UiO : Department of Technology Systems
University of Oslo

Lecture 6.3
Optimizing over poses

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Nonlinear state estimation

We have seen how we can find the MAP estimate of our unknown states given measurements

$$X^{MAP} = \underset{X}{\operatorname{argmax}}\ p(X \mid Z)$$

by representing it as
a nonlinear least squares problem

$$X^{*} = \underset{X}{\operatorname{argmin}} \sum_{i=1}^{m} \left\| h_i(X_i) - \mathbf{z}_i \right\|_{\boldsymbol{\Sigma}_i}^{2}$$

$\text{Choose a suitable inital estimate } X^{0}$

⇩

$\mathbf{A}, \mathbf{b} \leftarrow \text{Linearize at } X^{t}$

⇩

$\boldsymbol{\Delta}^{*} \leftarrow Solve\ \underset{\boldsymbol{\Delta}}{\operatorname{argmin}} \left\| \mathbf{A}\boldsymbol{\Delta} - \mathbf{b} \right\|^{2}$

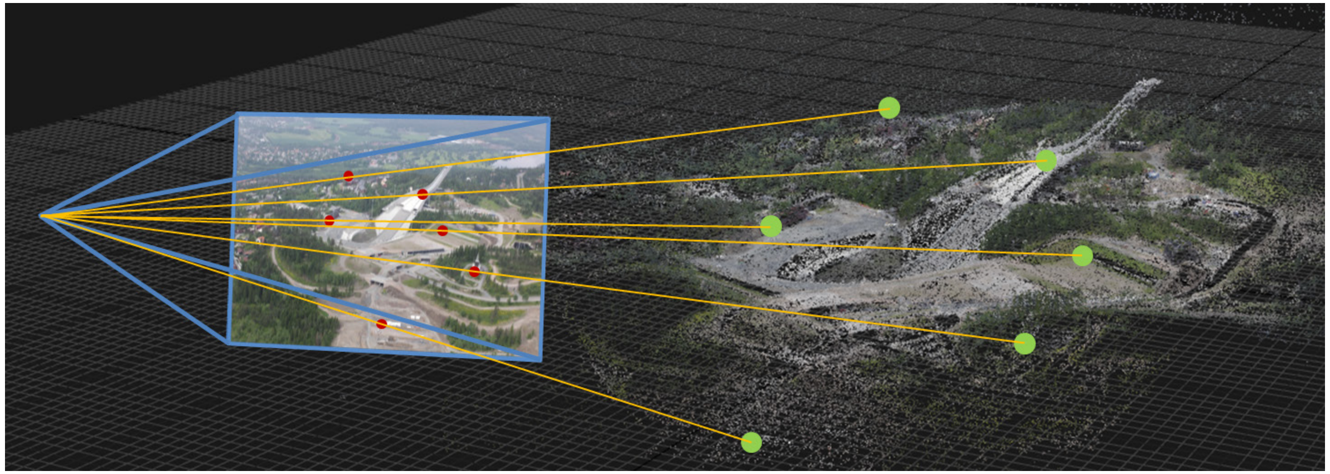
⇩

$X^{t+1} \leftarrow X^{t} + \boldsymbol{\Delta}^{*}$

The indirect tracking method

Minimize **geometric error** over the **camera pose**

$$\mathbf{T}_{cw}^{*} = \underset{\mathbf{T}_{cw}}{\operatorname{argmin}} \sum_{i} \left\| \pi(\mathbf{T}_{cw} \tilde{\mathbf{x}}_{i}^{w}) - \mathbf{u}_{i} \right\|^{2}$$



Rotations and poses are Lie groups

Rotations in 3D:

$$SO(3) = \{\mathbf{R} \in \mathbb{R}^{3\times 3} \mid \mathbf{R}\mathbf{R}^T = \mathbf{1}, \det \mathbf{R} = 1\}$$

Poses in 3D:

$$SE(3) = \left\{ \mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} \;\middle|\; \mathbf{R} = SO(3),\ \mathbf{t} \in \mathbb{R}^3 \right\}$$

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Rotations and poses are not vector spaces!

