

Lecture 6.4

Nonlinear pose estimation

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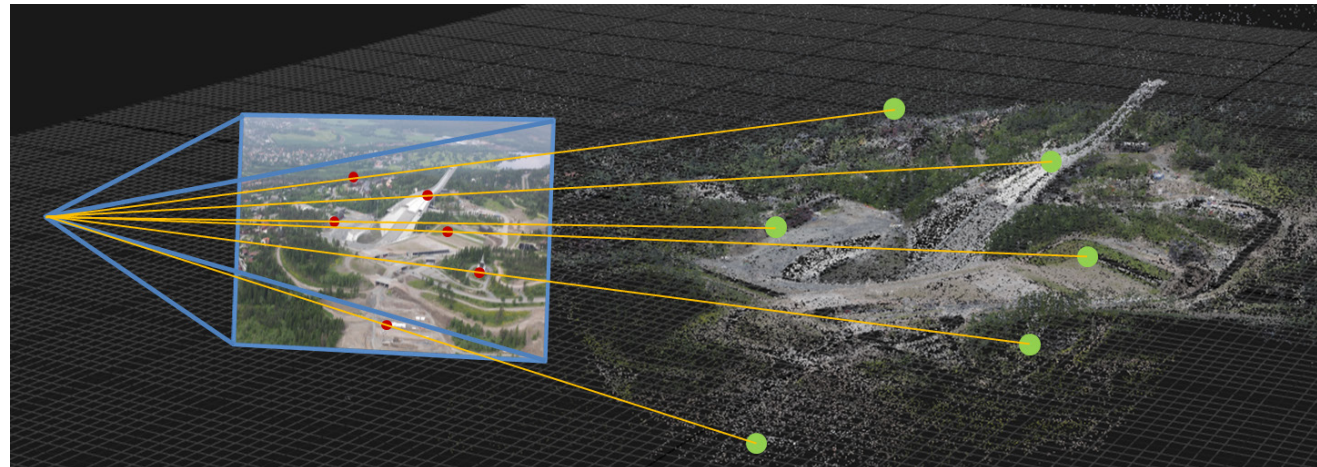


The indirect tracking method

Minimize **geometric error** over the **camera pose**

This is also sometimes called **Motion-Only Bundle Adjustment**

$$\mathbf{T}_{cw}^* = \operatorname{argmin}_{\mathbf{T}_{cw}} \sum_i \left\| \pi(\mathbf{T}_{cw} \tilde{\mathbf{x}}_i^w) - \mathbf{u}_i \right\|^2$$



Nonlinear state estimation

We have seen how we can find the MAP estimate of our unknown states given measurements

$$X^{MAP} = \operatorname{argmax}_X p(X | Z)$$

by representing it as a nonlinear least squares problem

$$X^* = \operatorname{argmin}_X \sum_{i=1}^m \|h_i(X_i) - \mathbf{z}_i\|_{\Sigma_i}^2$$

Choose a suitable initial estimate X^0



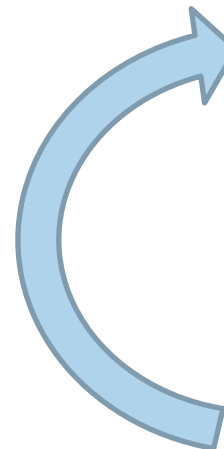
$\mathbf{A}, \mathbf{b} \leftarrow$ Linearize at X^t



$\Delta^* \leftarrow$ Solve $\operatorname{argmin}_{\Delta} \|\mathbf{A}\Delta - \mathbf{b}\|^2$



$X^{t+1} \leftarrow X^t + \Delta^*$



Objective function

Minimize error over the **state variable** $X = \mathbf{T}_{wc}$

The optimization problem is

$$X^* = \operatorname{argmin}_X \sum_j \left\| \pi(g(\mathbf{T}_{wc}, \mathbf{x}_j^w)) - \mathbf{x}_{n,j} \right\|_{\Sigma_j}^2$$

For simpler notation,
we assume that the measurements are pre-calibrated to normalized image coordinates

Objective function

Minimize error over the **state variable** $X = \mathbf{T}_{wc}$

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For simpler notation,
we assume that the measurements are pre-calibrated to normalized image coordinates

$$\mathbf{x}_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{K}^{-1} \begin{bmatrix} \mathbf{u} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{u - c_u}{f_u} \\ \frac{v - c_v}{f_v} \end{bmatrix} \quad (\text{and distortion...})$$

