Lecture 6.1
Pose from a known 3D map

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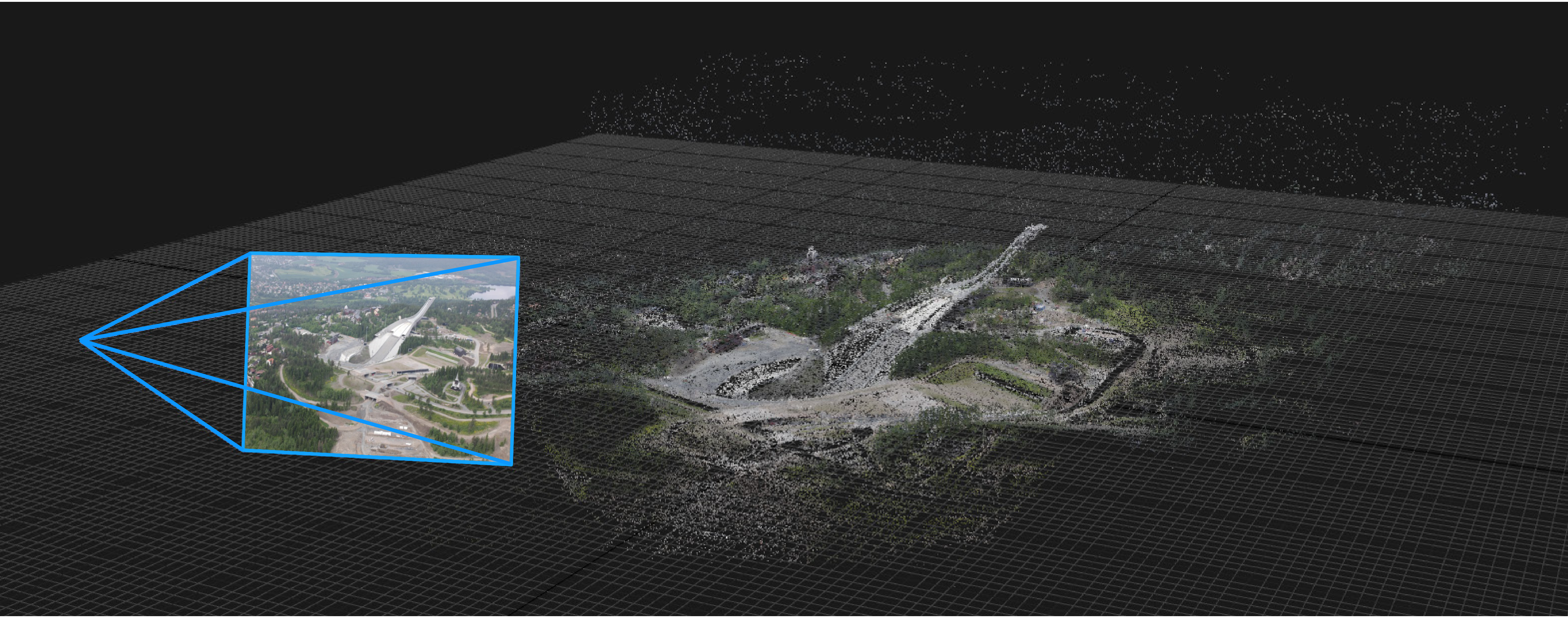
Pose estimation

- Pose estimation given a map is sometimes called **localization**
- In **visual localization**, this is sometimes also called **tracking**
  - Tracking the map in the image frames

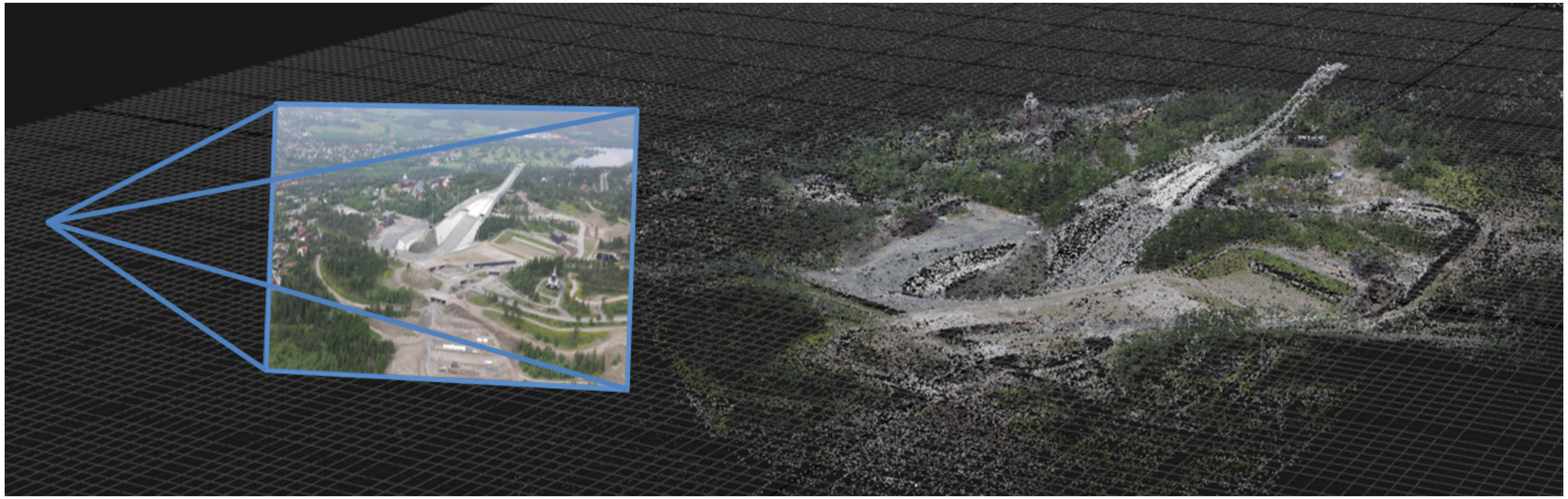
How can we track a map with a camera?



