UiO : Department of Technology Systems
University of Oslo

Lecture 8.1
Epipolar geometry

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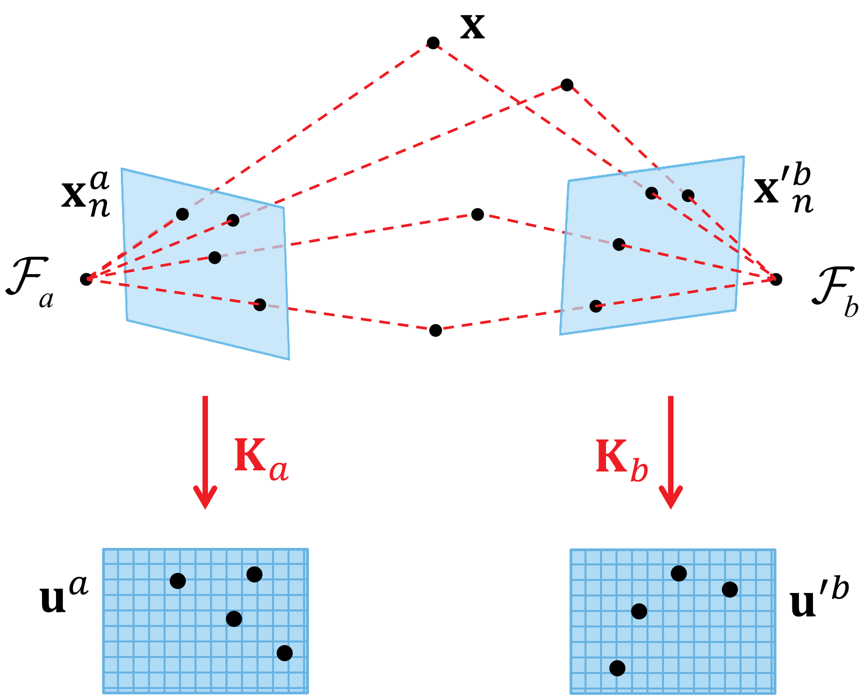
Introduction

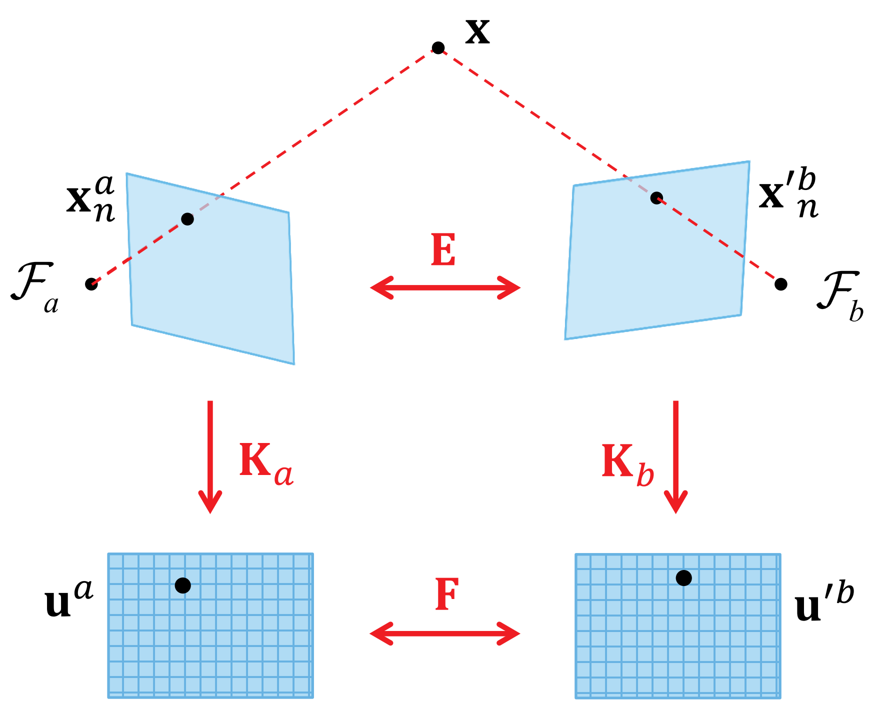
- Observing the same points with two cameras, $\mathcal{F}_a$ and $\mathcal{F}_b$, puts a strong geometrical constraint on the point correspondences

- This epipolar constraint can be represented by two 3×3 matrices

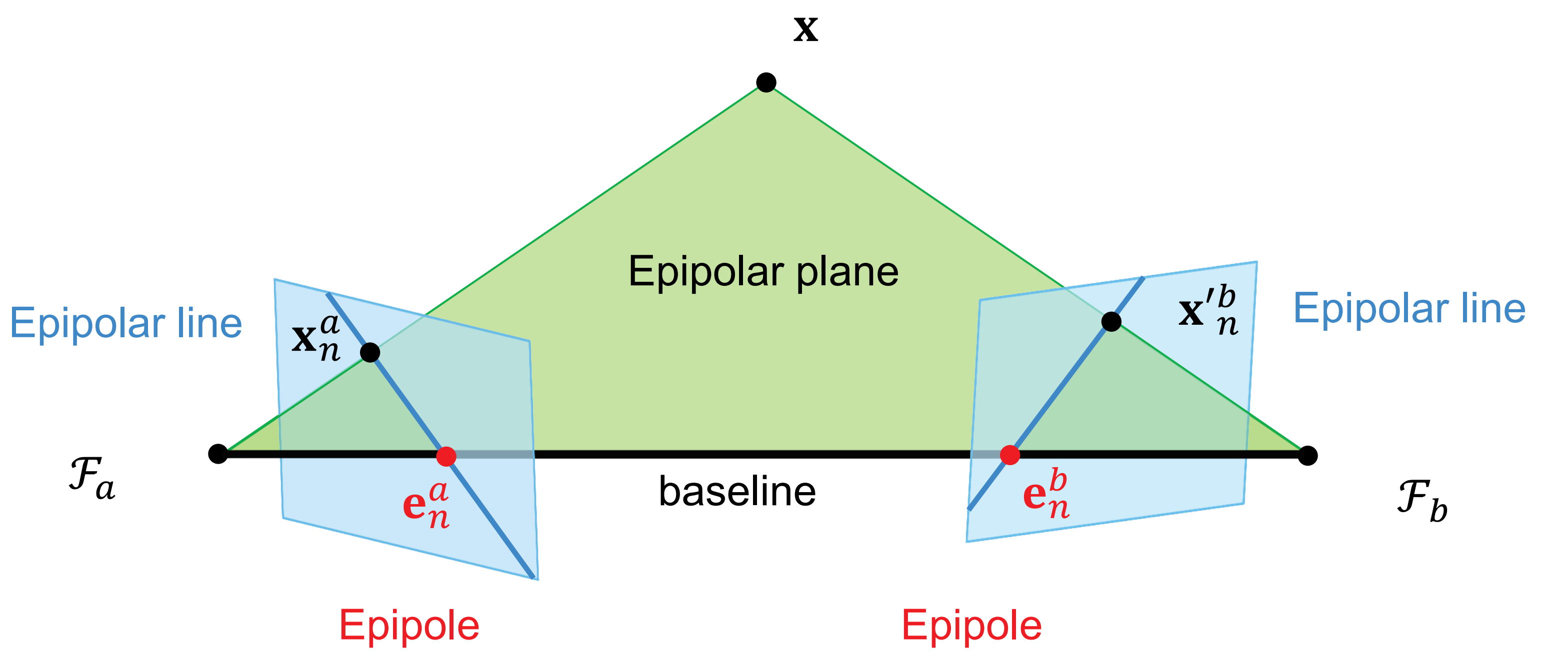


Introduction

- Observing the same points with two cameras, $\mathcal{F}_a$ and $\mathcal{F}_b$, puts a strong geometrical constraint on the point correspondences
- This epipolar constraint can be represented by two 3×3 matrices
- The **essential matrix** $\mathbf{E}$ represents the constraint for point correspondences $\mathbf{x}_n^a \leftrightarrow \mathbf{x}'^b_n$
- The **fundamental matrix** $\mathbf{F}$ represents the same constraint, but for point correspondences $\mathbf{u}^a \leftrightarrow \mathbf{u}'^b$



Introduction



- The **epipolar plane** is the plane containing $\mathbf{x}$ and the two camera centers of $\mathcal{F}_a$ and $\mathcal{F}_b$
- The **baseline** is the line joining $\mathcal{F}_a$ and $\mathcal{F}_b$
- The **epipolar lines** are where the epipolar plane intersect the image planes
- The **epipoles** are where the baseline intersects the two image planes
- Epipoles and epipolar lines can be represented in the normalized image plane as well as in the image

Exploring the epipolar geometry

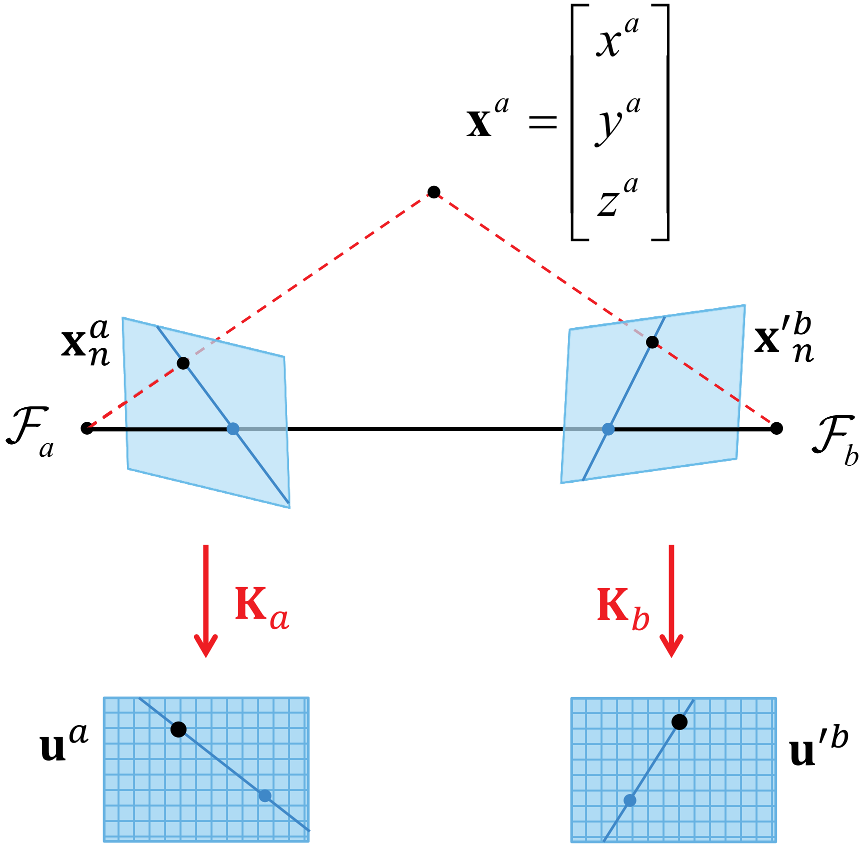
Let us consider two cameras, $\mathcal{F}_a$ and $\mathcal{F}_b$, and let $\mathcal{F}_a$ to be our "world frame"

Then we have the camera projection matrices

$$\mathbf{P}_a = \mathbf{K}_a \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}$$

$$\mathbf{P}_b = \mathbf{K}_b \begin{bmatrix} \mathbf{R}_{ba} & \mathbf{t}_{ba}^b \end{bmatrix}$$

Assume that the two cameras project a 3D world point $\mathbf{x}^a = [x^a \quad y^a \quad z^a]^T$ to $\mathbf{u}^a$ and $\mathbf{u}'^b$ correspondingly