Measurement prediction

This gives us the **measurement prediction function**

$$\hat{\mathbf{x}}_n = h(\mathbf{T}_{wc}; \mathbf{x}^w) = \pi(g(\mathbf{T}_{wc}, \mathbf{x}^w))$$

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where

$$g(\mathbf{T}_{wc}, \mathbf{x}^w) = \mathbf{R}_{wc}^T (\mathbf{x}^w - \mathbf{t}_{wc}^w) = \begin{bmatrix} x^c \\ y^c \\ z^c \end{bmatrix} = \mathbf{x}^c \quad \text{(Coordinate transformation)}$$

$$\pi(\mathbf{x}^c) = \frac{1}{z^c} \begin{bmatrix} x^c \\ y^c \end{bmatrix} = \begin{bmatrix} \hat{x}_n \\ \hat{y}_n \end{bmatrix} = \hat{\mathbf{x}}_n \quad \text{(Camera model)}$$

Linearization

We can **linearize** the measurement prediction function with a local Taylor expansion

$$h(\mathbf{T}_{wc} \exp(\hat{\xi}_{\Delta}); \mathbf{x}^w) \approx h(\mathbf{T}_{wc}; \mathbf{x}^w) + \mathbf{F} \xi_{\Delta}$$

where $\exp(\hat{\xi}_{\Delta})$ is a small perturbation in the camera frame.
The **measurement Jacobian** is given by

$$\mathbf{F} = \left. \frac{\partial h(\mathbf{T}_{wc} \exp(\hat{\xi}); \mathbf{x}^w)}{\partial \xi} \right|_{\xi=0} = \left. \frac{\partial \pi(\mathbf{x}^c)}{\partial \mathbf{x}^c} \right|_{\mathbf{x}^c = g(\mathbf{T}_{wc}, \mathbf{x}^w)} \left. \frac{\partial g(\mathbf{T}_{wc} \exp(\hat{\xi}), \mathbf{x}^w)}{\partial \xi} \right|_{\xi=0}$$

Jacobians

$$\left. \frac{\partial g(\mathbf{T}_{wc} \exp(\hat{\xi}), \mathbf{x}^w)}{\partial \xi} \right|_{\xi=0}$$

$$g(\mathbf{T}_{wc}, \mathbf{x}^w) = \mathbf{R}_{wc}^T (\mathbf{x}^w - \mathbf{t}_{wc}^w) = \mathbf{x}^c$$

Jacobians

$$\left. \frac{\partial g(\mathbf{T}_{wc} \exp(\hat{\xi}), \mathbf{x}^w)}{\partial \hat{\xi}} \right|_{\hat{\xi}=0} = \left. \frac{\partial (\mathbf{T}_{wc} \exp(\hat{\xi}))^{-1} \oplus \mathbf{x}^w}{\partial \hat{\xi}} \right|_{\hat{\xi}=0}$$

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Jacobians

$$\begin{aligned} \frac{\partial g(\mathbf{T}_{wc} \exp(\hat{\xi}), \mathbf{x}^w)}{\partial \hat{\xi}} \Big|_{\hat{\xi}=0} &= \frac{\partial (\mathbf{T}_{wc} \exp(\hat{\xi}))^{-1} \oplus \mathbf{x}^w}{\partial \hat{\xi}} \Big|_{\hat{\xi}=0} \\ &= \frac{\partial (\exp(-\hat{\xi}) \mathbf{T}_{wc}^{-1}) \oplus \mathbf{x}^w}{\partial \hat{\xi}} \Big|_{\hat{\xi}=0} \end{aligned}$$

Jacobians

$$\begin{aligned} \frac{\partial g(\mathbf{T}_{wc} \exp(\hat{\xi}), \mathbf{x}^w)}{\partial \hat{\xi}} \Big|_{\hat{\xi}=0} &= \frac{\partial (\mathbf{T}_{wc} \exp(\hat{\xi}))^{-1} \oplus \mathbf{x}^w}{\partial \hat{\xi}} \Big|_{\hat{\xi}=0} \\ &= \frac{\partial (\exp(-\hat{\xi}) \mathbf{T}_{wc}^{-1}) \oplus \mathbf{x}^w}{\partial \hat{\xi}} \Big|_{\hat{\xi}=0} \\ &= - \frac{\partial (\exp(\hat{\xi}) \mathbf{T}_{cw}) \oplus \mathbf{x}^w}{\partial \hat{\xi}} \Big|_{\hat{\xi}=0} \end{aligned}$$