Pose from known 3D surface

Minimize **photometric error**

$$\mathbf{T}_{cw}^{*} = \underset{\mathbf{T}_{cw}}{\operatorname{argmin}} \sum_{i} \left\| I_c\left(\pi(\mathbf{T}_{cw}\tilde{\mathbf{x}}_i^w)\right) - I_i \right\|^2$$



Pose from known 3D surface

Minimize **photometric error** (**direct tracking**)

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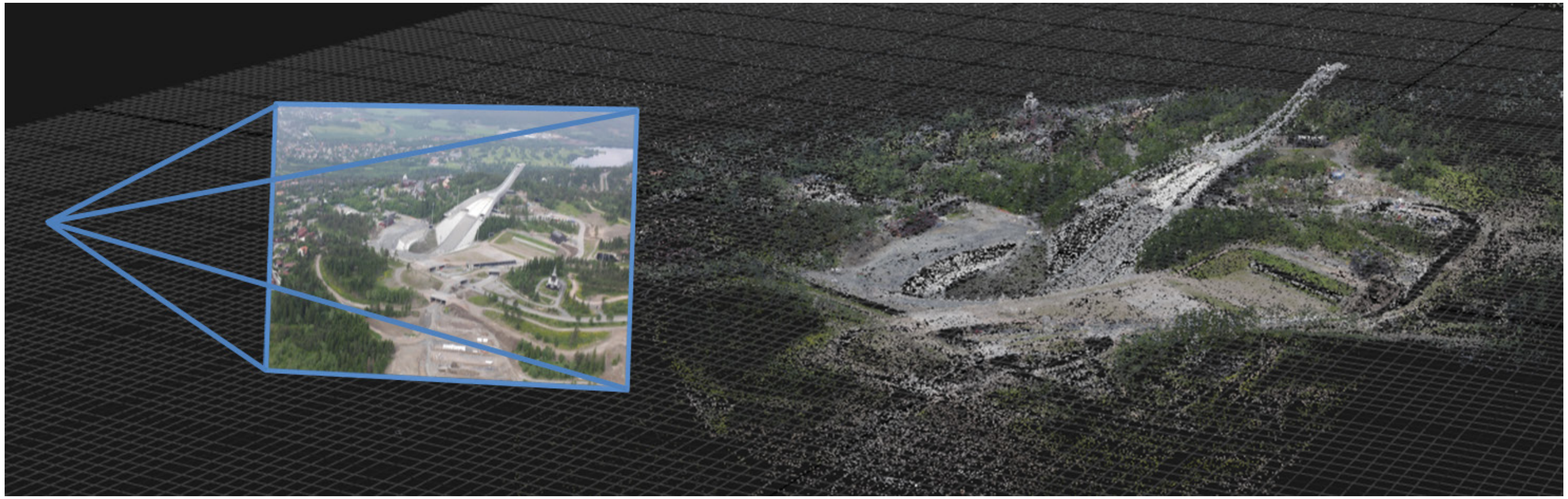
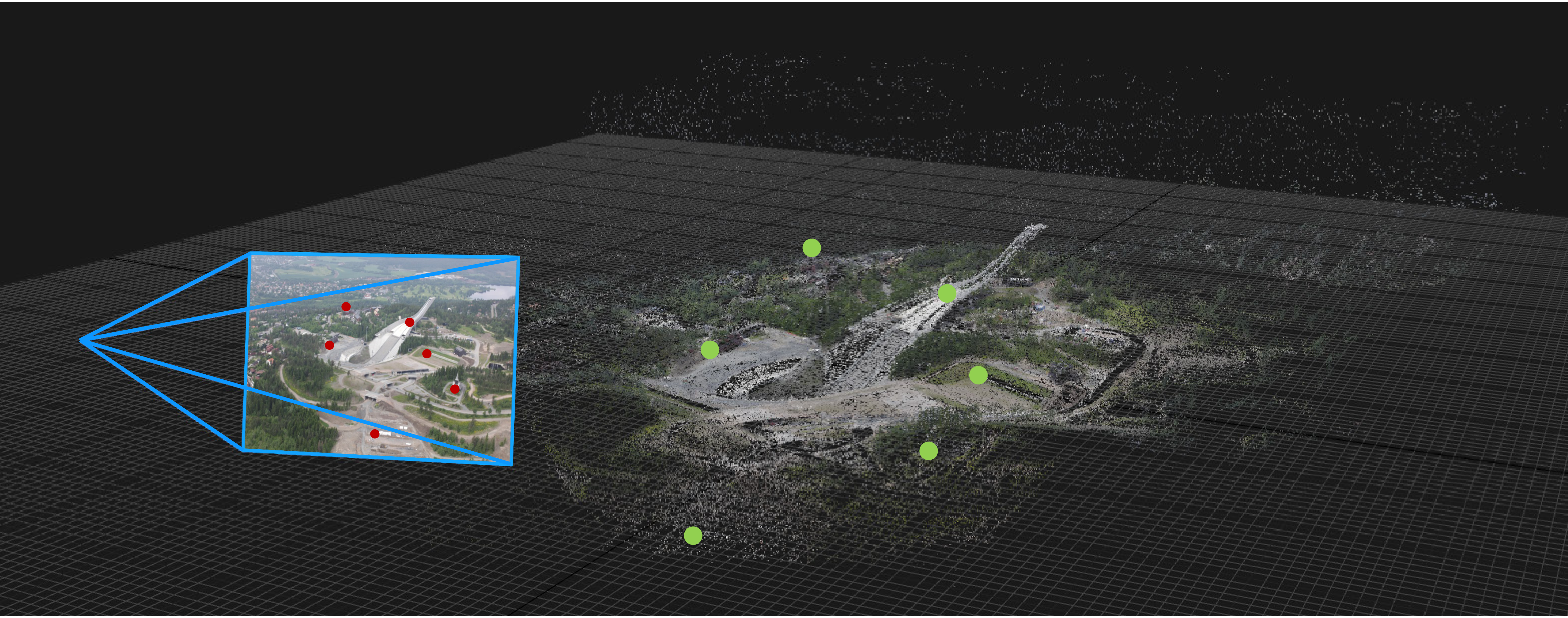