Exploring the epipolar geometry

Projecting $\mathbf{x}$ into the first image yields

$$\tilde{\mathbf{u}}^a = \mathbf{K}_a [\mathbf{I} \quad \mathbf{0}] \tilde{\mathbf{x}}^a$$

$$\tilde{\mathbf{u}}^a = \mathbf{K}_a \mathbf{x}^a$$

$$\tilde{\mathbf{x}}^a = \begin{bmatrix} \mathbf{x}^a \\ 1 \end{bmatrix}$$

So up to scale we know that

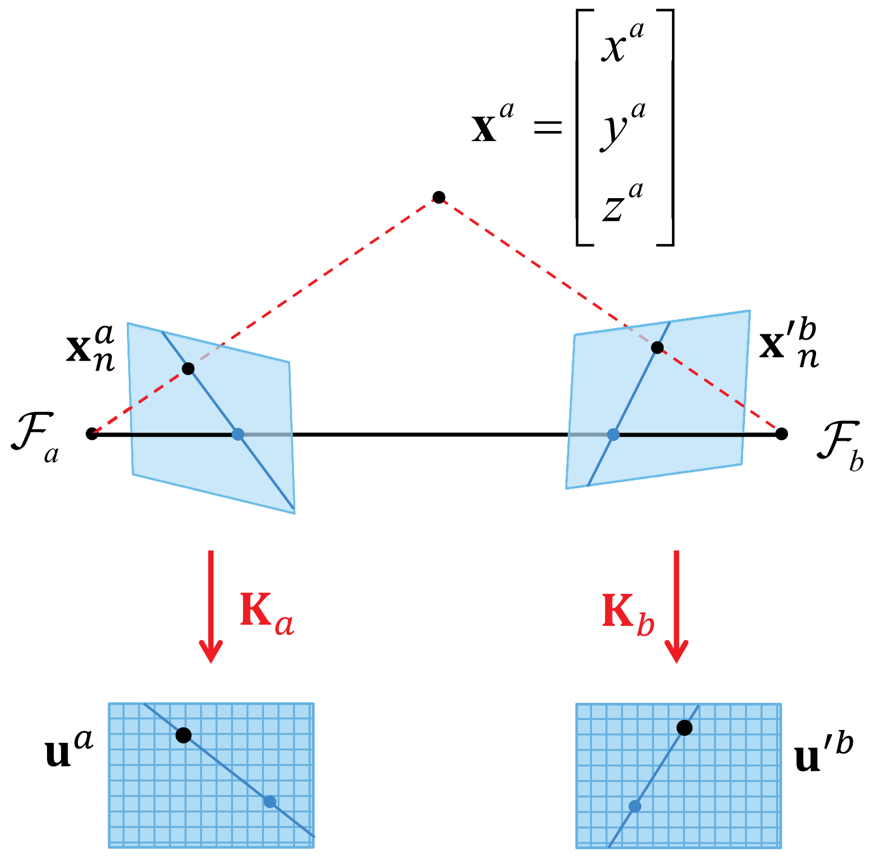
$$\mathbf{x}^a = \mathbf{K}_a^{-1} \tilde{\mathbf{u}}^a$$

equal up to scale

But given that $\mathbf{x}^a = [x^a \quad y^a \quad z^a]^T$, we also know the scale

$$\mathbf{x}^a = z^a \mathbf{K}_a^{-1} \tilde{\mathbf{u}}^a$$

truly equal



Exploring the epipolar geometry

Projecting $\mathbf{x}$ into the second image yields

$$\tilde{\mathbf{u}}'^b = \mathbf{K}_b \begin{bmatrix} \mathbf{R}_{ba} & \mathbf{t}_{ba}^b \end{bmatrix} \tilde{\mathbf{x}}^a$$

$$\tilde{\mathbf{x}}^a = \begin{bmatrix} \mathbf{x}^a \\ 1 \end{bmatrix}$$

$$\tilde{\mathbf{u}}'^b = \mathbf{K}_b \left(\mathbf{R}_{ba} \mathbf{x}^a + \mathbf{t}_{ba}^b \right)$$

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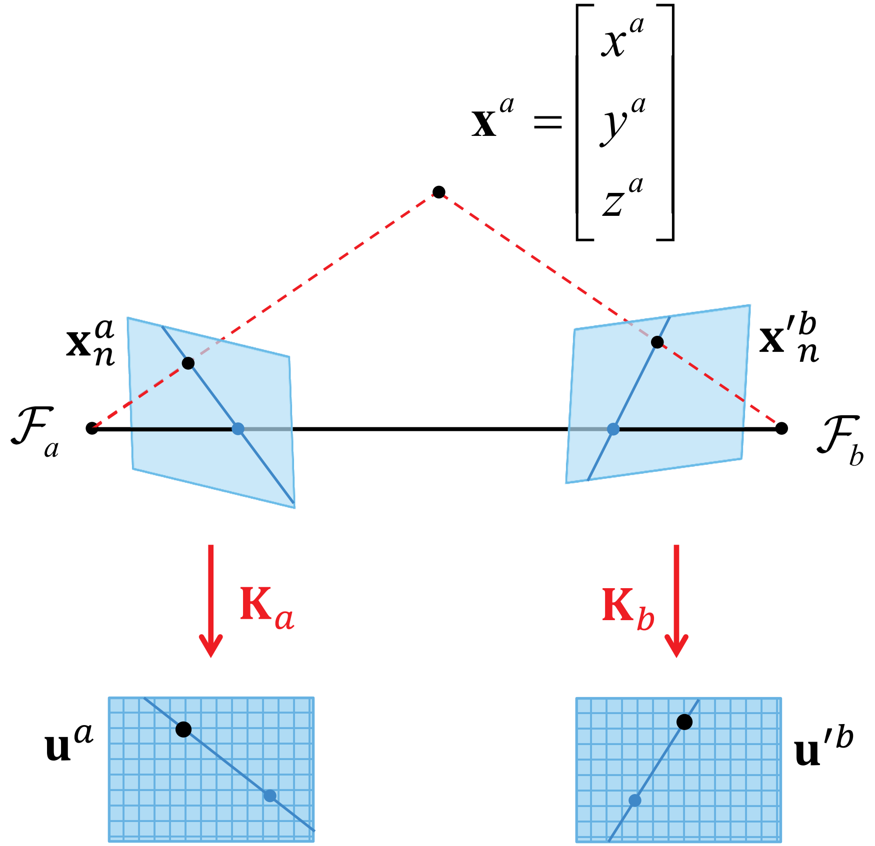
equal up to scale

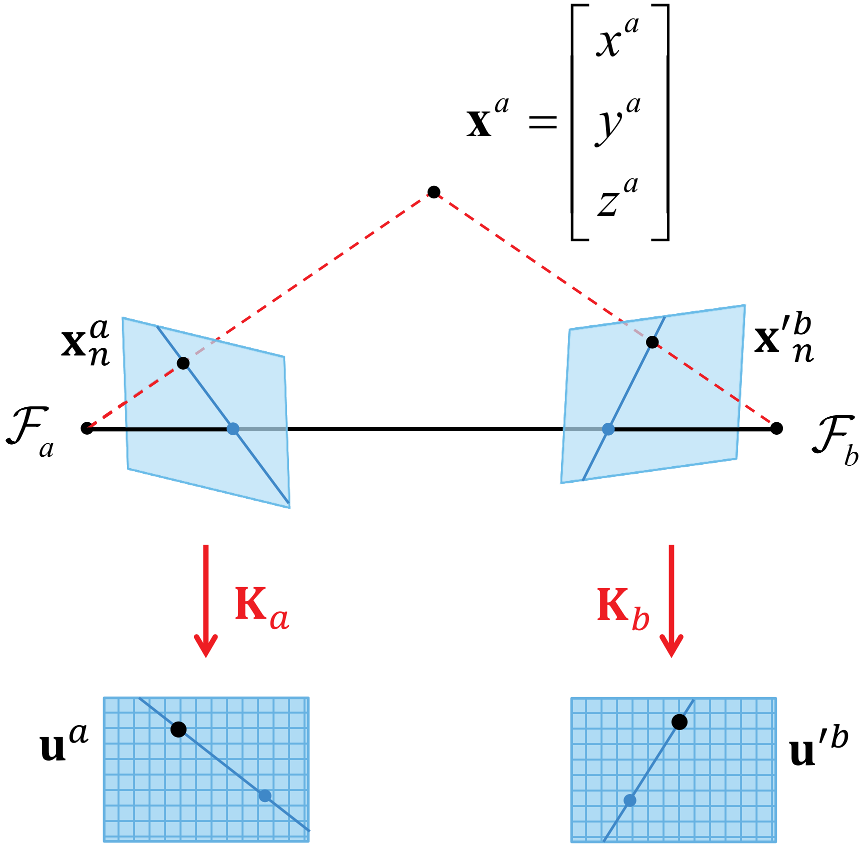


Exploring the epipolar geometry

Combining these two results gives us

$$\tilde{\mathbf{u}}'^{b} = \mathbf{K}_b \mathbf{R}_{ba} \mathbf{K}_a^{-1} \tilde{\mathbf{u}}^{a} + \frac{1}{z^a} \mathbf{K}_b \mathbf{t}_{ba}^{b}$$

equal up to scale



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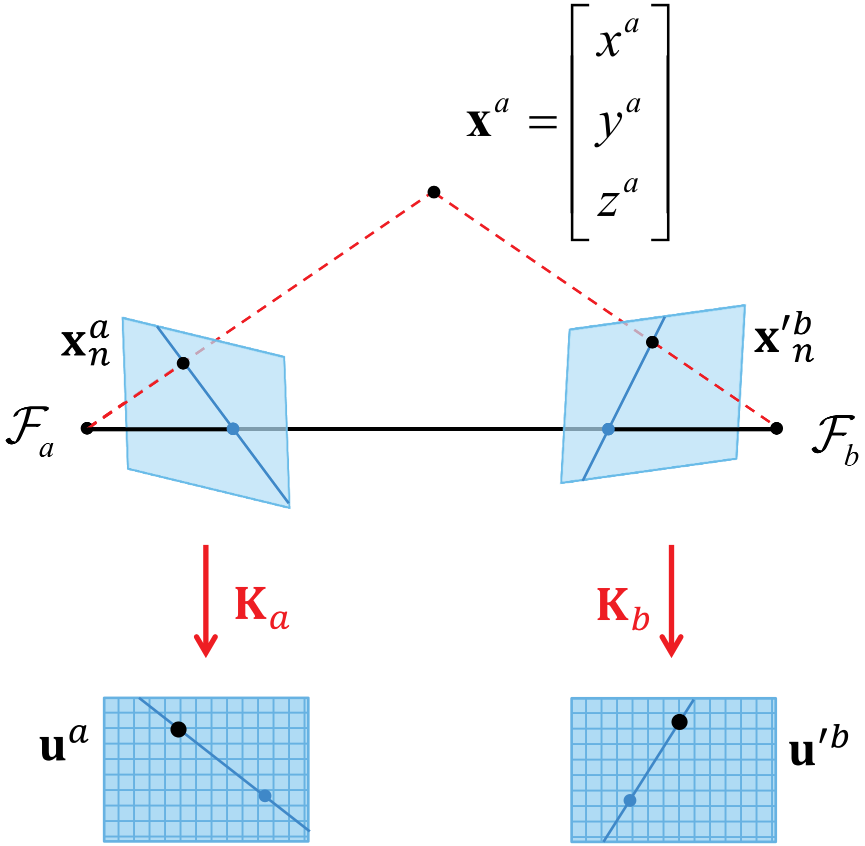
This describes how the position of $\mathbf{u}'^b$ on the epipolar line varies with the depth z^a of the observed world point $\mathbf{x}^a$



Exploring the epipolar geometry

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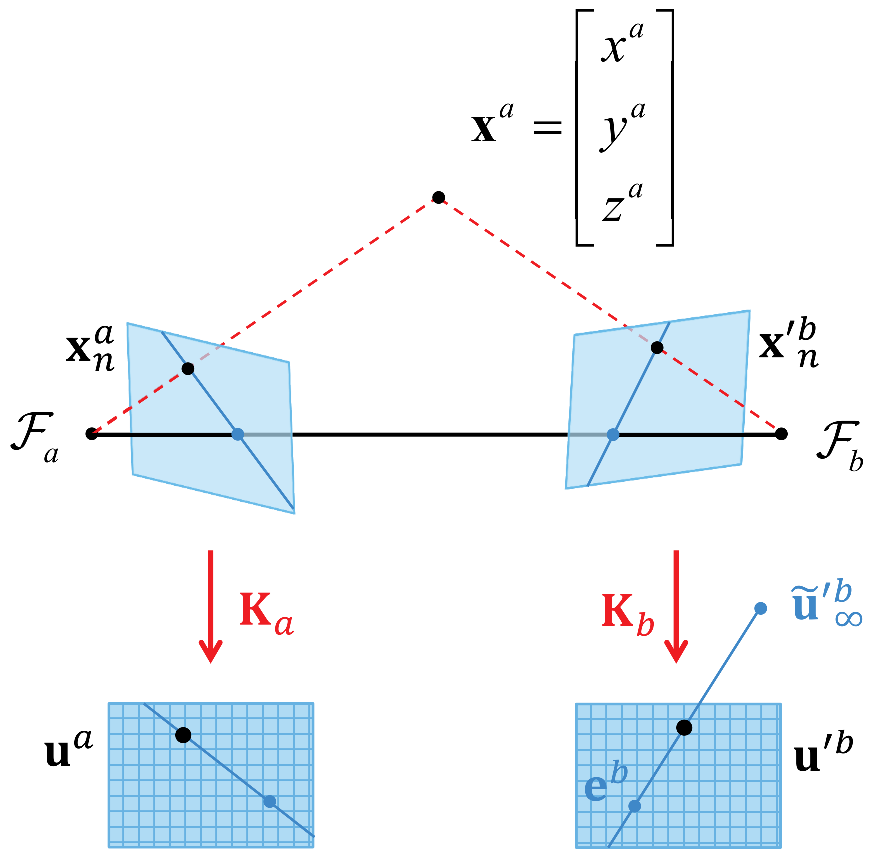
It is clear that $\mathbf{u}'^{b}$ is restricted to the epipolar line, but also that it is restricted to an interval of this line with the epipole $\mathbf{e}^b$ on one side and "infinity" $\tilde{\mathbf{u}}'^{b}_{\infty} = \mathbf{K}_b \mathbf{R}_{ba} \mathbf{K}_a^{-1} \tilde{\mathbf{u}}^{a}$ on the other