Jacobians

$$\begin{aligned}
 \left. \frac{\partial g(\mathbf{T}_{wc} \exp(\hat{\xi}), \mathbf{x}^w)}{\partial \hat{\xi}} \right|_{\hat{\xi}=0} &= \left. \frac{\partial (\mathbf{T}_{wc} \exp(\hat{\xi}))^{-1} \oplus \mathbf{x}^w}{\partial \hat{\xi}} \right|_{\hat{\xi}=0} \\
 &= \left. \frac{\partial (\exp(-\hat{\xi}) \mathbf{T}_{wc}^{-1}) \oplus \mathbf{x}^w}{\partial \hat{\xi}} \right|_{\hat{\xi}=0} \\
 &= - \left. \frac{\partial (\exp(\hat{\xi}) \mathbf{T}_{cw}) \oplus \mathbf{x}^w}{\partial \hat{\xi}} \right|_{\hat{\xi}=0} \\
 &= - \begin{bmatrix} \mathbf{I}_{3 \times 3} & -[\mathbf{T}_{cw} \oplus \mathbf{x}^w]^\wedge \end{bmatrix}
 \end{aligned}$$

Jacobians

$$\begin{aligned}
 \left. \frac{\partial g(\mathbf{T}_{wc} \exp(\hat{\xi}), \mathbf{x}^w)}{\partial \hat{\xi}} \right|_{\hat{\xi}=0} &= \left. \frac{\partial (\mathbf{T}_{wc} \exp(\hat{\xi}))^{-1} \oplus \mathbf{x}^w}{\partial \hat{\xi}} \right|_{\hat{\xi}=0} \\
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 &= - \begin{bmatrix} \mathbf{I}_{3 \times 3} & -[\mathbf{T}_{cw} \oplus \mathbf{x}^w]^\wedge \end{bmatrix} \\
 &= \begin{bmatrix} -\mathbf{I}_{3 \times 3} & [\mathbf{x}^c]^\wedge \end{bmatrix}
 \end{aligned}$$

Jacobians

$$\left. \frac{\partial \pi(\mathbf{x}^c)}{\partial \mathbf{x}^c} \right|_{\mathbf{x}^c = g(\mathbf{T}_{wc}, \mathbf{x}^w)}$$

$$\pi(\mathbf{x}^c) = \frac{1}{z^c} \begin{bmatrix} x^c \\ y^c \end{bmatrix}$$

Jacobians

$$\frac{\partial \pi(\mathbf{x}^c)}{\partial \mathbf{x}^c} \bigg|_{\mathbf{x}^c = g(\mathbf{T}_{wc}, \mathbf{x}^w)} = \frac{1}{z^c} \begin{bmatrix} 1 & 0 & -x^c/z^c \\ 0 & 1 & -y^c/z^c \end{bmatrix}$$

$$\pi(\mathbf{x}^c) = \frac{1}{z^c} \begin{bmatrix} x^c \\ y^c \end{bmatrix}$$

Jacobians

$$\begin{aligned}\frac{\partial \pi(\mathbf{x}^c)}{\partial \mathbf{x}^c} \bigg|_{\mathbf{x}^c = g(\mathbf{T}_{wc}, \mathbf{x}^w)} &= \frac{1}{z^c} \begin{bmatrix} 1 & 0 & -x^c/z^c \\ 0 & 1 & -y^c/z^c \end{bmatrix} \\ &= d \begin{bmatrix} 1 & 0 & -x_n \\ 0 & 1 & -y_n \end{bmatrix}\end{aligned}$$

$$d = \frac{1}{z^c}$$

Jacobians

$$\mathbf{F} = \left. \frac{\partial h(\mathbf{T}_{wc} \exp(\hat{\xi}); \mathbf{x}^w)}{\partial \hat{\xi}} \right|_{\hat{\xi}=0} = \left. \frac{\partial \pi(\mathbf{x}^c)}{\partial \mathbf{x}^c} \right|_{\mathbf{x}^c = g(\mathbf{T}_{wc}, \mathbf{x}^w)} \left. \frac{\partial g(\mathbf{T}_{wc} \exp(\hat{\xi}), \mathbf{x}^w)}{\partial \hat{\xi}} \right|_{\hat{\xi}=0}$$