Pose from known 3D points



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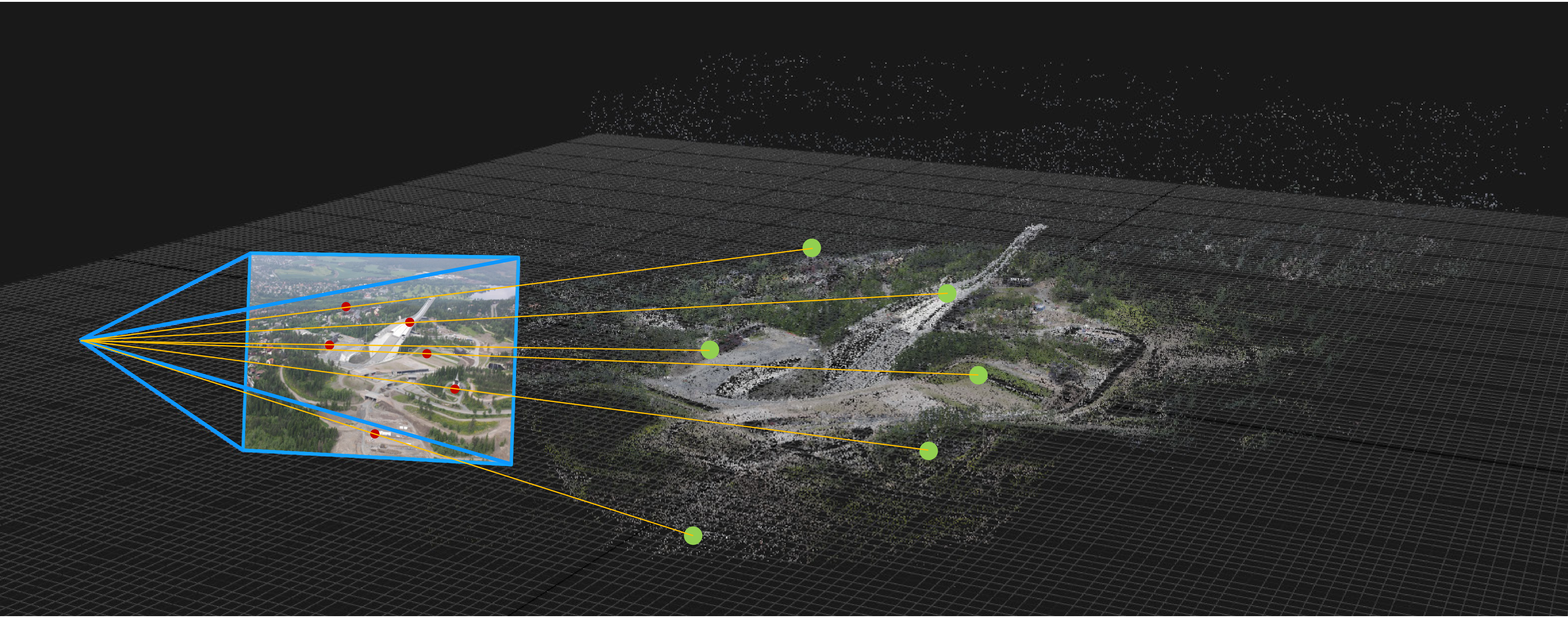
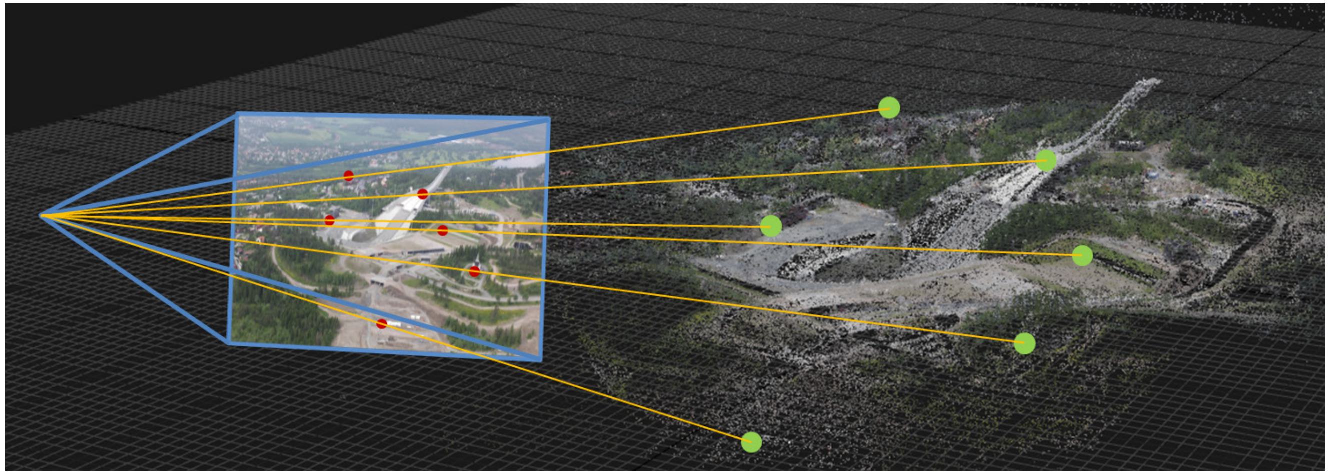


