c_3 & s_1s_2c_3 - c_1s_3 & c_1s_2c_3 + s_1s_3 \\ c_2s_3 & s_1s_2s_3 + c_1c_3 & c_1s_2s_3 - s_1c_3 \\ -s_2 & s_1c_2 & c_1c_2 \end{bmatrix}$$

Other representations – Euler angles

- $(roll, pitch, yaw)$ is practical for vehicles not meant to experience $\theta_2 = \frac{\pi}{2}$
  - Most airplanes, cars and ships

- $(roll, pitch, yaw)$ provides an intuitive understanding about the orientation



Other representations – Axis angle

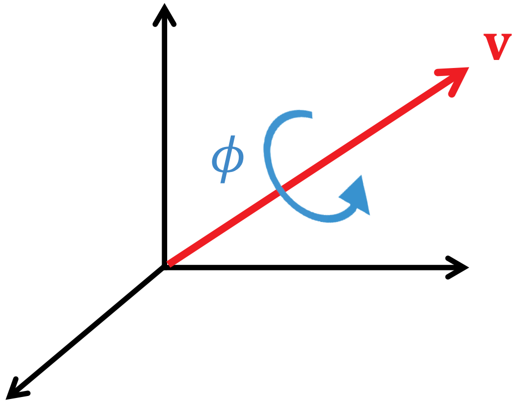
- Euler's rotation theorem states that the most general motion of a rigid body with one point fixed is a rotation about an axis through that point

- So we can represent any orientation by a pair $(\mathbf{v}, \phi)$, where $\mathbf{v} = [v_1, v_2, v_3]^T$ is the axis of rotation and ϕ is the angle of rotation

- This representation is intuitive, but typically not used for computations

- The corresponding rotation matrix is

$$\mathbf{R}_{ab} = \cos\phi \mathbf{1} + (1 - \cos\phi)\mathbf{v}\mathbf{v}^T + \sin\phi \mathbf{v}^\wedge$$



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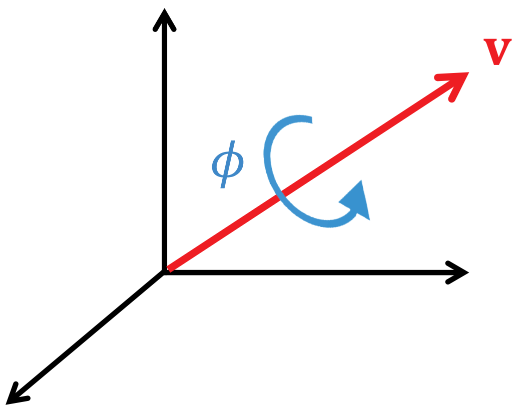


Recall that:

$$\mathbf{v}^\wedge = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$$

Other representations – Unit quaternions

- Quaternions are 4D complex numbers

$$q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}$$

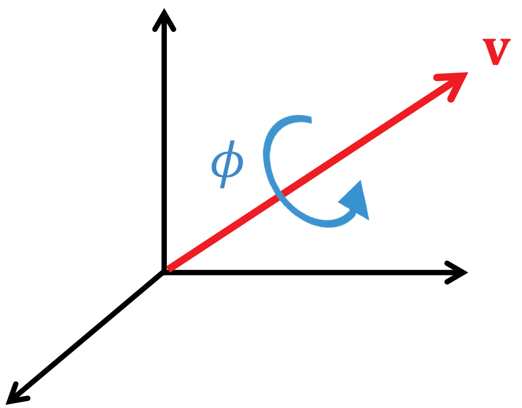
defined by $i^2 = j^2 = k^2 = ijk = -1$

- Norm

$$\|q\| = \sqrt{{q_0}^2 + {q_1}^2 + {q_2}^2 + {q_3}^2}$$

- Unit quaternions ($\|q\| = 1$) is a popular representation for orientation/rotation

- The complex terms are closely related to the axis of rotation, while the real term is closely related to the angle of rotation



$$q_0 = \cos\left(\frac{\phi}{2}\right)$$

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \sin\left(\frac{\phi}{2}\right)\mathbf{v}$$

Other representations – Unit quaternions

- Composition $q_{ac} = q_{ab} q_{bc}$ is very efficient
  - 16 multiplications and 12 additions
  - Matrix multiplication: 27 multiplications and 18 additions
  - Limited numerical precision ⇒ Normalization (divide by $\|q\|$)

- Inverse of unit quaternions

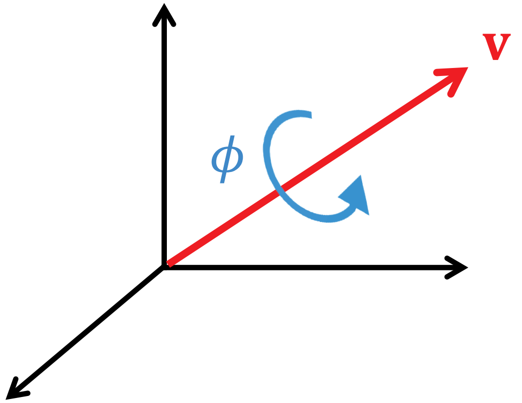
$$q^{-1} = q^* = q_0 - q_1 i - q_2 j - q_3 k$$