(They lie on manifolds)

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How do we optimize?

Choose a suitable inital estimate X^{0}

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The corresponding Lie algebra

Rotations in 3D:

$$\mathfrak{so}(3) = \left\{ \boldsymbol{\Omega} = \boldsymbol{\omega}^\wedge \in \mathbb{R}^{3\times 3} \;\middle|\; \boldsymbol{\omega} \in \mathbb{R}^3 \right\}$$

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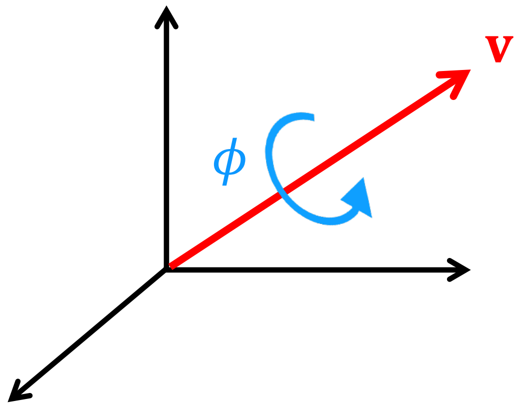
Remember the axis-angle representation:

$$\mathbf{R}_{ab} = \cos\phi \mathbf{I} + (1 - \cos\phi)\mathbf{v}\mathbf{v}^T + \sin\phi \mathbf{v}^\wedge$$



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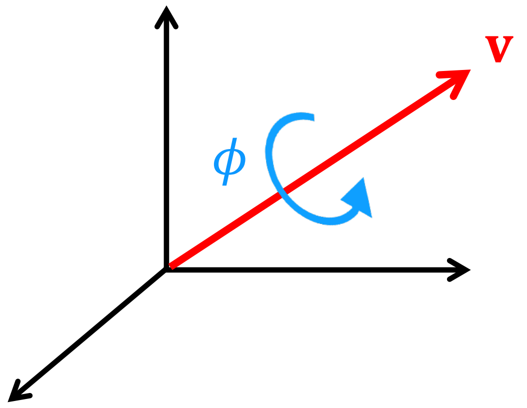
When ϕ is small:
$\cos(\phi) \approx 1$
$\sin(\phi) \approx \phi$



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Remember the axis-angle representation:

$$\mathbf{R}_{ab} = \cos\phi \mathbf{I} + (1 - \cos\phi)\mathbf{v}\mathbf{v}^T + \sin\phi \mathbf{v}^\wedge$$
$$\approx \mathbf{I} + \phi \mathbf{v}^\wedge = \mathbf{I} + \boldsymbol{\omega}^\wedge$$

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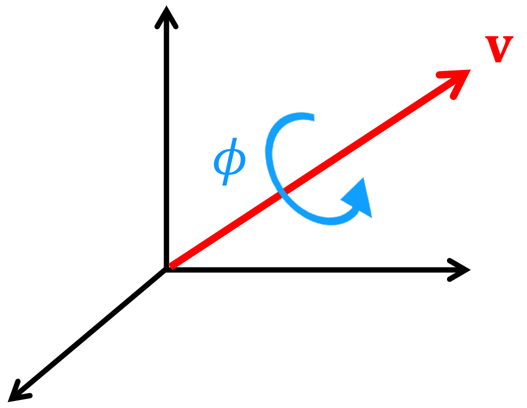

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Poses in 3D:

$$\mathfrak{se}(3) = \left\{ \boldsymbol{\Xi} = \boldsymbol{\xi}^\wedge \in \mathbb{R}^{4\times 4} \;\middle|\; \boldsymbol{\xi} \in \mathbb{R}^6 \right\}$$

$$\boldsymbol{\xi}^\wedge = \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\omega} \end{bmatrix}^\wedge = \begin{bmatrix} \boldsymbol{\omega}^\wedge & \mathbf{v} \\ \mathbf{0}^T & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}, \;\; \mathbf{v}, \boldsymbol{\omega} \in \mathbb{R}^3$$

The corresponding Lie algebra

The corresponding Lie algebras are vector spaces!