Pose from known 3D points

Minimize **geometric error**

$$\mathbf{T}_{cw}^{*} = \underset{\mathbf{T}_{cw}}{\operatorname{argmin}} \sum_{i} \left\| \pi(\mathbf{T}_{cw} \tilde{\mathbf{x}}_{i}^{w}) - \mathbf{u}_{i} \right\|^{2}$$



Pose from known 3D points

Minimize **geometric error** (**indirect tracking**)

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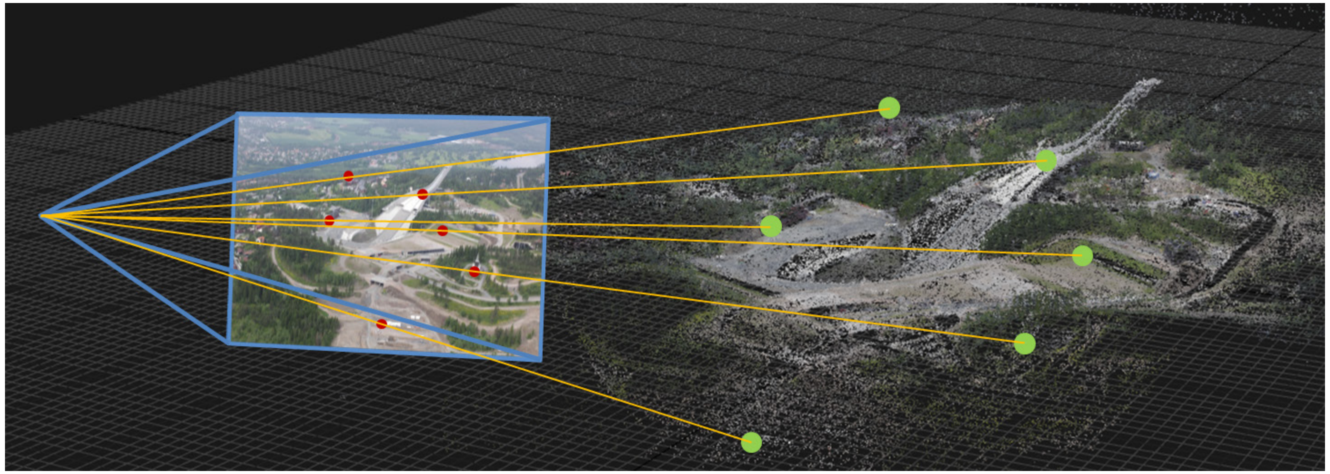


Pose estimation

- We need a few more tools to be able to solve

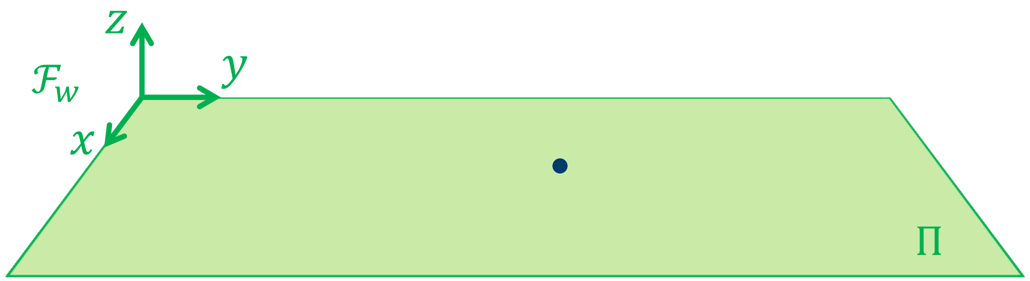
$$\mathbf{T}_{cw}^{*} = \underset{\mathbf{T}_{cw}}{\operatorname{argmin}} \sum_{i} \left\| \pi(\mathbf{T}_{cw}\tilde{\mathbf{x}}_i^w) - \mathbf{u}_i \right\|^2$$

- But we are ready to track a world plane!

Pose estimation relative to a world plane

Choose the world coordinate system so that the xy-plane corresponds to a plane Π in the scene

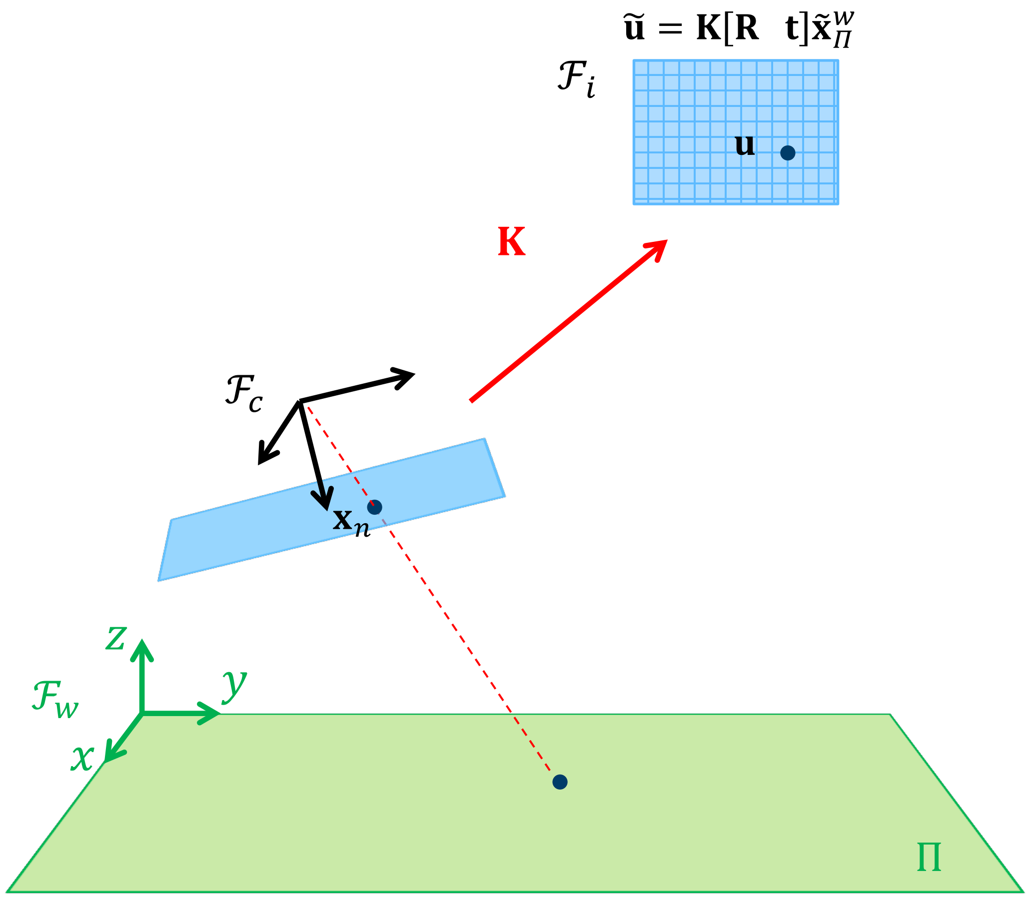
$$\mathbf{x}_{\Pi}^{w} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \mathbf{x}^{\Pi} = \begin{bmatrix} x \\ y \end{bmatrix}$$