Exploring the epipolar geometry

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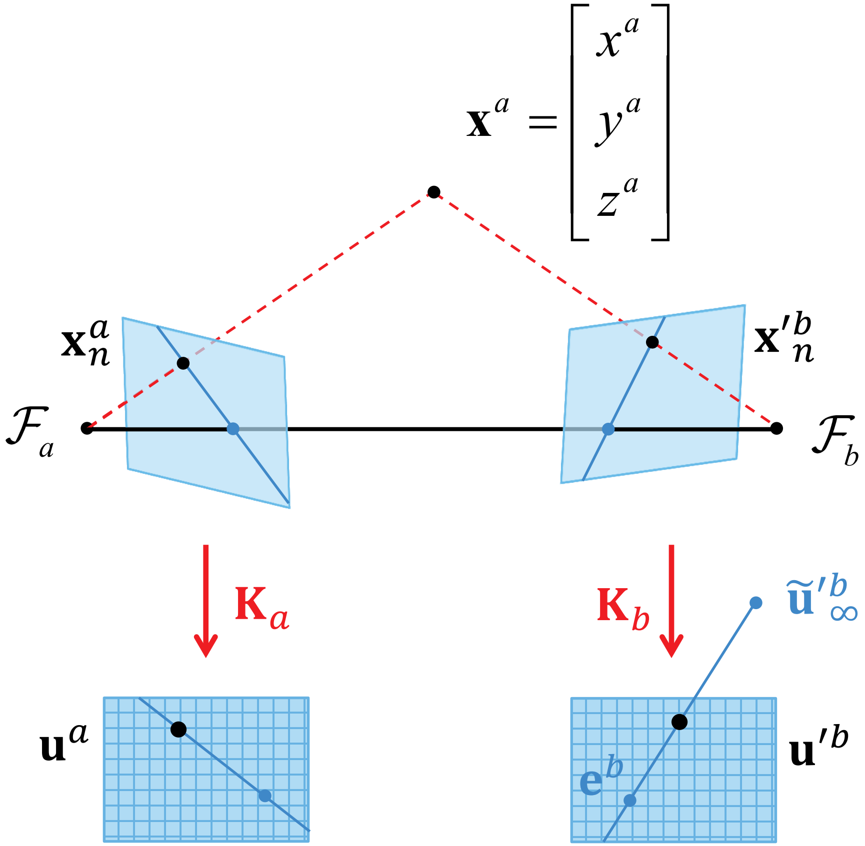


$$\text{disparity} = \left\| \tilde{\mathbf{u}}'^{b} - \mathbf{K}_b \mathbf{R}_{ba} \mathbf{K}_a^{-1} \tilde{\mathbf{u}}^{a} \right\|$$

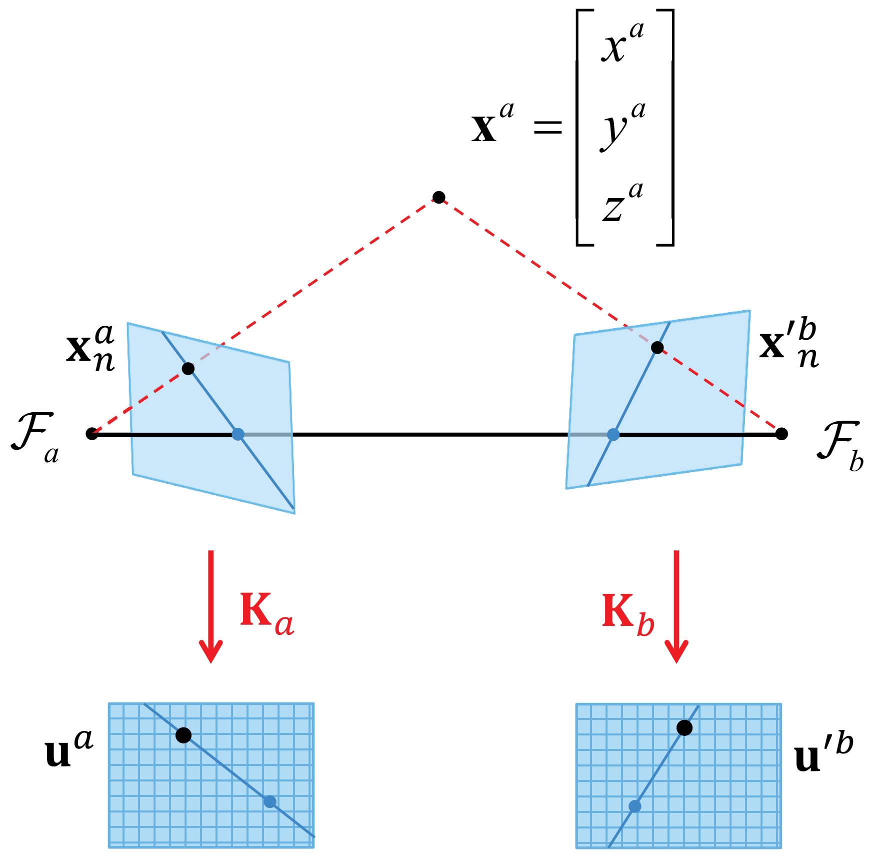
Exploring the epipolar geometry

Another observation is that all correspondences $\mathbf{u}^a \leftrightarrow \mathbf{u}'^b$ for far away 3D points $\mathbf{x}^a$ must be related by the same homography

$$\mathbf{H} = \mathbf{K}_b \mathbf{R}_{ba} \mathbf{K}_a^{-1}$$

The same is obviously true for absolutely all correspondences when $\mathcal{F}_b$ is just a rotation of $\mathcal{F}_a$, i.e. when $\mathbf{t}_{ba}^b = \mathbf{0}$

This explains why it is easy to coregister images of distant scenes even when the camera motion is not a pure rotation



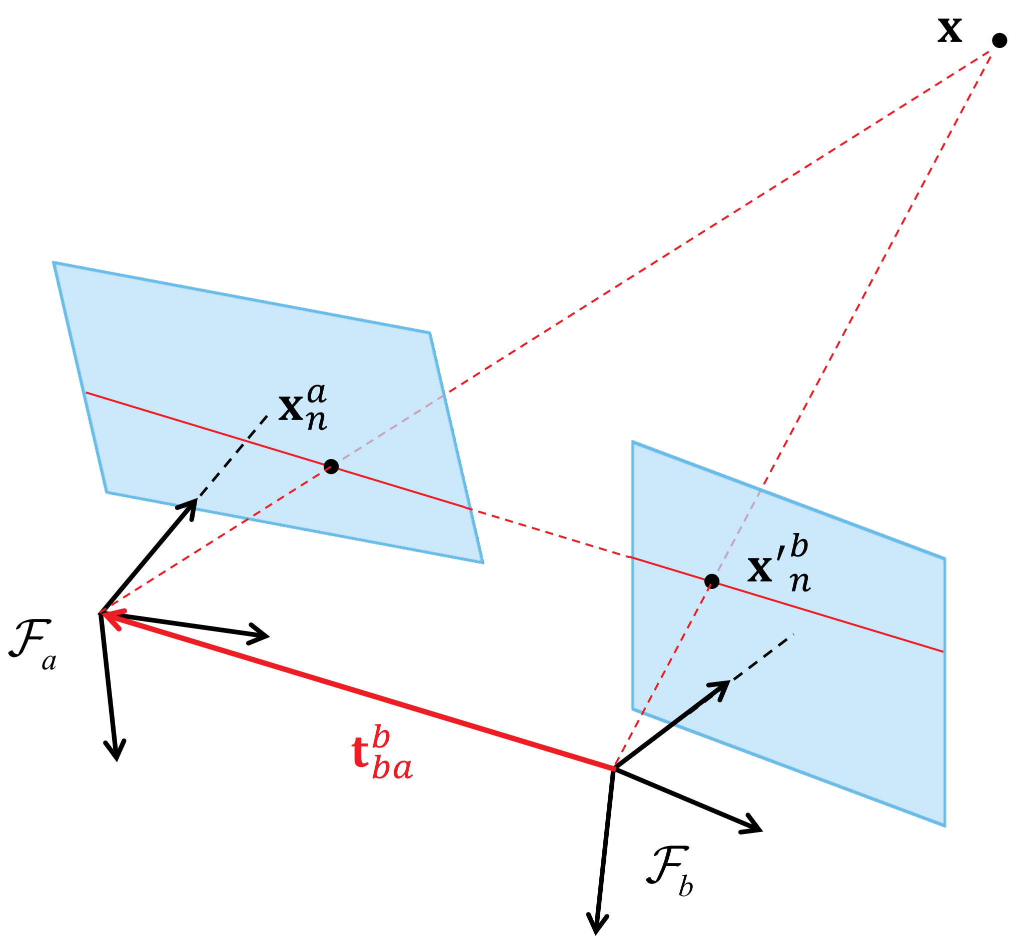
"far away" $\Leftrightarrow$ $z^a \gg \left\| \mathbf{K}_b \mathbf{t}_{ba}^b \right\|$

Describing the epipolar geometry

Let $\mathbf{x}$ project to $\mathbf{x}_n^a$ in the normalized image plane of $\mathcal{F}_a$ and $\mathbf{x'}_n^b$ in that of $\mathcal{F}_b$

Let the pose of $\mathcal{F}_a$ relative to $\mathcal{F}_b$ be

$$\mathbf{T}_{ba} = \begin{bmatrix} \mathbf{R}_{ba} & \mathbf{t}_{ba}^b \\ \mathbf{0} & 1 \end{bmatrix}$$



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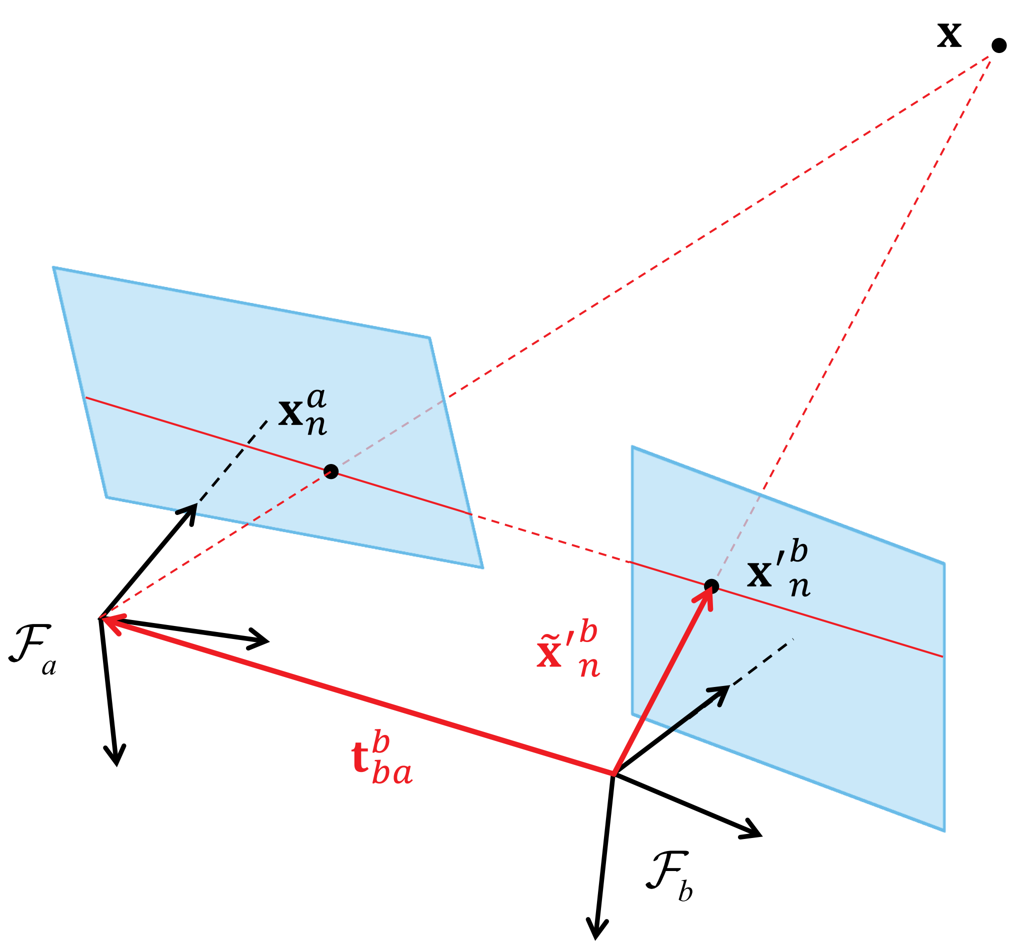
$$\mathbf{T}_{ba} = \begin{bmatrix} \mathbf{R}_{ba} & \mathbf{t}_{ba}^b \\ \mathbf{0} & 1 \end{bmatrix}$$

Observe that $\tilde{\mathbf{x}}'^b_n$, in addition to being a homogeneous representation of the 2D point $\mathbf{x}'^b_n$, is a 3D vector