Rotations in 3D:

$$\mathfrak{so}(3) = \{\boldsymbol{\Omega} = \boldsymbol{\omega}^\wedge \in \mathbb{R}^{3\times 3} \mid \boldsymbol{\omega} \in \mathbb{R}^3\}$$

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Relation between group and algebra

We can relate the group and algebra through the **matrix exponential** and **matrix logarithm**

$$\exp : \mathfrak{so}(3) \mapsto SO(3)$$
$$\boldsymbol{\omega} \mapsto \mathbf{R}$$

$$\mathbf{R} = \exp(\boldsymbol{\omega}^\wedge) = \mathbf{I} + \frac{1-\cos\phi}{\phi^2}(\boldsymbol{\omega}^\wedge)^2 + \frac{\sin\phi}{\phi}\boldsymbol{\omega}^\wedge$$
$$\phi = |\boldsymbol{\omega}|$$

$$\log : SO(3) \mapsto \mathfrak{so}(3)$$
$$\mathbf{R} \mapsto \boldsymbol{\omega}$$

$$\log(\mathbf{R}) = \frac{\phi}{2\sin\phi}\left(\mathbf{R} - \mathbf{R}^T\right)$$
$$\phi = \arccos\frac{\mathrm{tr}(\mathbf{R}) - 1}{2}$$
$$\boldsymbol{\omega} = \log(\mathbf{R})^\vee$$

Relation between group and algebra

We can relate the group and algebra through the **matrix exponential** and **matrix logarithm**

$$\begin{aligned}\exp : \mathfrak{se}(3) &\mapsto SE(3)\\ \boldsymbol{\xi} &\mapsto \mathbf{T}\end{aligned}$$

$$\mathbf{T} = \exp(\boldsymbol{\xi}^\wedge) = \mathbf{I} + \boldsymbol{\xi}^\wedge + \frac{1-\cos\phi}{\phi^2}(\boldsymbol{\xi}^\wedge)^2 + \frac{\phi - \sin\phi}{\phi^3}(\boldsymbol{\xi}^\wedge)^3$$

$$\phi = |\boldsymbol{\omega}|$$

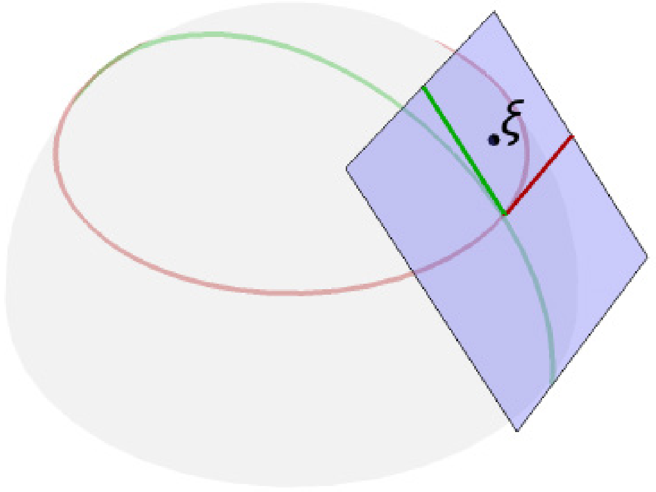
$$\begin{aligned}\log : SE(3) &\mapsto \mathfrak{se}(3)\\ \mathbf{T} &\mapsto \boldsymbol{\xi}\end{aligned}$$

$$\boldsymbol{\xi} = \log(\mathbf{T})^\vee = \begin{bmatrix}\mathbf{V}^{-1}\mathbf{v}\\ \log(\mathbf{R})^\vee\end{bmatrix}$$

$$\mathbf{V}^{-1} = \mathbf{I} - \frac{1}{2}\boldsymbol{\omega}^\wedge + \frac{\left(1 - \frac{\phi\cos(\phi/2)}{2\sin(\phi/2)}\right)}{\phi^2}(\boldsymbol{\omega}^\wedge)^2$$