Pose estimation relative to a world plane

We can map points on the world plane into image coordinates by using the perspective camera model

$$\tilde{\mathbf{u}} = \mathbf{K}\left[\mathbf{R}\ \mathbf{t}\right]\tilde{\mathbf{x}}_{\Pi}^{w} \qquad \mathbf{T}_{cw} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$$



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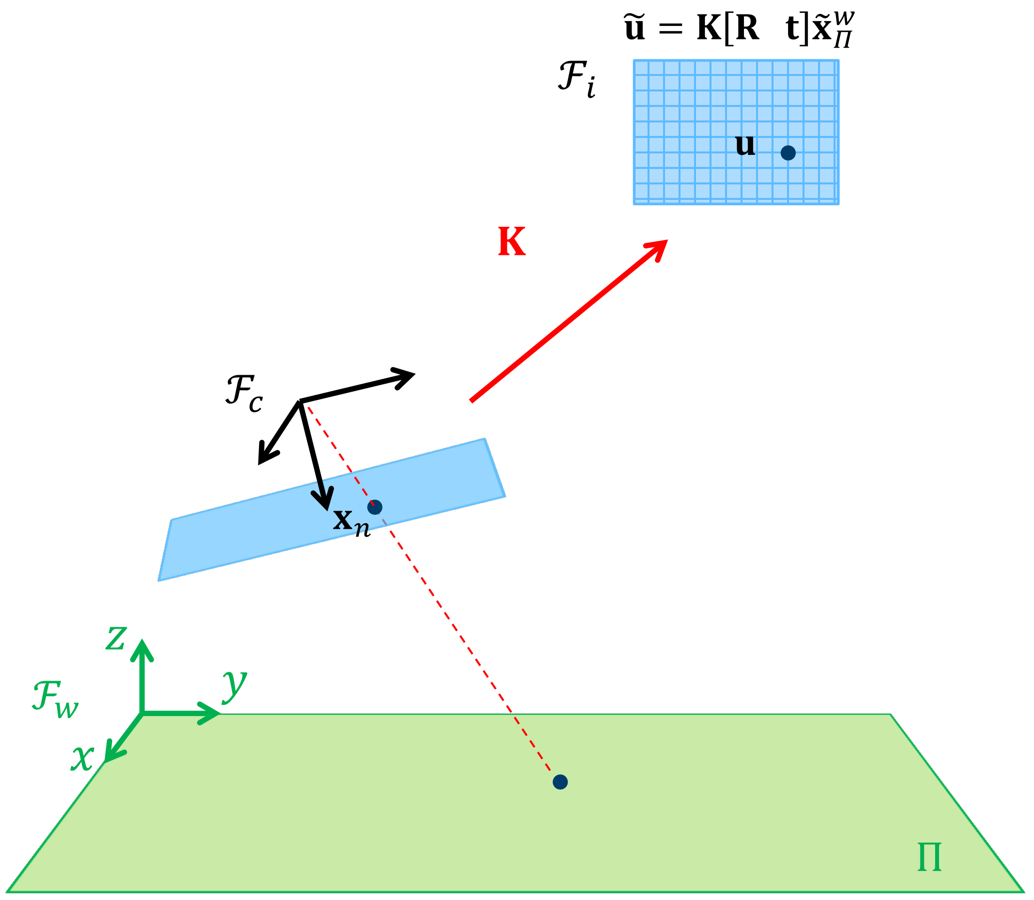
$$= \mathbf{K}\left[\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{t}\right]\begin{bmatrix} x \\ y \\ 0 \\ 1 \end{bmatrix}$$