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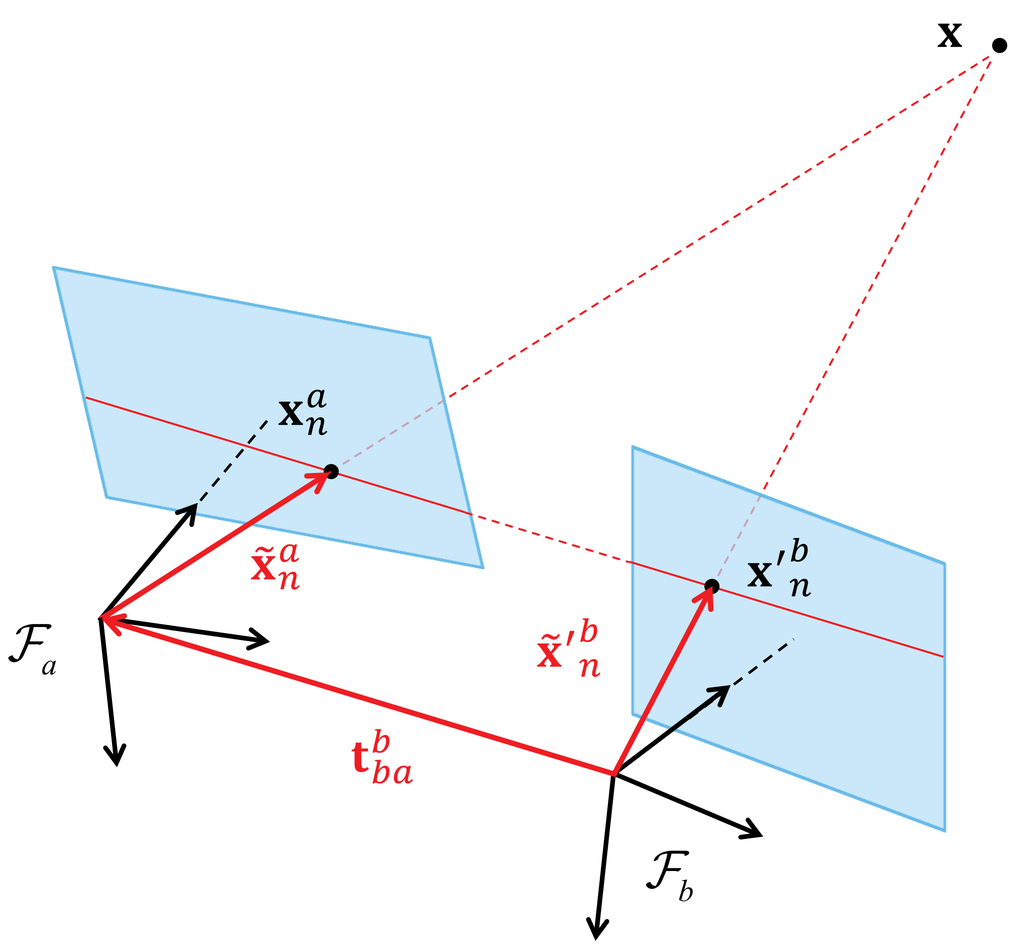
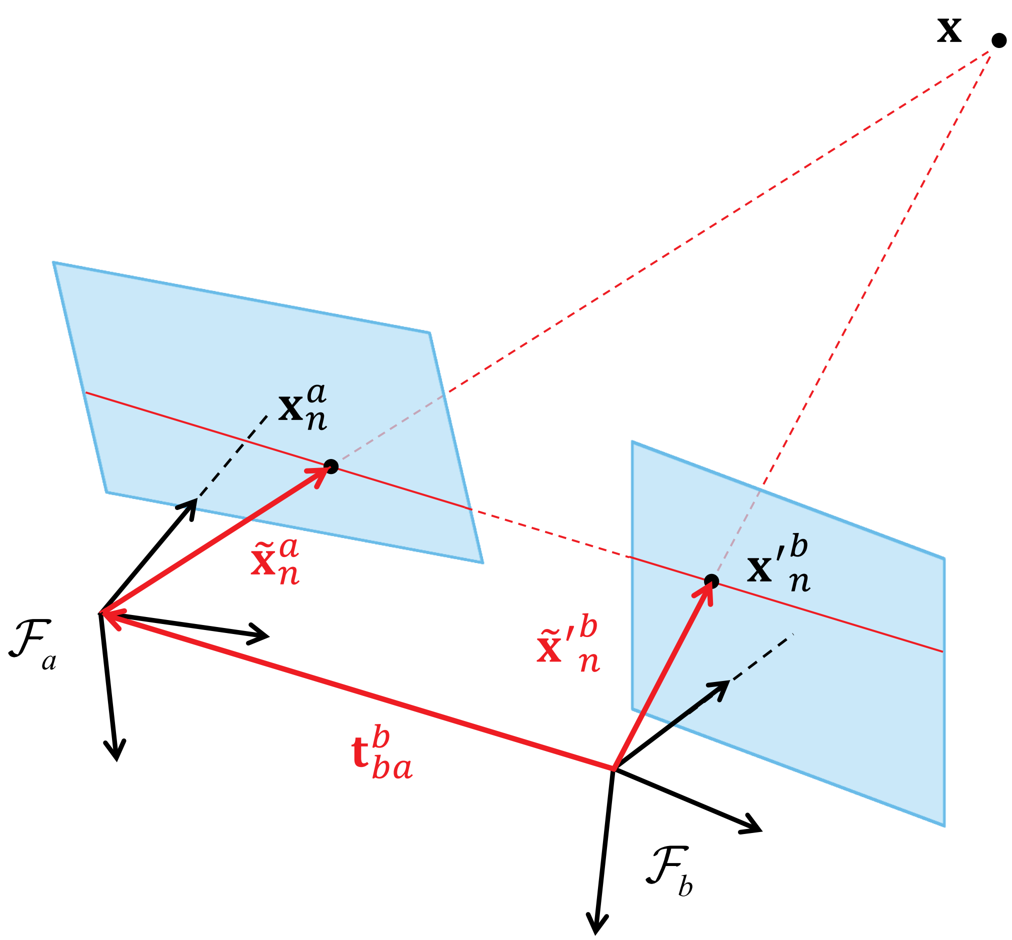
The same goes for $\tilde{\mathbf{x}}_n^a$



Describing the epipolar geometry

Since both vectors $\tilde{\mathbf{x}}_n^a$ and $\tilde{\mathbf{x}}'^b_n$ lie in the epipolar plane, they must satisfy the equation

$$\left(\tilde{\mathbf{x}}'^b_n \times \mathbf{t}^b_{ba}\right) \cdot \left(\mathbf{R}_{ba} \tilde{\mathbf{x}}^a_n\right) = 0$$



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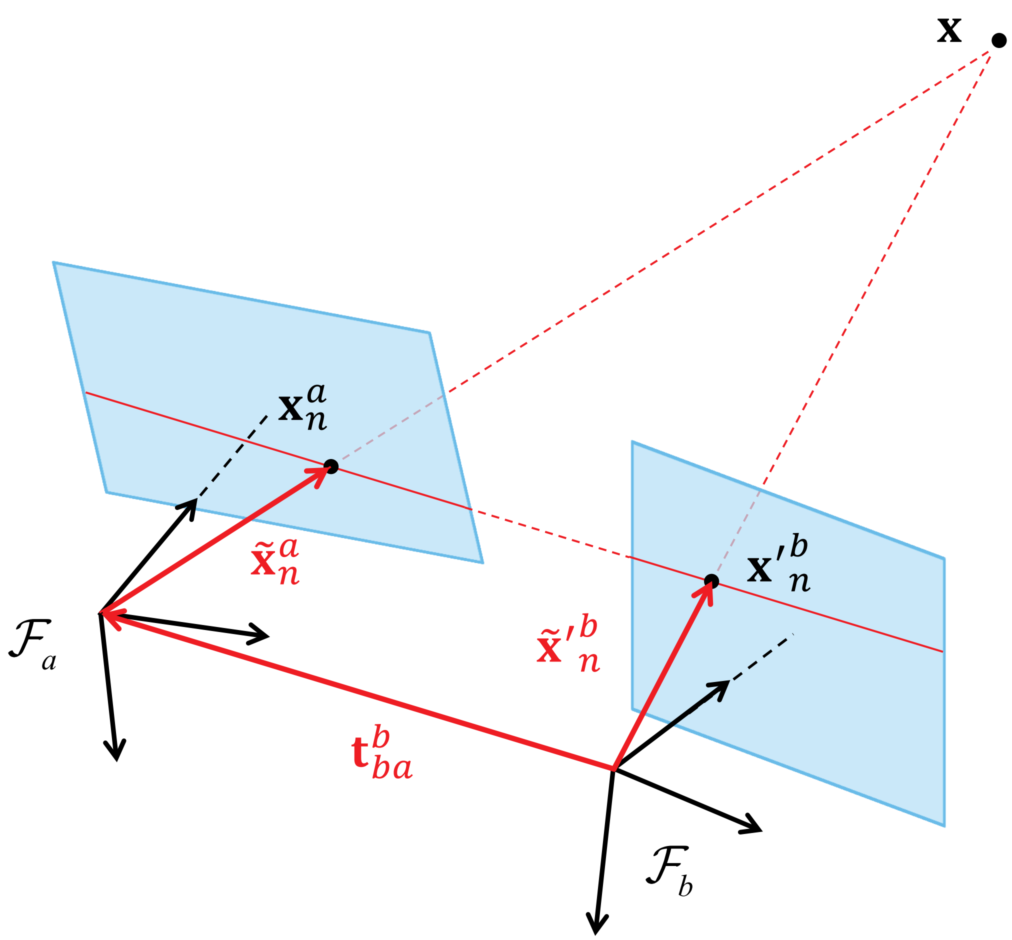
normal vector of the epipolar plane

$\tilde{\mathbf{x}}_n^a$ transformed to $\mathcal{F}_b$



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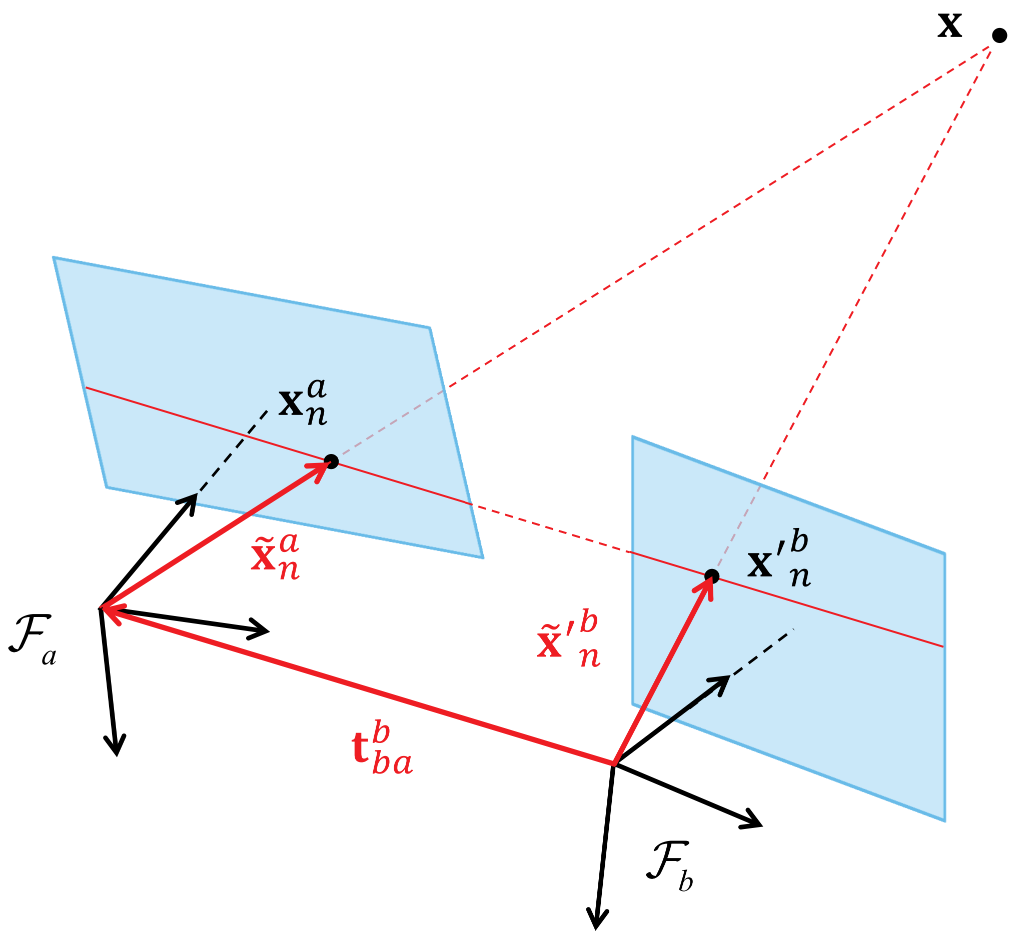
$$\left(\tilde{\mathbf{x}}'^b_n \times \mathbf{t}^b_{ba}\right) \cdot \left(\mathbf{R}_{ba}\tilde{\mathbf{x}}^a_n\right) = 0$$