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 &= d \begin{bmatrix} 1 & 0 & -x_n \\ 0 & 1 & -y_n \end{bmatrix} \begin{bmatrix} -\mathbf{I}_{3 \times 3} & [\mathbf{x}^c]^\wedge \end{bmatrix} \\
 &= \begin{bmatrix} -d & 0 & dx_n & x_n y_n & -1 - x_n^2 & y_n \\ 0 & -d & dy_n & 1 + y_n^2 & -x_n y_n & -x_n \end{bmatrix}
 \end{aligned}$$

Linear least-squares

We can then obtain a **linear least-squares problem**

$$\xi_{\Delta}^* = \operatorname{argmin}_{\xi_{\Delta}} \sum_j \left\| h(\mathbf{T}_{wc}; \mathbf{x}_j^w) + \mathbf{F}_j \xi_{\Delta} - \mathbf{x}_{n,j} \right\|_{\Sigma_j}^2$$

Linear least-squares

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Linear least-squares

We can then obtain a **linear least-squares problem**

$$\begin{aligned}\xi_{\Delta}^* &= \operatorname{argmin}_{\xi_{\Delta}} \sum_j \left\| h(\mathbf{T}_{wc}; \mathbf{x}_j^w) + \mathbf{F}_j \xi_{\Delta} - \mathbf{x}_{n,j} \right\|_{\Sigma_j}^2 \\ &= \operatorname{argmin}_{\xi_{\Delta}} \sum_j \left\| \mathbf{F}_j \xi_{\Delta} - \left\{ \mathbf{x}_{n,j} - h(\mathbf{T}_{wc}; \mathbf{x}_j^w) \right\} \right\|_{\Sigma_j}^2 \\ &= \operatorname{argmin}_{\xi_{\Delta}} \sum_j \left\| \mathbf{A}_j \xi_{\Delta} - \mathbf{b}_j \right\|^2\end{aligned}$$

Linear least-squares

We can then obtain a **linear least-squares problem**

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Linear least-squares

With the **state update vector**

$$\xi_{\Delta}$$

For an example with *three points*,
the **measurement Jacobian** \mathbf{A} and the **prediction error** \mathbf{b} are

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \mathbf{A}_3 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix}$$

Solution to the linearized problem

The solution can be found by solving **the normal equations**

$$(\mathbf{A}^T \mathbf{A}) \xi_{\Delta}^* = \mathbf{A}^T \mathbf{b}$$

Choose a suitable initial estimate X^0



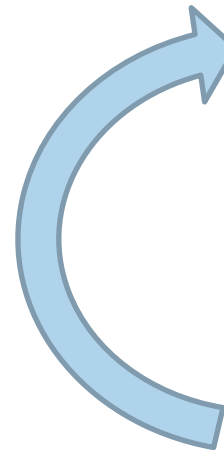
$\mathbf{A}, \mathbf{b} \leftarrow$ Linearize at X^t



$\Delta^* \leftarrow$ Solve $\underset{\Delta}{\operatorname{argmin}} \|\mathbf{A}\Delta - \mathbf{b}\|^2$



$X^{t+1} \leftarrow X^t + \Delta^*$



Gauss-Newton optimization

Given a good initial estimate \mathbf{T}_{wc}^0 .