- Action on point a $\mathbf{x}^a$ can be expressed as a product

$$p^b = q_{ab} p^a q^*_{ab}$$

where points are represented as quaternions with zero real term

$$\mathbf{x}^a = [x, y, z]^T \quad \mapsto \quad p^a = 0 + xi + yj + zk$$



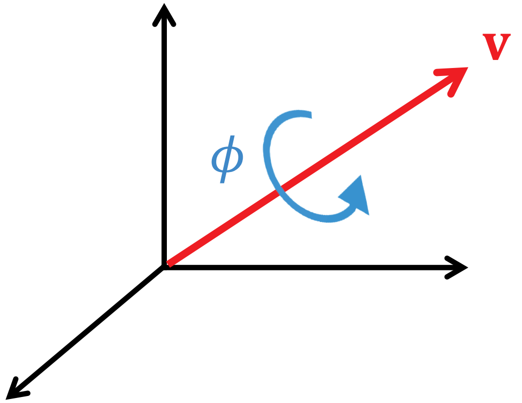
$$q_0 = \cos\left(\frac{\phi}{2}\right)$$

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \sin\left(\frac{\phi}{2}\right)\mathbf{v}$$

Other representations – Unit quaternions

- The rotation matrix corresponding to the unit quaternion $q = q_0 + q_1 i + q_2 j + q_3 k$ is

$$\mathbf{R} = \begin{bmatrix} 1-2\left(q_3^2+q_4^2\right) & 2\left(q_2 q_3 - q_1 q_4\right) & 2\left(q_2 q_4 + q_1 q_3\right) \\ 2\left(q_2 q_3 + q_1 q_4\right) & 1-2\left(q_2^2+q_4^2\right) & 2\left(q_3 q_4 - q_1 q_2\right) \\ 2\left(q_2 q_4 - q_1 q_3\right) & 2\left(q_3 q_4 + q_1 q_2\right) & 1-2\left(q_2^2+q_3^2\right) \end{bmatrix}$$



$$q_0 = \cos\left(\frac{\phi}{2}\right)$$

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \sin\left(\frac{\phi}{2}\right)\mathbf{v}$$

WARNING

Some like to order the quaternion terms differently

$$q = q_0 i + q_1 j + q_2 k + q_3$$

Pros and cons

Rotation matrix $\mathbf{R} \in SO(3)$
- 9 parameters
- Interpretation
- Composition, normalization
- Action on points

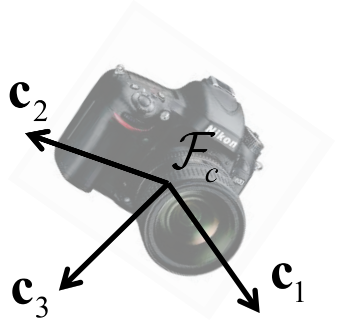
Axis-angle $(\mathbf{v}, \phi)$
- 4 parameters
- Interpretation
- Composition
- Action on points

Euler angles $\boldsymbol{\theta} = [\theta_1, \theta_2, \theta_3]^T$
- 3 parameters
- Interpretation
- Composition
- Action on points

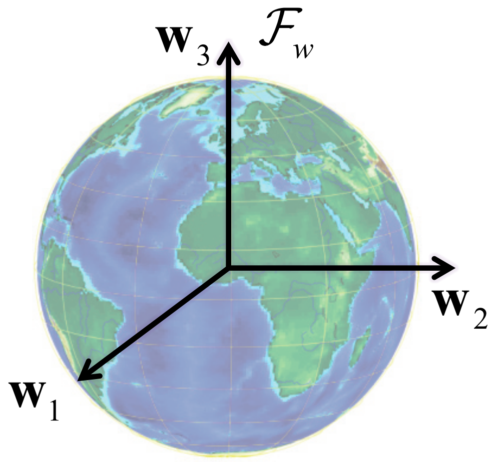
Unit quaternions $q = q_1 + q_2 i + q_3 j + q_4 k$
- 4 parameters
- Interpretation
- Composition, normalization
- Action on points

Summary

- Orientation of a frame $\mathcal{F}_b$ relative to a frame $\mathcal{F}_a$ has several representations
  - Rotation matrix $\mathbf{R} \in SO(3)$ — Main representation for us!
  - Euler angles $\boldsymbol{\theta} = [\theta_1, \theta_2, \theta_3]^T$ — Minimal representation
  - Axis-angle $(\mathbf{v}, \phi) = \{[v_1, v_2, v_3]^T, \phi\}$ — We will not us this
  - Unit quaternion $\mathbf{q} = q_1 + q_2 i + q_3 j + q_4 k$ — We will use this indirectly



- Important properties
  - Inverse: $\mathbf{R}_{ba} = \mathbf{R}_{ab}^{-1}$
  - Composition: $\mathbf{R}_{ac} = \mathbf{R}_{ab}\mathbf{R}_{bc}$
  - Action on points: $\mathbf{x}^b = \mathbf{R}_{ba}\mathbf{x}^a$



Further reading

- Do you want to know more?

- Online book by Richard Szeliski – Computer Vision: Algorithms and Applications
  http://szeliski.org/Book/drafts/SzeliskiBook_20100903_draft.pdf
  - Chapter 2.14 covers 3D rotations

- Online book by Timothy D. Barfoot – State Estimation for Robotics
  http://asrl.utias.utoronto.ca/~tdb/bib/barfoot_ser17.pdf
  - Chapter 6.2 covers rotations