Using the matrix representation of the cross product, we can rewrite this as

$$\left(\tilde{\mathbf{x}}'^b_n\right)^T \left(\mathbf{t}^b_{ba}\right)^\wedge \mathbf{R}_{ba}\tilde{\mathbf{x}}^a_n = 0$$

$$\mathbf{v}^\wedge = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$$



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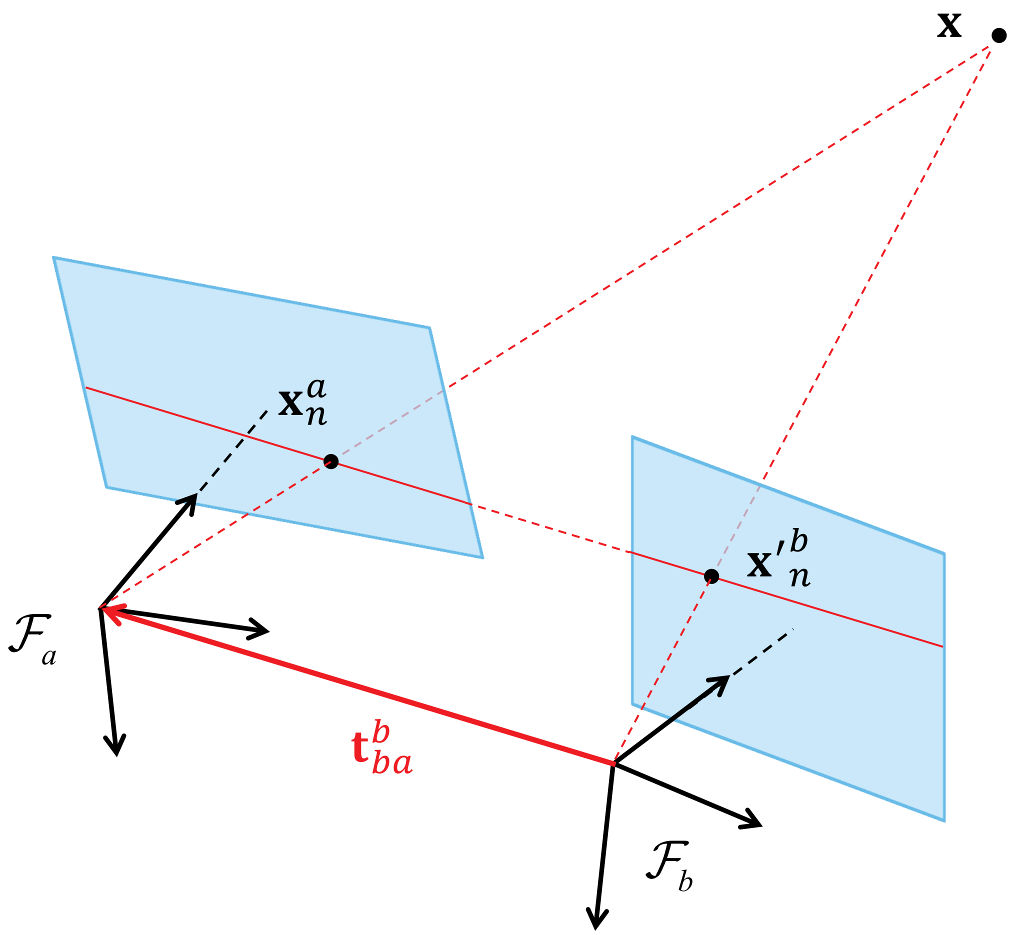
$$\left(\tilde{\mathbf{x}}'^b_n\right)^T \left(\mathbf{t}^b_{ba}\right)^{\wedge} \mathbf{R}_{ba}\tilde{\mathbf{x}}^a_n = 0$$

This equation embodies the epipolar constraint on the correspondence $\mathbf{x}^a_n \leftrightarrow \mathbf{x}'^b_n$ and we denote the matrix $\left(\mathbf{t}^b_{ba}\right)^{\wedge}\mathbf{R}_{ba}$ by $\mathbf{E}_{ba}$ and call it the **essential matrix**

We can derive the essential matrix $\mathbf{E}_{ab}$ similarly



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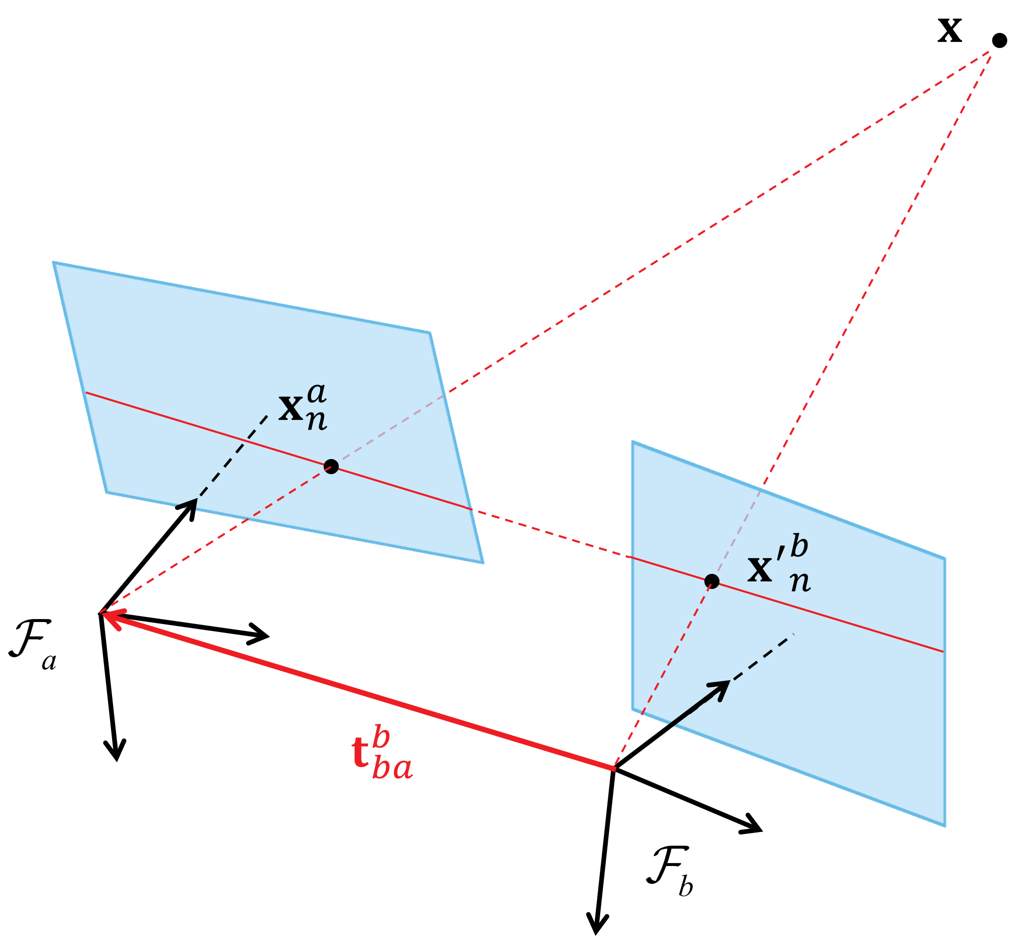
We can derive the essential matrix $\mathbf{E}_{ab}$ similarly

Notice that this derivation is independent of $\left\|\mathbf{t}^b_{ba}\right\|$, so that $\mathbf{E}_{ab}$ and $\mathbf{E}_{ba}$ are homogeneous by nature



The essential matrix E

For a correspondence $\mathbf{x}_n^a \leftrightarrow \mathbf{x'}_n^b$ to be geometrically viable, it must satisfy the equations

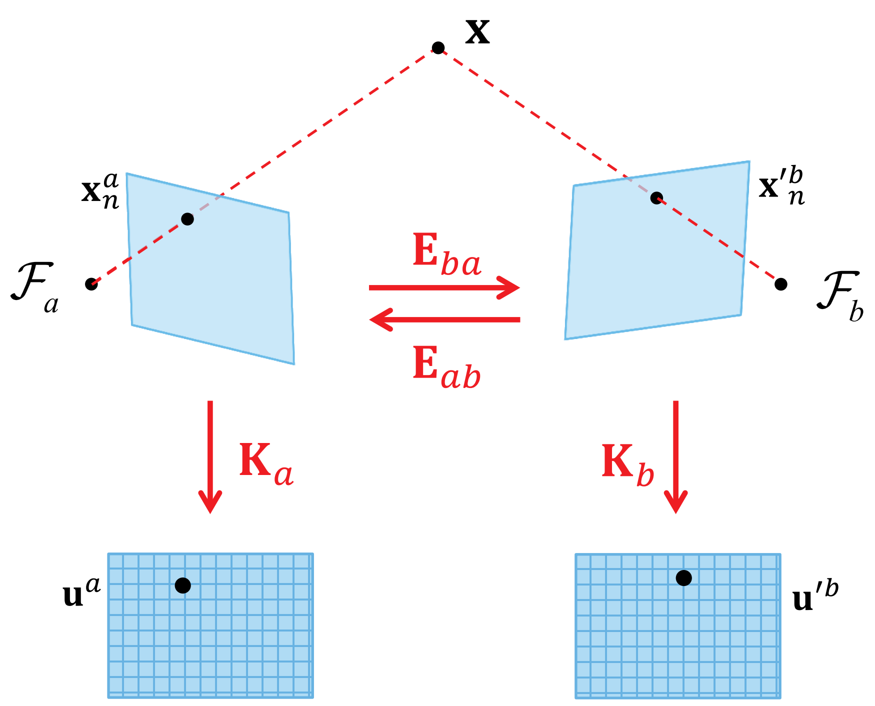
$$\left(\tilde{\mathbf{x}}_n^a\right)^T \mathbf{E}_{ab}\tilde{\mathbf{x}}'^b_n = 0$$

$$\left(\tilde{\mathbf{x}}'^b_n\right)^T \mathbf{E}_{ba}\tilde{\mathbf{x}}_n^a = 0$$

where the essential matrices $\mathbf{E}_{ab}$ and $\mathbf{E}_{ba}$ are homogeneous and given (up to scale) by

$$\mathbf{E}_{ab} = \left(\mathbf{t}_{ab}^a\right)^\wedge \mathbf{R}_{ab}$$

$$\mathbf{E}_{ba} = \left(\mathbf{t}_{ba}^b\right)^\wedge \mathbf{R}_{ba}$$



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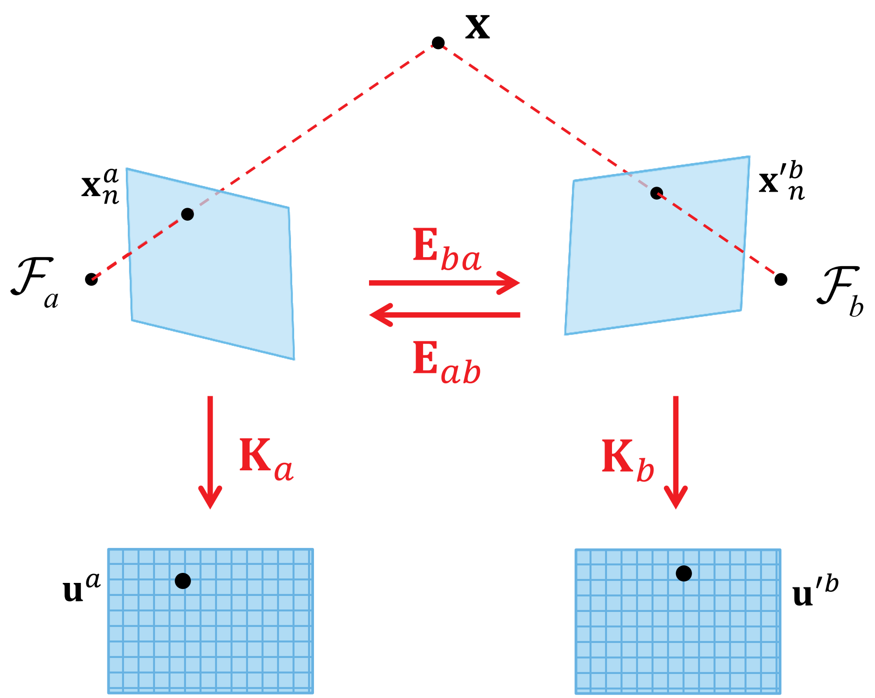
From the equations it is clear that $\mathbf{E}_{ba} = \mathbf{E}_{ab}^T$, so that the two equations are equivalent representations of the same constraint

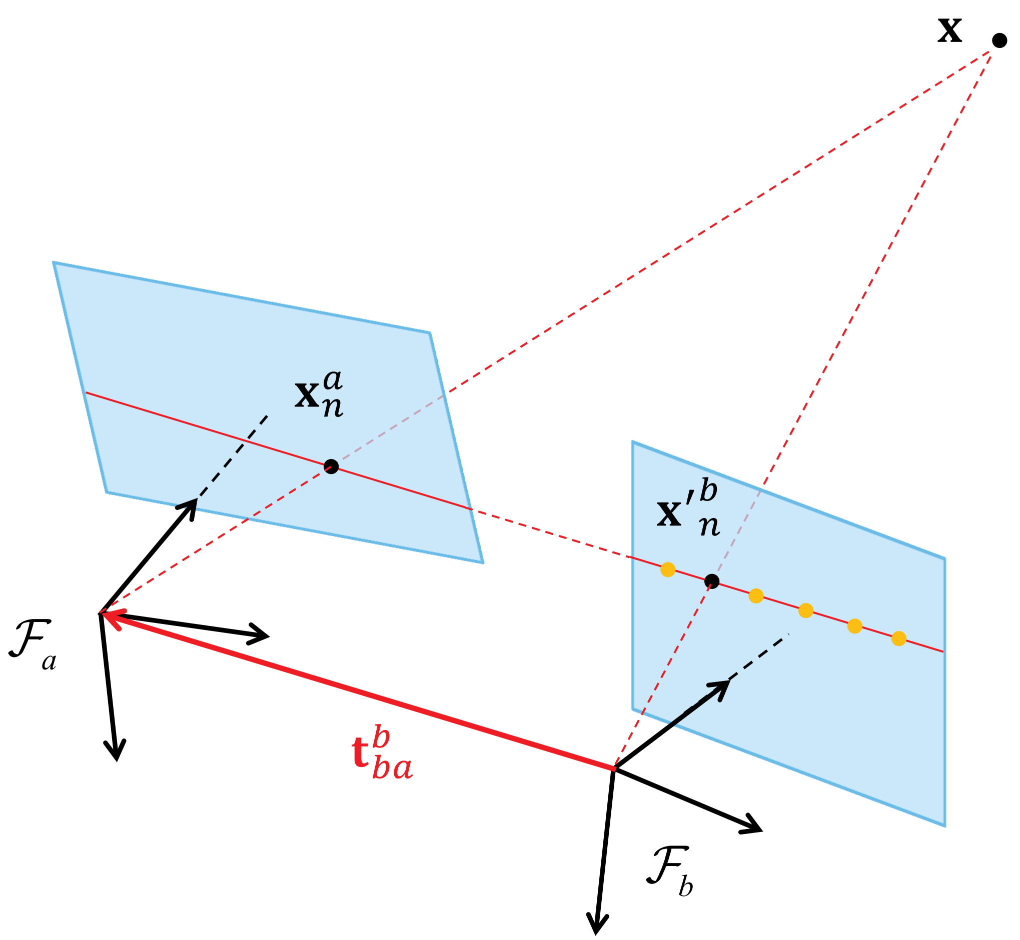


The essential matrix E

Note

Although $\left(\tilde{\mathbf{x}}'^b_n\right)^T \mathbf{E}_{ba}\tilde{\mathbf{x}}^a_n = 0$ is a *necessary* requirement for the correspondence $\mathbf{x}^a_n \leftrightarrow \mathbf{x}'^b_n$ to be correct, it is *not sufficient* to guarantee its correctness