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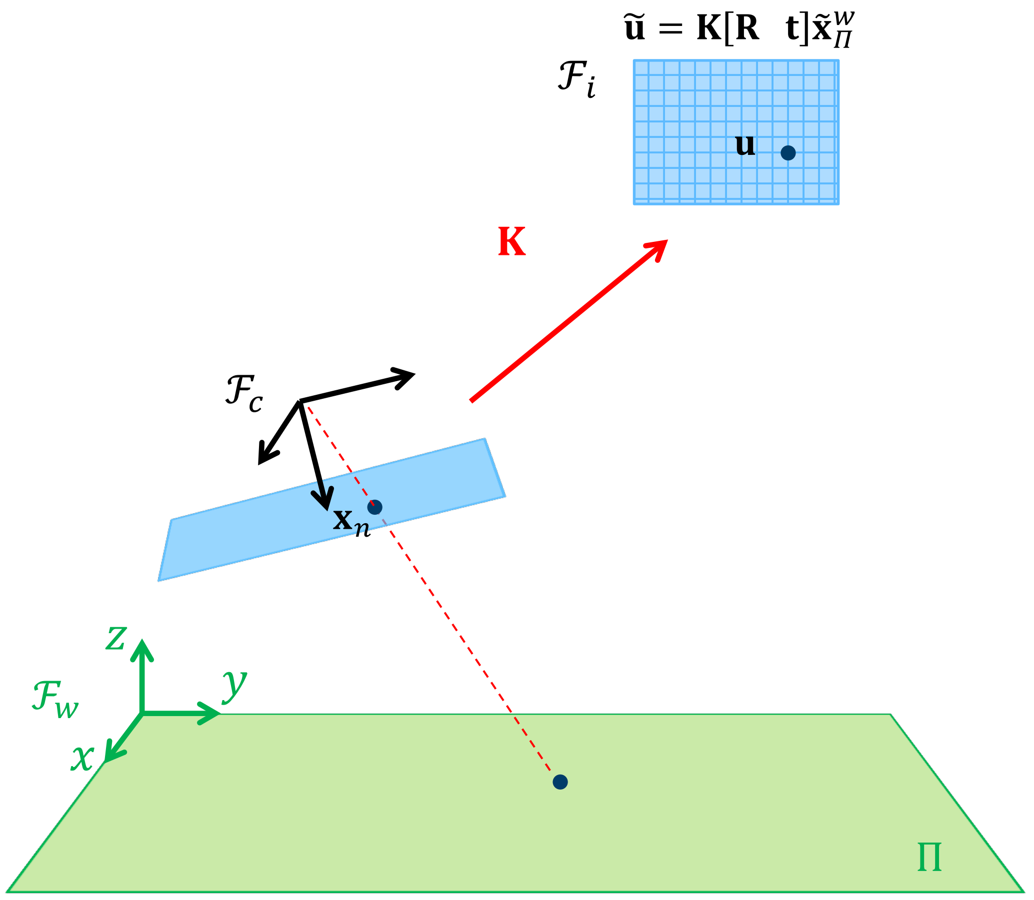

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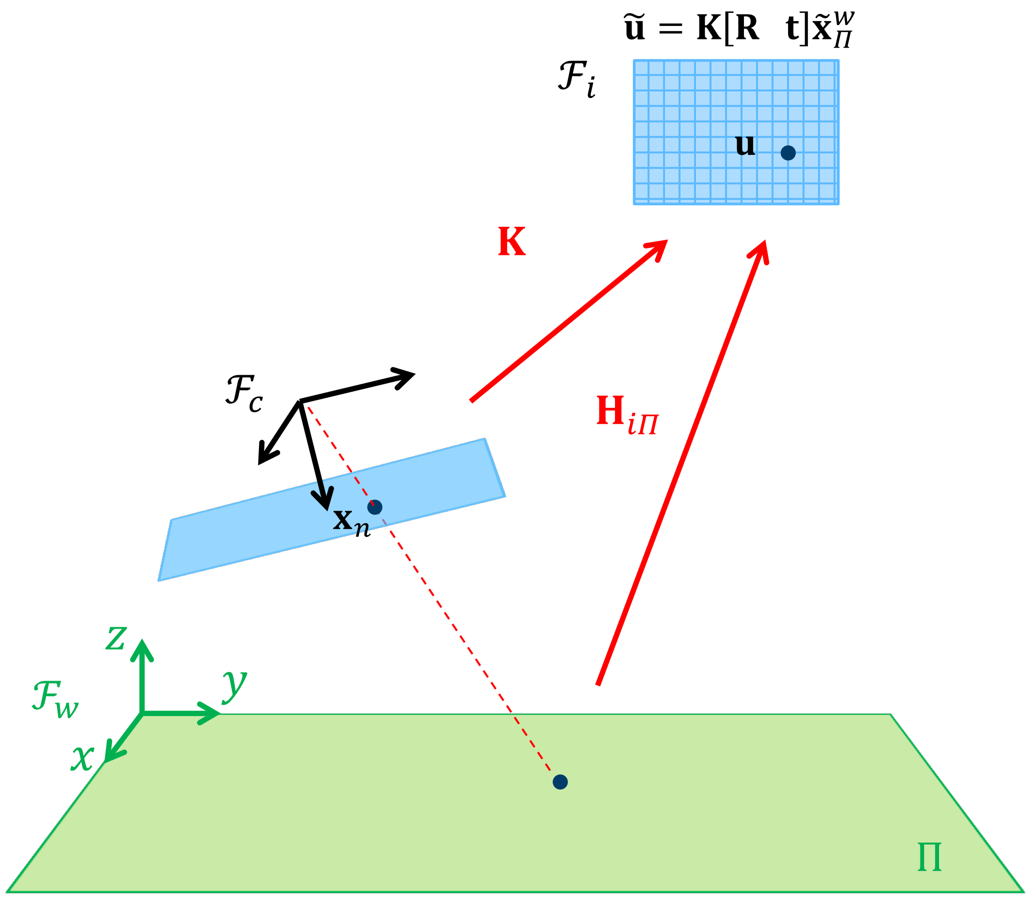

Pose estimation relative to a world plane

⇒ For a calibrated camera,
we have a relation between the camera pose
and the homography between the world plane and the image!