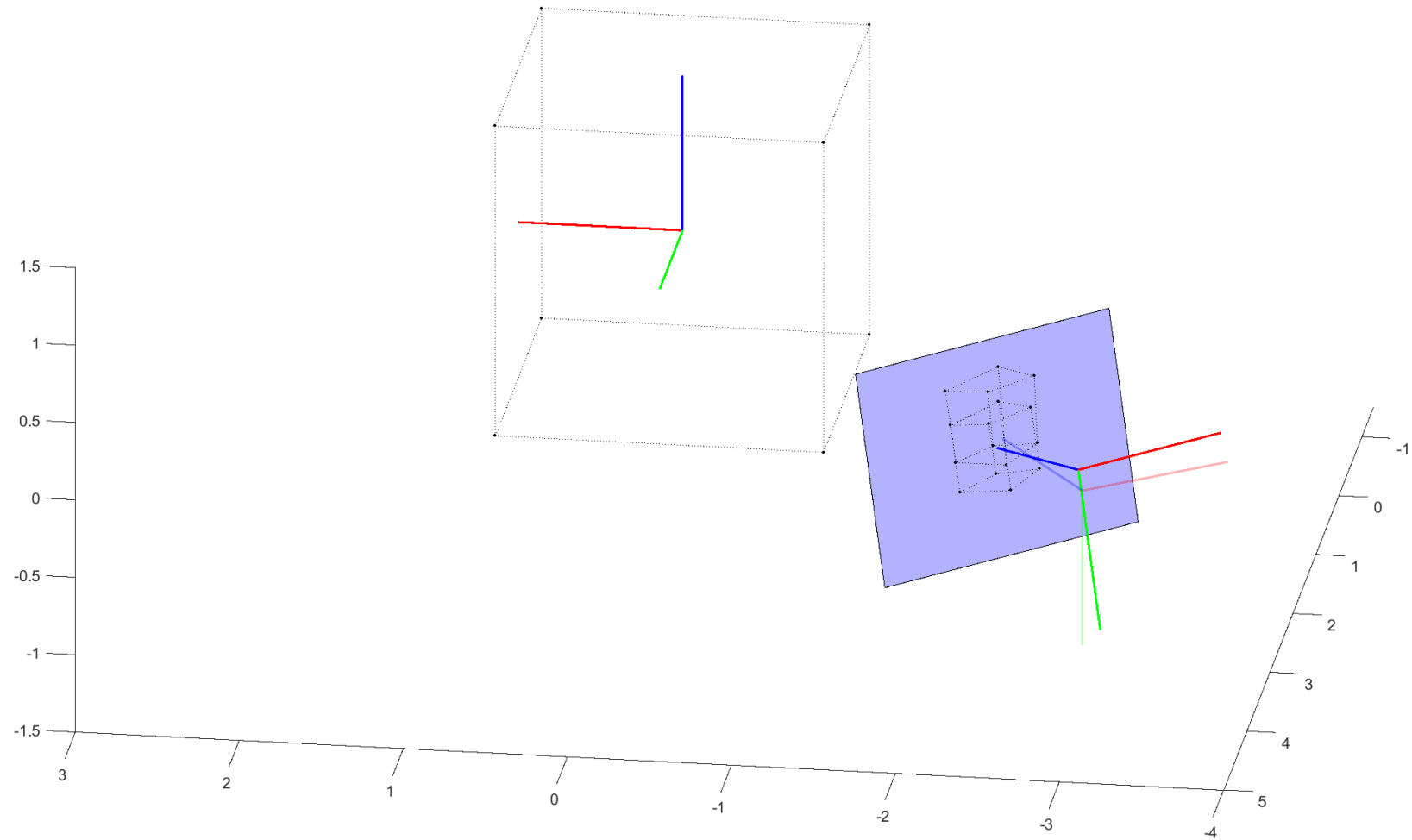
For $t = 0, 1, \dots, t^{max}$

$\mathbf{A}, \mathbf{b} \leftarrow$ Linearize at \mathbf{T}_{wc}^t

$\xi_{\Delta}^* \leftarrow$ Solve the linearized problem with $(\mathbf{A}^T \mathbf{A}) \xi_{\Delta}^* = \mathbf{A}^T \mathbf{b}$