It only guarantees that the two points lie in the same epipolar plane



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Note

Although $\left(\tilde{\mathbf{x}}'^{b}_{n}\right)^{T}\mathbf{E}_{ba}\tilde{\mathbf{x}}^{a}_{n} = 0$ is a *necessary* requirement for the correspondence $\mathbf{x}^{a}_{n} \leftrightarrow \mathbf{x}'^{b}_{n}$ to be correct, it is *not sufficient* to guarantee its correctness

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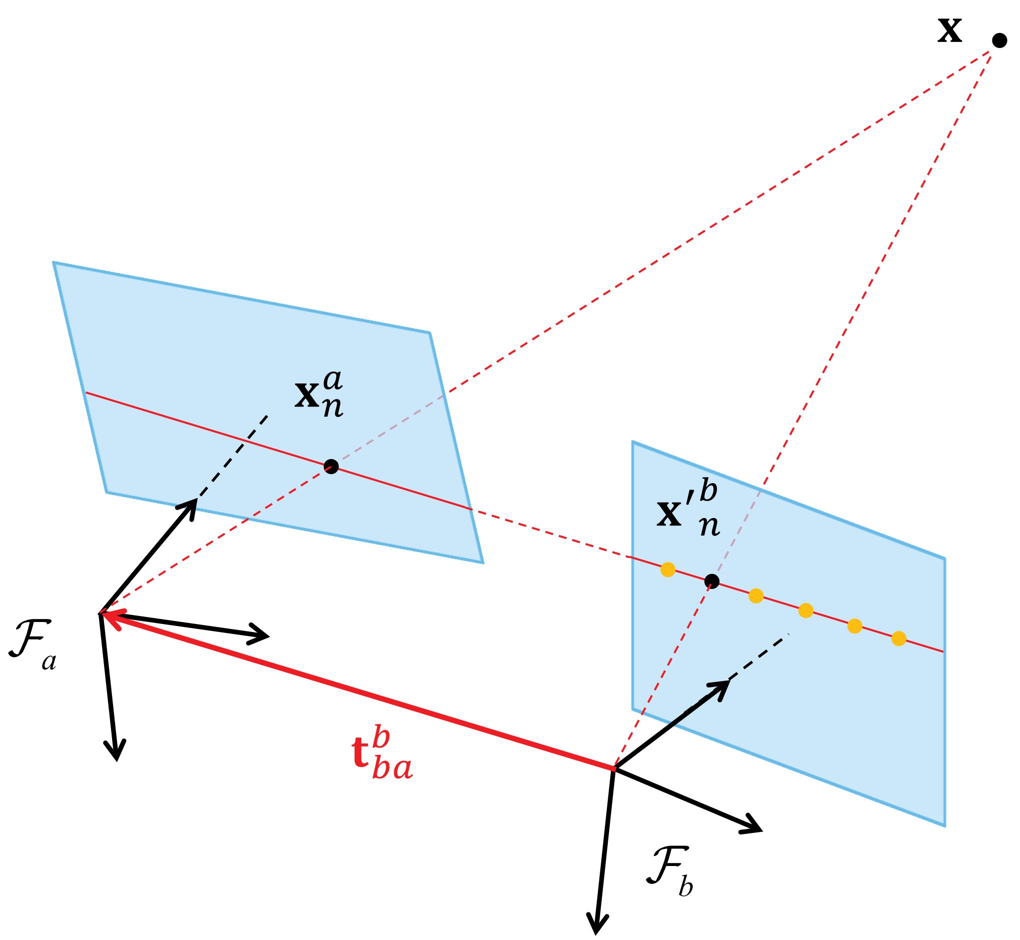
The expression

$$\tilde{\mathbf{u}}'^{b} = \mathbf{K}_{b}\mathbf{R}_{ba}\mathbf{K}_{a}^{-1}\tilde{\mathbf{u}}^{a} + \frac{1}{z^{a}}\mathbf{K}_{b}\mathbf{t}^{b}_{ba}$$

$$\tilde{\mathbf{x}}'^{b}_{n} = \mathbf{R}_{ba}\tilde{\mathbf{x}}^{a}_{n} + \frac{1}{z^{a}}\mathbf{t}^{b}_{ba}$$

can be used to constrain the correspondence further



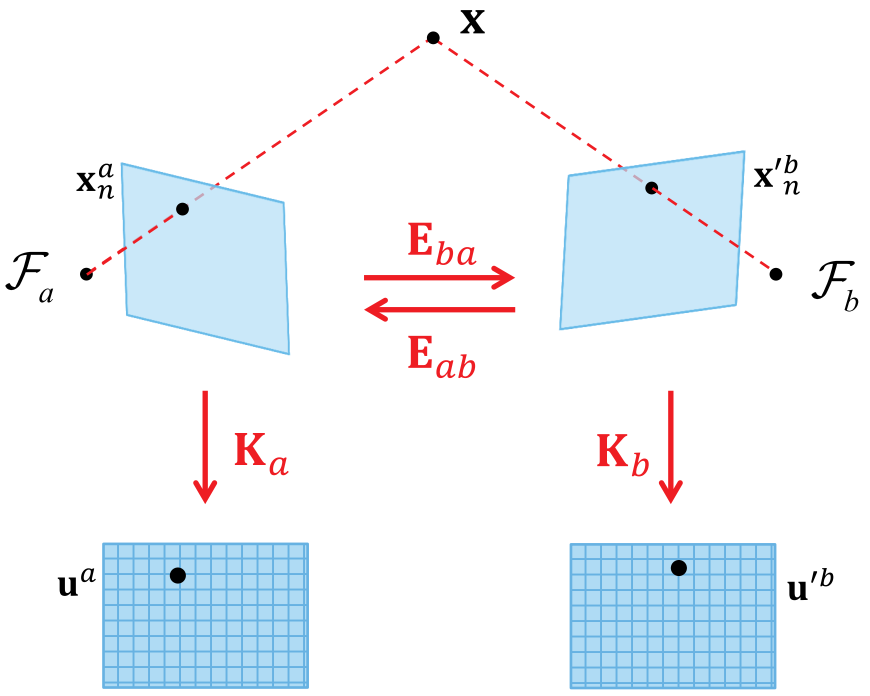
Properties of E

- $\mathbf{E} = \mathbf{t}^{\wedge}\mathbf{R}$
- $\mathbf{E}$ is homogeneous
- $\mathrm{rank}(\mathbf{E}) = 2$
- $\det(\mathbf{E}) = 0$

- If $\mathbf{x}_n$ and $\mathbf{x}'_n$ correspond to the same 3D point then

$$(\mathbf{x}_n)^T \mathbf{E} \mathbf{x}'_n = 0$$

- $\mathbf{E} = \mathbf{t}^{\wedge}\mathbf{R}$ has five degrees of freedom
  - $\mathbf{R} \Rightarrow 3$, $\mathbf{t} \Rightarrow 3$, homogeneous $\Rightarrow -1$
  - It can be estimated from as little as five point correspondences $\mathbf{x'}^b_n \leftrightarrow \mathbf{x}^a_n$

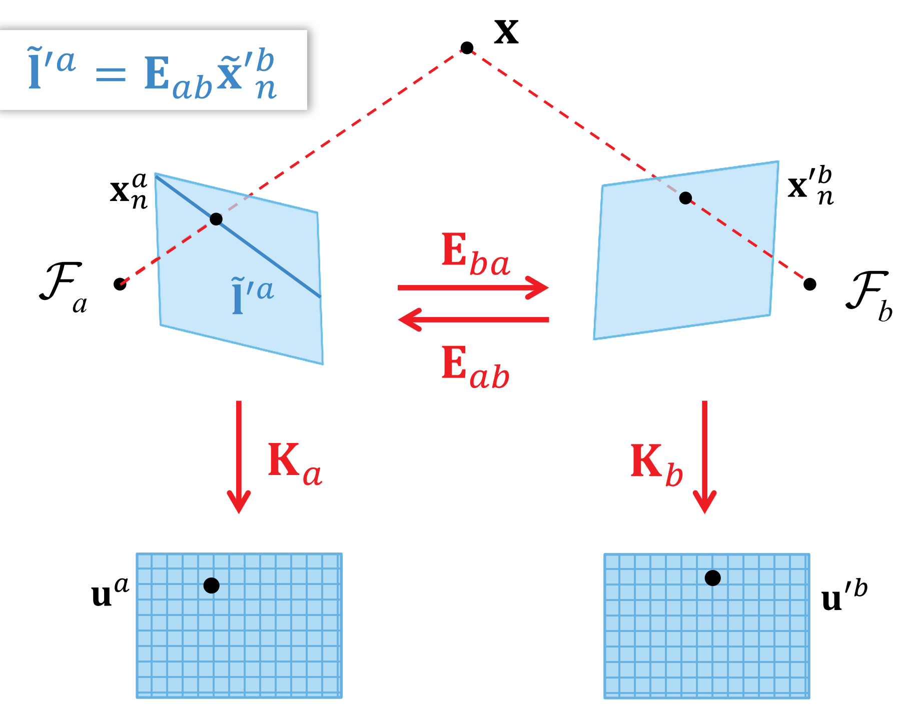


$$\left(\tilde{\mathbf{x}}^a_n\right)^T \mathbf{E}_{ab} \tilde{\mathbf{x}}'^b_n = 0$$

$$\left(\tilde{\mathbf{x}}'^b_n\right)^T \mathbf{E}_{ba} \tilde{\mathbf{x}}^a_n = 0$$

Properties of E

- $\tilde{\mathbf{l}}'^{a} = \mathbf{E}_{ab}\tilde{\mathbf{x}}'^{b}_{n}$ is the homogeneous representation of the epipolar line in the normalized image plane of $\mathcal{F}_a$ corresponding to the point $\mathbf{x}'^{b}_{n}$



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Line in $\mathbb{R}^2$:

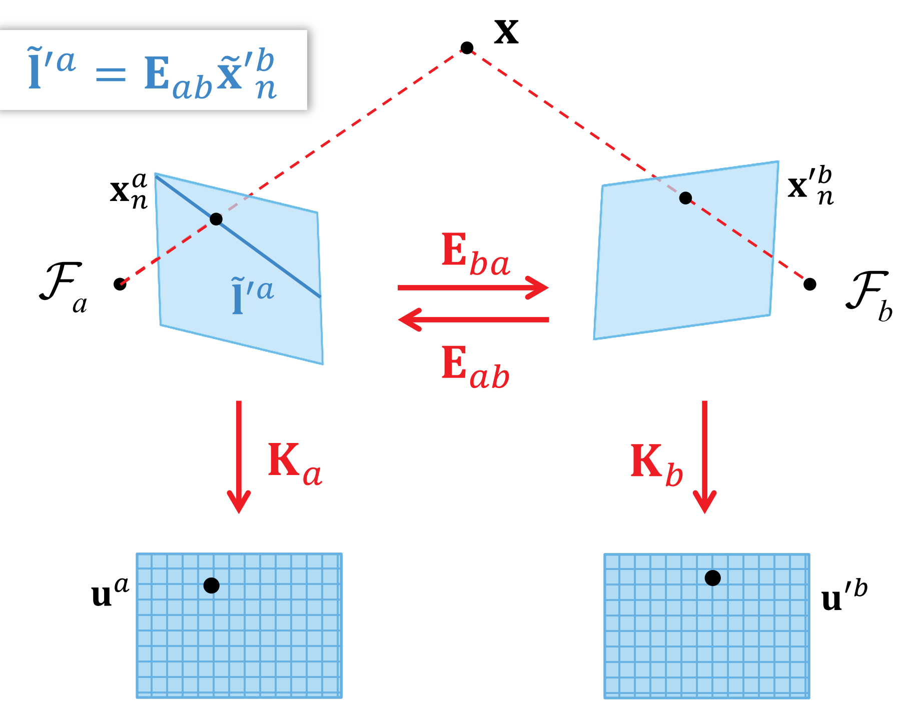
$$ax + by + c = 0$$

Line in $\mathbb{P}^2$:

$$\tilde{\mathbf{x}}^T \tilde{\mathbf{l}} = 0$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}^T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\tilde{\mathbf{l}}'^a = \mathbf{E}_{ab}\tilde{\mathbf{x}}'^b_n$$