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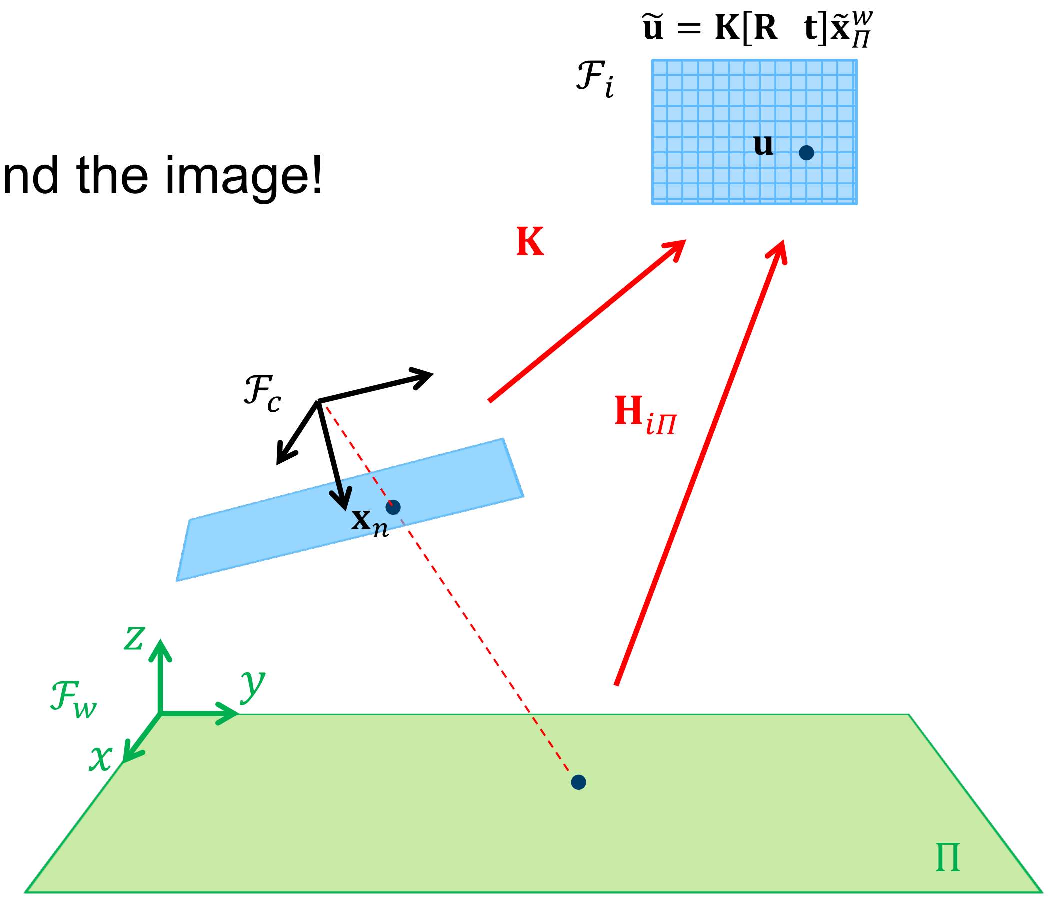

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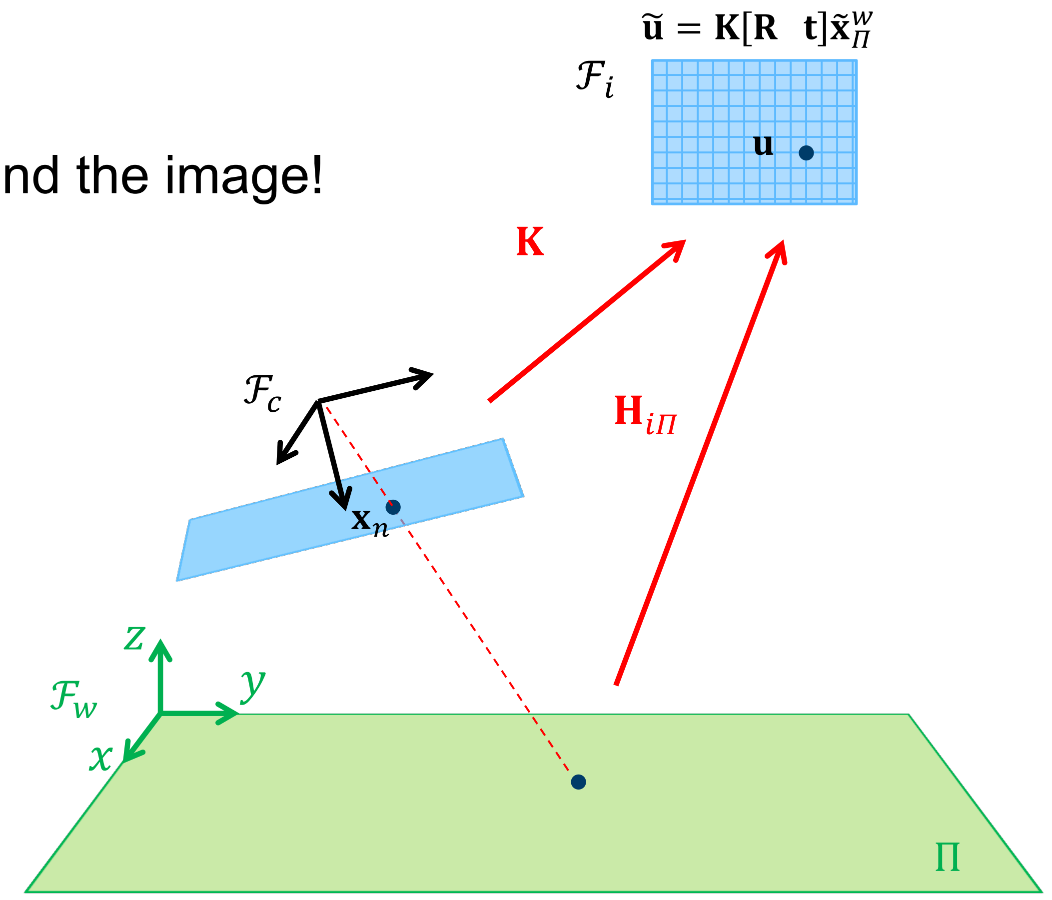
How can we use this to
estimate camera pose
given such a homography?