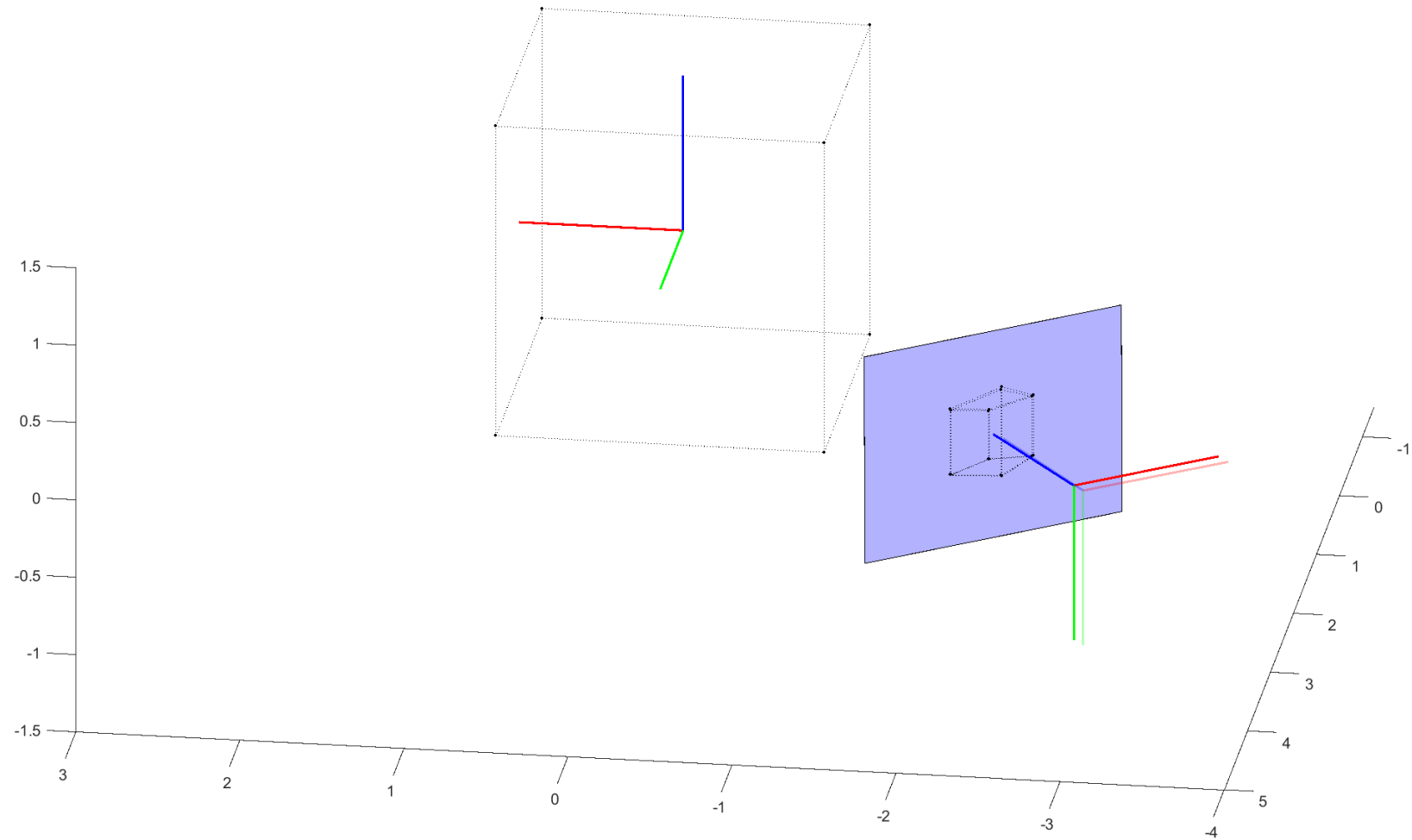
$\mathbf{T}_{wc}^{t+1} \leftarrow \mathbf{T}_{wc}^t \exp(\xi_{\Delta}^{*\wedge})$