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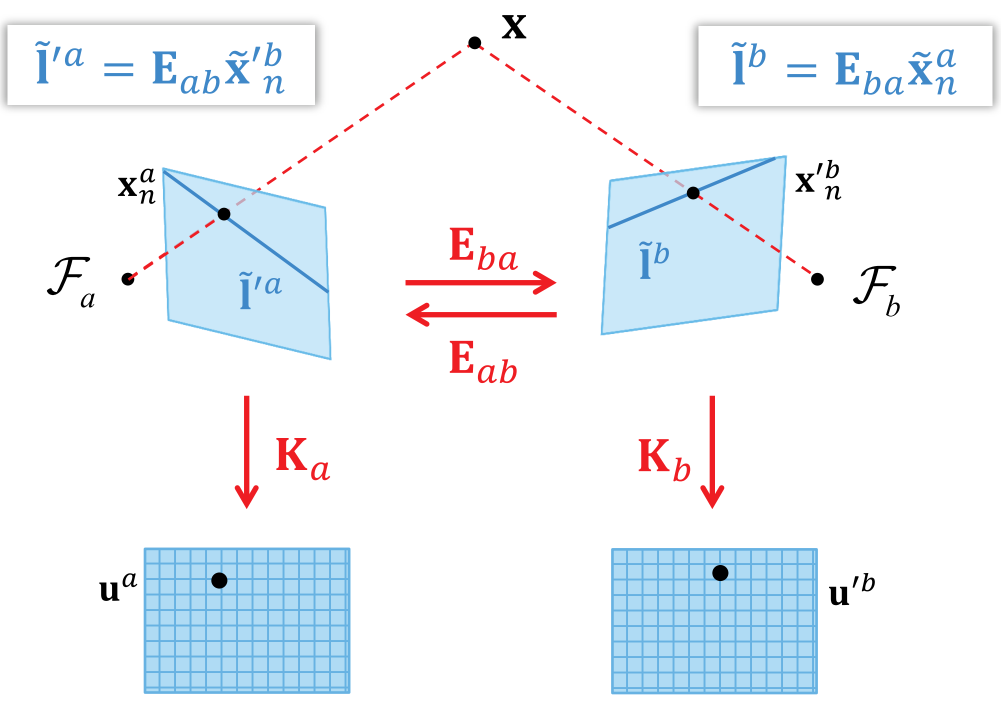
- $\tilde{\mathbf{l}}^{b} = \mathbf{E}_{ba}\tilde{\mathbf{x}}^{a}_{n}$ is the epipolar line in the normalized image plane of $\mathcal{F}_b$ corresponding to the point $\mathbf{x}^{a}_{n}$



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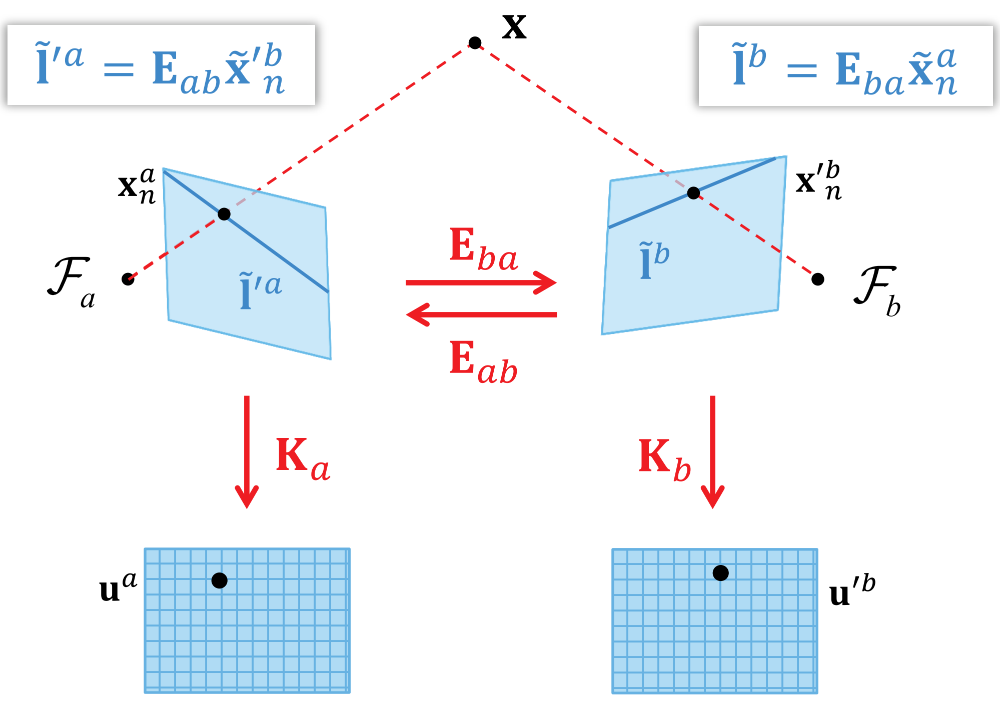
- $\tilde{\mathbf{l}}^{b} = \mathbf{E}_{ba}\tilde{\mathbf{x}}^{a}_{n}$ is the epipolar line in the normalized image plane of $\mathcal{F}_b$ corresponding to the point $\mathbf{x}^{a}_{n}$

- It is possible to determine $\mathbf{R}$ and $\mathbf{t}$ (up to scale) by decomposing $\mathbf{E}$



$$\left(\tilde{\mathbf{x}}^{a}_{n}\right)^{T}\mathbf{E}_{ab}\tilde{\mathbf{x}}'^{b}_{n} = 0$$

$$\left(\tilde{\mathbf{x}}'^{b}_{n}\right)^{T}\mathbf{E}_{ba}\tilde{\mathbf{x}}^{a}_{n} = 0$$

The fundamental matrix F

The epipolar constraint extends naturally to point correspondences $\mathbf{u}^a \leftrightarrow \mathbf{u}'^b$ via the camera calibration matrices $\mathbf{K}_a$ and $\mathbf{K}_b$

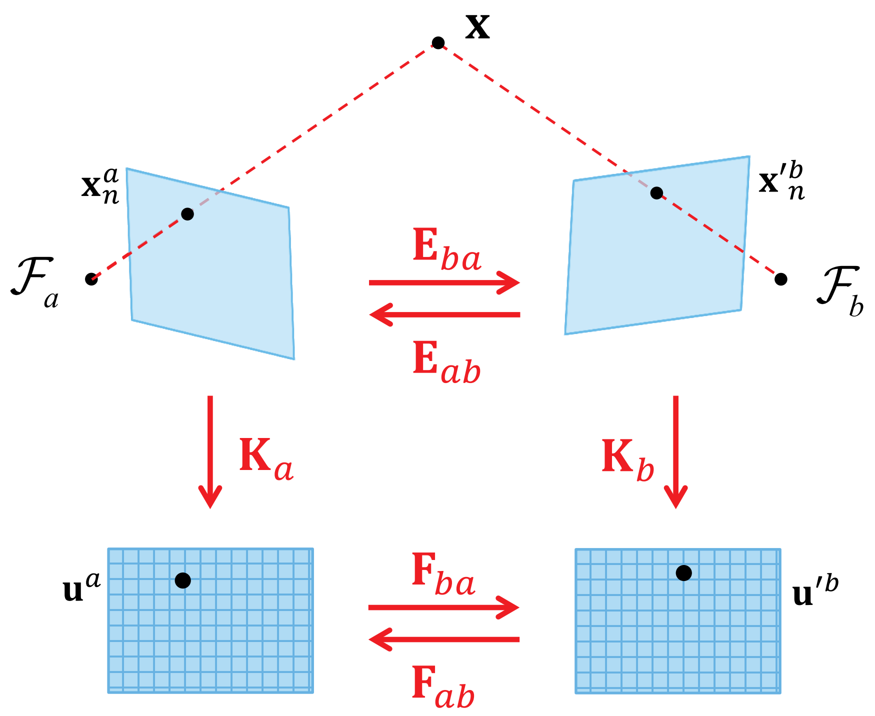
$$\left(\tilde{\mathbf{u}}^a\right)^T \mathbf{F}_{ab}\tilde{\mathbf{u}}'^b = 0$$

$$\left(\tilde{\mathbf{u}}'^b\right)^T \mathbf{F}_{ba}\tilde{\mathbf{u}}^a = 0$$

where the fundamental matrices $\mathbf{F}_{ab}$ and $\mathbf{F}_{ba}$ are given by

$$\mathbf{F}_{ab} = \mathbf{K}_a^{-T}\mathbf{E}_{ab}\mathbf{K}_b^{-1}$$

$$\mathbf{F}_{ba} = \mathbf{K}_b^{-T}\mathbf{E}_{ba}\mathbf{K}_a^{-1}$$



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$$\Downarrow \begin{cases} \tilde{\mathbf{x}}_n^a = \mathbf{K}_a^{-1}\tilde{\mathbf{u}}^a \\ \tilde{\mathbf{x}}_n^b = \mathbf{K}_b^{-1}\tilde{\mathbf{u}}^b \end{cases}$$

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From the equations it is clear that $\mathbf{F}_{ba} = \mathbf{F}_{ab}^T$, so that the two equations are equivalent representations of the same constraint