Pose estimation relative to a world plane

Assume a perfect, noise-free homography between the world plane and the image:

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Then, because of scale ambiguity:

$$\left[\mathbf{r}_1, \mathbf{r}_2, \mathbf{t}\right] \sim \mathbf{K}^{-1}\mathbf{H}_{i\Pi} = \mathbf{M}$$

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Since the columns of rotation matrices have unit norm,
we can also find a scale factor λ so that the first two columns of $\mathbf{M}$ get unit norm.
We then have the two possible solutions:

$$\left[\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, \hat{\mathbf{t}}\right] = \pm\lambda\mathbf{M}$$

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The last column in $\hat{\mathbf{R}}$ is given by the cross product of the two first columns:

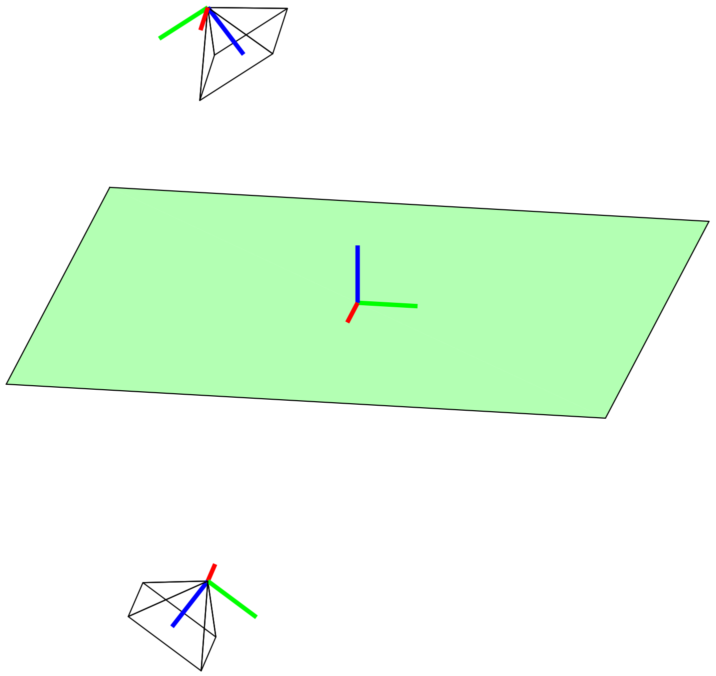
$\hat{\mathbf{r}}_3 = \pm\left(\hat{\mathbf{r}}_1 \times \hat{\mathbf{r}}_2\right)$, where the sign is chosen so that $\det\left(\hat{\mathbf{R}}\right) = 1$

Pose estimation relative to a world plane

We are now able to reconstruct the camera pose in the world coordinate system for each of the two solutions:

$$\hat{\mathbf{T}}_{wc} = \hat{\mathbf{T}}_{cw}^{-1} = \begin{bmatrix} \hat{\mathbf{R}}^T & -\hat{\mathbf{R}}^T\hat{\mathbf{t}} \\ \mathbf{0} & 1 \end{bmatrix}$$

where $\hat{\mathbf{R}} = \left[\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2, \hat{\mathbf{r}}_3\right]$



It is in practice simple find the correct solution because only one side of the plane is typically visible

Pose estimation relative to a world plane

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- But it is possible to find the closest rotation matrix
  (in the Frobenius-norm sense)
  with SVD!

$$\hat{\mathbf{R}} \rightarrow \hat{\mathbf{R}}^* \in SO(3)$$

Pose estimation relative to a world plane

Let $\bar{\mathbf{M}}$ be the matrix with the two first columns of $\mathbf{M}$:

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The first two columns $\bar{\mathbf{R}}^*$ of a proper $\hat{\mathbf{R}}^*$ is then

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The corresponding scale λ can be computed as:

$$\lambda = \frac{\operatorname{trace}\left(\bar{\mathbf{R}}^{*T}\bar{\mathbf{M}}\right)}{\operatorname{trace}\left(\bar{\mathbf{M}}^T\bar{\mathbf{M}}\right)} = \frac{\sum_{i=1}^{3}\sum_{j=1}^{2} r_{ij}^* m_{ij}}{\sum_{i=1}^{3}\sum_{j=1}^{2} m_{ij}^2}$$

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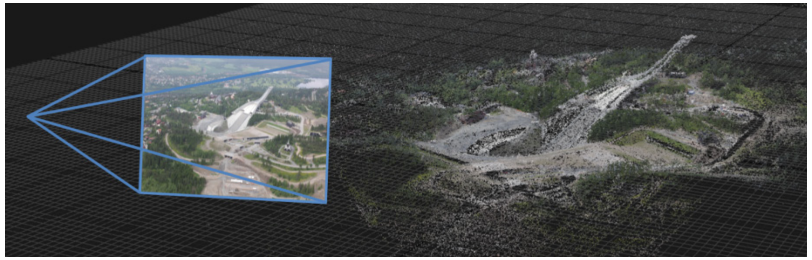
The corresponding pose with ambiguity can then be found as before

Summary

3D-2D pose estimation:

- Direct methods based on minimizing photometric error

$$\mathbf{T}_{cw}^{*} = \underset{\mathbf{T}_{cw}}{\operatorname{argmin}} \sum_{i} \left\| I_c\left(\pi(\mathbf{T}_{cw}\tilde{\mathbf{x}}_i^w)\right) - I_i \right\|^2$$



- Indirect methods based on minimizing geometric error

$$\mathbf{T}_{cw}^{*} = \underset{\mathbf{T}_{cw}}{\operatorname{argmin}} \sum_{i} \left\| \pi(\mathbf{T}_{cw}\tilde{\mathbf{x}}_i^w) - \mathbf{u}_i \right\|^2$$

- Homography-based method