Tangent space

The Lie algebra is the tangent space
around the identity element of the group



Dellaert, F., & Kaess, M. (2017). Factor Graphs for Robot Perception

- The tangent space is the “optimal” space
  in which to represent differential quantities related to the group

- The tangent space is a vector space
  with the same dimension as the number of degrees of freedom of the group transformations

Perturbations

We can represent steps and uncertainty as perturbations in the tangent space

$$\mathbf{R} = \exp(\boldsymbol{\omega}^{\wedge})\bar{\mathbf{R}}$$

$$\mathbf{T} = \exp(\boldsymbol{\xi}^{\wedge})\bar{\mathbf{T}}$$

Jacobians for perturbations on SO(3)

Group action on points: $\mathbf{R} \oplus \mathbf{x} = \mathbf{R}\mathbf{x}$

$$\frac{\partial \left(\exp(\boldsymbol{\omega}^\wedge)\mathbf{R} \right) \oplus \mathbf{x}}{\partial \mathbf{x}} = \frac{\partial \mathbf{R} \oplus \mathbf{x}}{\partial \mathbf{x}} = \mathbf{R}$$

$$\left. \frac{\partial \left(\exp(\boldsymbol{\omega}^\wedge)\mathbf{R} \right) \oplus \mathbf{x}}{\partial \boldsymbol{\omega}} \right|_{\boldsymbol{\omega}=\mathbf{0}} = -[\mathbf{R} \oplus \mathbf{x}]^\wedge$$

Jacobians for perturbations on SE(3)

Group action on points: $\mathbf{T} \oplus \mathbf{x} = \mathbf{R}\mathbf{x} + \mathbf{t}$

$$\frac{\partial \left(\exp(\boldsymbol{\xi}^{\wedge}) \mathbf{T} \right) \oplus \mathbf{x}}{\partial \mathbf{x}} = \frac{\partial \mathbf{T} \oplus \mathbf{x}}{\partial \mathbf{x}} = \mathbf{R}$$

$$\left. \frac{\partial \left(\exp(\boldsymbol{\xi}^{\wedge}) \mathbf{T} \right) \oplus \mathbf{x}}{\partial \boldsymbol{\xi}} \right|_{\boldsymbol{\xi} = \mathbf{0}} = \begin{bmatrix} \mathbf{I}_{3\times3} & -[\mathbf{T} \oplus \mathbf{x}]^{\wedge} \end{bmatrix}$$

Summary

- Updates on rotations and poses as perturbations using Lie algebra

$$\mathbf{R} = \exp(\boldsymbol{\omega}^{\wedge})\bar{\mathbf{R}}$$

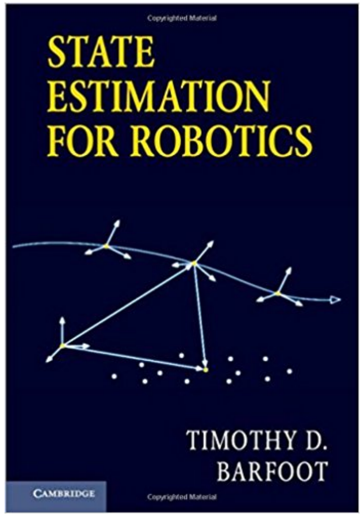
$$\mathbf{T} = \exp(\boldsymbol{\xi}^{\wedge})\bar{\mathbf{T}}$$

- Jacobians for these perturbations

- We are ready to solve

$$\mathbf{T}_{cw}^{*} = \underset{\mathbf{T}_{cw}}{\operatorname{argmin}} \sum_{i} \left\| \pi(\mathbf{T}_{cw}\tilde{\mathbf{x}}_{i}^{w}) - \mathbf{u}_{i} \right\|^{2}$$

Supplementary material



Chapter 7

- Ethan Eade, "Lie Groups for 2D and 3D transformations"
- José Luis Blanco Claraco, "A tutorial on SE(3) transformation parameterizations and on-manifold optimization"