Example



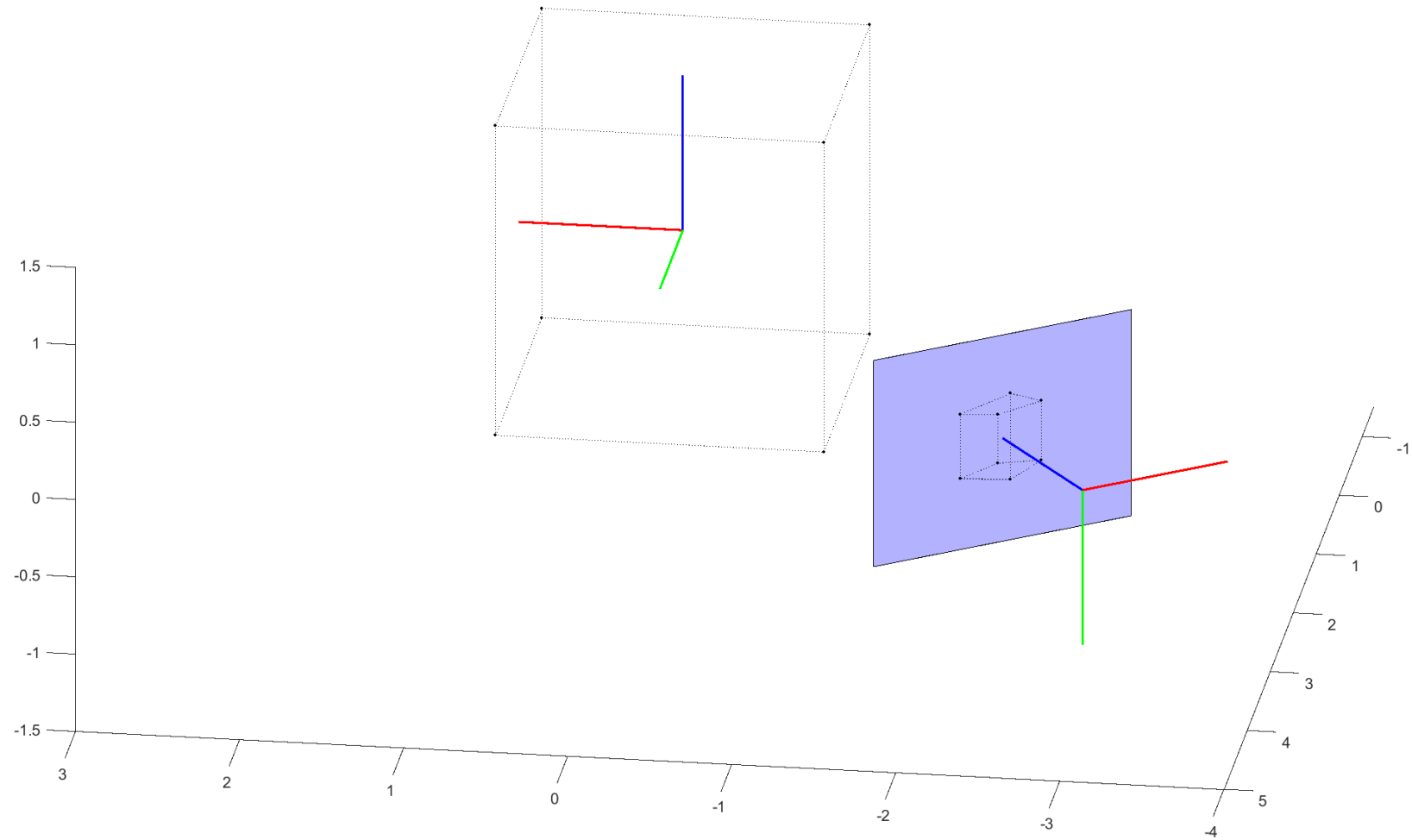
TEK5030

Example



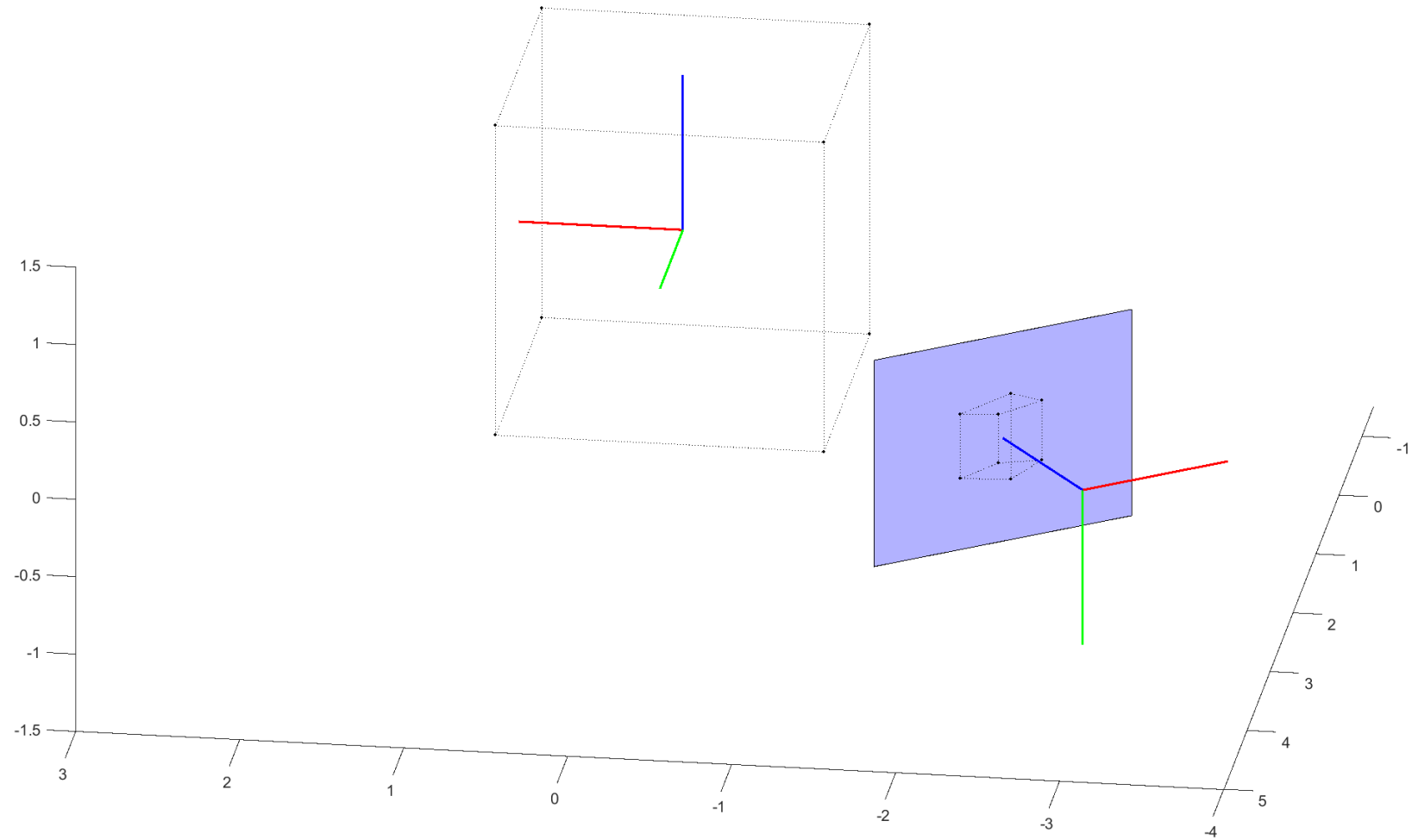
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Example



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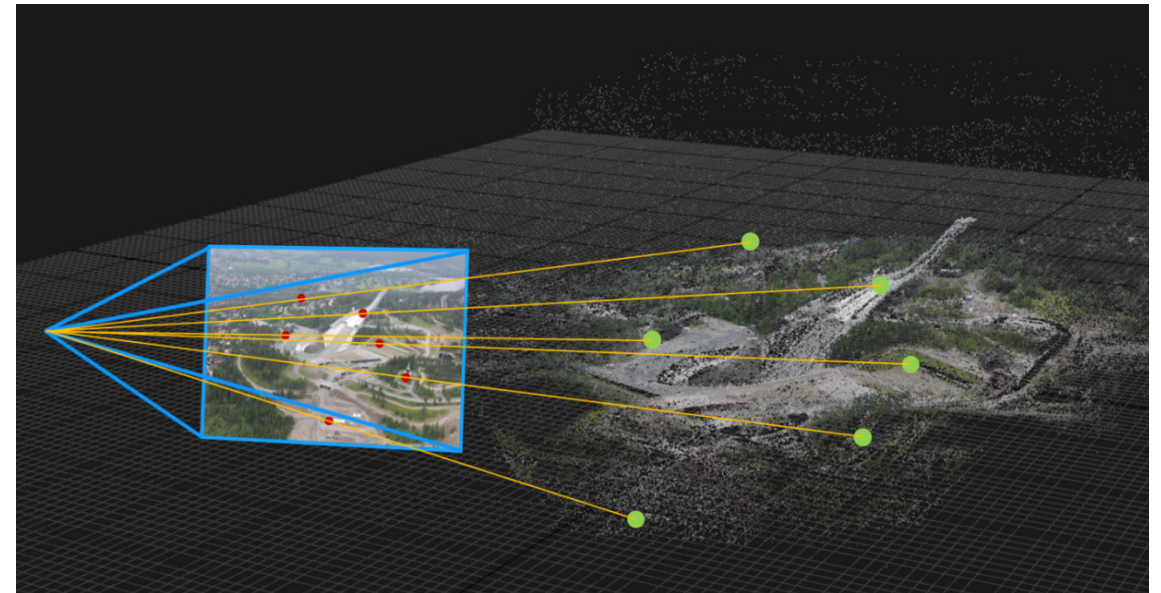
Example



TEK5030

Pose estimation relative to known 3D points

- n -Point Pose Problem (P_nP)
 - Typically fast non-iterative methods
 - Minimal in number of points
 - Accuracy comparable to iterative methods
 - Good initial estimates
- Examples
 - P3P, EPnP
 - P4Pf
 - Estimate pose and focal length
 - P6P
 - Estimates \mathbf{P} with DLT
 - R6P
 - Estimate pose with rolling shutter



Summary

- Pose estimation relative to a world plane
 - Pose from homography
- Nonlinear optimization over poses
- Pose estimation relative to known 3D points
 - Iterative methods
 - PnP