Properties of F

- $\mathbf{F}$ is homogeneous

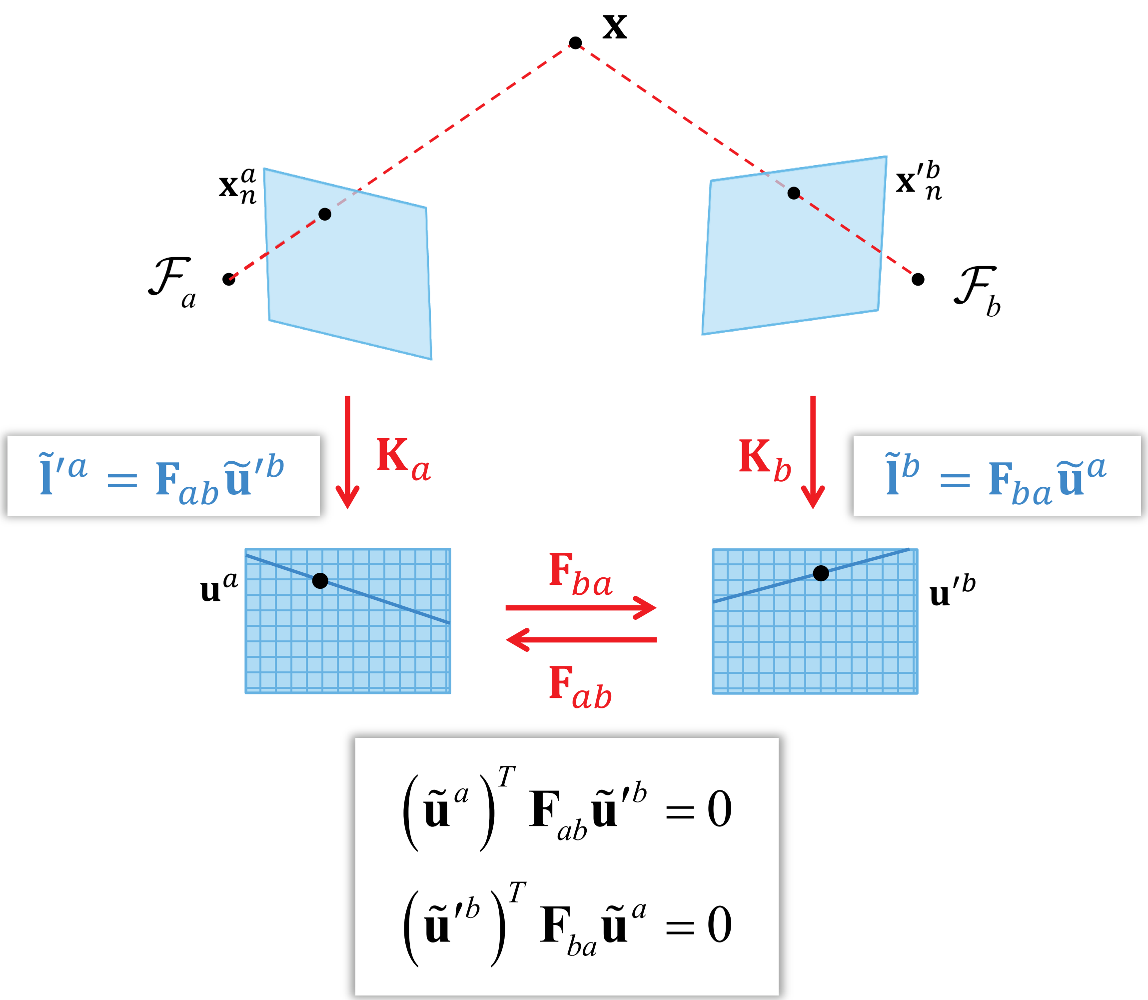
- $\text{rank}(\mathbf{F}) = 2$

- $\det(\mathbf{F}) = 0$

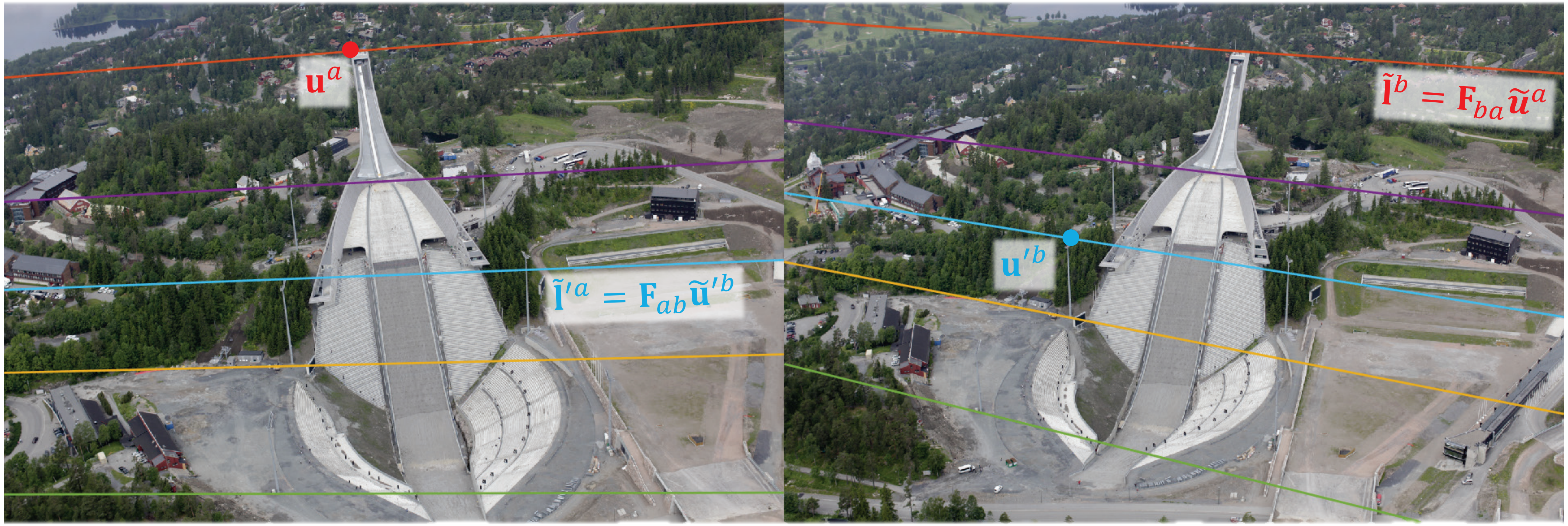
- $\mathbf{F}$ has seven degrees of freedom
  - It can be estimated from as little as seven point correspondences $\mathbf{u}^a \leftrightarrow \mathbf{u}'^b$

- Epipolar line corresponding to $\mathbf{u}'^b$ is
$$\tilde{\mathbf{l}}'^a = \mathbf{F}_{ab}\tilde{\mathbf{u}}'^b$$

- Epipolar line corresponding to $\tilde{\mathbf{u}}^a$ is
$$\tilde{\mathbf{l}}^b = \mathbf{F}_{ba}\tilde{\mathbf{u}}^a$$



Example



img_a img_b

Estimating F

Several algorithms can be used

- Linear: 7-pt, 8-pt
- Non-linear: Minimize total epipolar distance

Due to potential erroneous correspondences, it is natural to begin with a RANSAC estimation

The 8-pt algorithm is very similar to the homography estimation we have already seen

To simplify notations let us consider the correspondence $\mathbf{u} \leftrightarrow \mathbf{u}'$ and the fundamental matrix

$$\tilde{\mathbf{u}}'^T \mathbf{F} \tilde{\mathbf{u}} = 0$$

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For each point correspondence $\mathbf{u} \leftrightarrow \mathbf{u}'$ we have that

$$\tilde{\mathbf{u}}'^T \mathbf{F} \tilde{\mathbf{u}} = 0$$

$$\begin{bmatrix} u' & v' & 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} uu' & vu' & u' & uv' & vv' & v' & u & v & 1 \end{bmatrix} \mathbf{f} = 0$$

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From several correspondences, we get a system of linear equations that we can solve by SVD

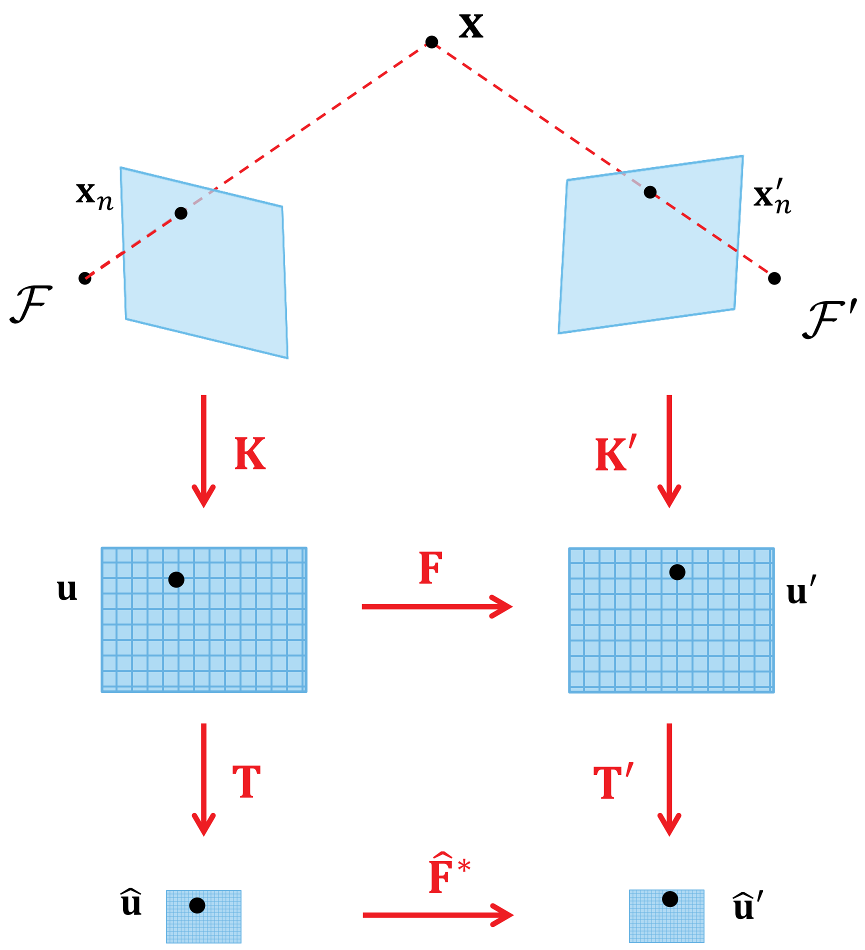
$$\begin{bmatrix} u_1 u_1' & v_1 u_1' & u_1' & u_1 v_1' & v_1 v_1' & v_1' & u_1 & v_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_k u_k' & v_k u_k' & u_k' & u_k v_k' & v_k v_k' & v_k' & u_k & v_k & 1 \end{bmatrix} \mathbf{f} = 0$$

$$\mathbf{A}\mathbf{f} = 0$$

Estimating F – The 8-point algorithm

Given eight (or more) correspondences $\mathbf{u}_i \leftrightarrow \mathbf{u}_i{}'$

1. Normalize point sets $\{\mathbf{u}_i\}$ and $\{\mathbf{u}_i{}'\}$ using similarity transforms $\mathbf{T}$ and $\mathbf{T}'$
2. Build matrix $\mathbf{A}$ from point-correspondences and compute its SVD
3. Extract the estimate $\hat{\mathbf{F}}$ from the right singular vector corresponding to the smallest singular value
4. Perform SVD on $\hat{\mathbf{F}}$:
$$\hat{\mathbf{F}} = \mathbf{USV}^T$$
5. Enforce zero determinant by setting the smallest singular value (s_{33} in $\mathbf{S}$) to zero and compute a proper fundamental matrix
$$\hat{\mathbf{F}}^* = \mathbf{USV}^T$$
6. Denormalize
$$\mathbf{F} = \mathbf{T}'^T \hat{\mathbf{F}}^* \mathbf{T}$$



Estimating F – The 7-point algorithm

Given seven correspondences $\mathbf{u}_i \leftrightarrow \mathbf{u}_i{}'$

- The matrix $\mathbf{A}$ is a 7×9 matrix, so in general $\mathrm{rank}(\mathbf{A}) = 7$ and the null space of $\mathbf{A}$ is 2-dimensional

- Then the fundamental matrix must be a linear combination of the two right singular vectors of $\mathbf{A}$ which correspond to the two singular values that are zero

$$\mathbf{F}(\alpha) = \alpha\mathbf{F}_1 + (1 - \alpha)\mathbf{F}_2$$

- The additional constraint $\det(\mathbf{A}) = 0$ leads to a cubic polynomial equation in α which has one or three solutions

- Hence the 7-pt algorithm returns one or three possible fundamental matrices

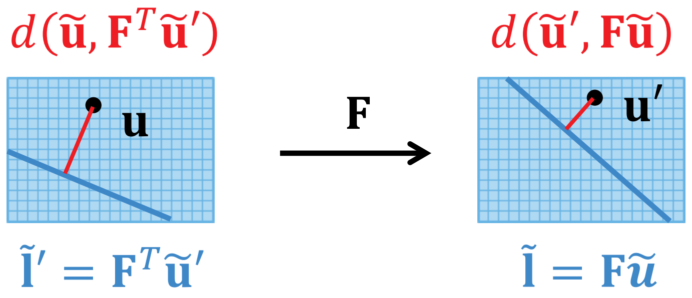
- In a RANSAC scheme, the 7-pt algorithm is better than the 8-pt algorithm
  - It is minimal, since we only need to sample seven random correspondences per iteration
  - Each sampled set of correspondences can return three fundamental matrices for testing

Estimating F – Beyond linear estimation

Improved estimates of $\mathbf{F}$ can be obtained using iterative methods

One possibility is to determine the matrix $\mathbf{F}$ that minimizes the total squared **epipolar distance**

$$\varepsilon = \sum_i d\left(\tilde{\mathbf{u}}_i', \mathbf{F}\tilde{\mathbf{u}}_i\right)^2 + d\left(\tilde{\mathbf{u}}_i, \mathbf{F}^T\tilde{\mathbf{u}}_i'\right)^2$$



The distance between a homogeneous point $\tilde{\mathbf{u}}$ and a homogeneous line $\tilde{\mathbf{l}} = [\tilde{l}_1 \quad \tilde{l}_2 \quad \tilde{l}_3]^T$ is

$$d\left(\tilde{\mathbf{u}}, \tilde{\mathbf{l}}\right) = \frac{\tilde{\mathbf{u}}^T\tilde{\mathbf{l}}}{\sqrt{\tilde{l}_1^2 + \tilde{l}_2^2}}$$

Iterative methods typically achieve a noticeably better estimate than the linear methods

But linear methods typically provide quite good estimates

Estimating E

For calibrated cameras we can first estimate $\mathbf{F}$ and then compute $\mathbf{E}$ by

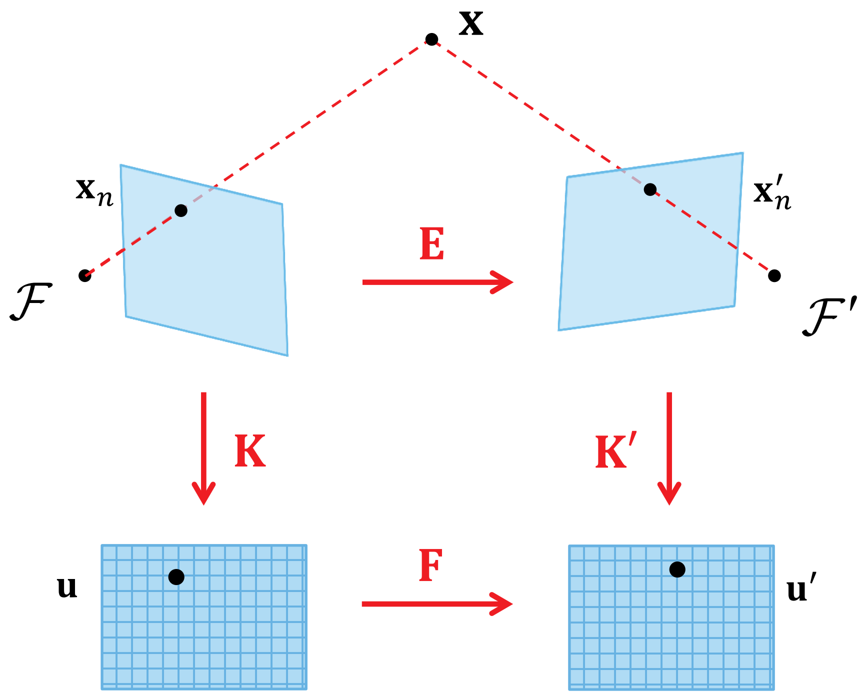
$$\mathbf{E} = \mathbf{K}'^T \mathbf{F} \mathbf{K}$$

One can also estimate $\mathbf{E}$ directly from five normalized point correspondences $\mathbf{x}_n \leftrightarrow \mathbf{x}'_n$ using an algorithm called the **5-pt algorithm**

- Involves finding the roots of a 10^{th} degree polynomial

In a RANSAC scheme, the 5-pt algorithm is the preferred alternative

- To achieve 99% confidence with 50% outliers, requires 145 tests with using the 5-pt algorithm versus 1177 tests using the 8-pt algorithm



Summary

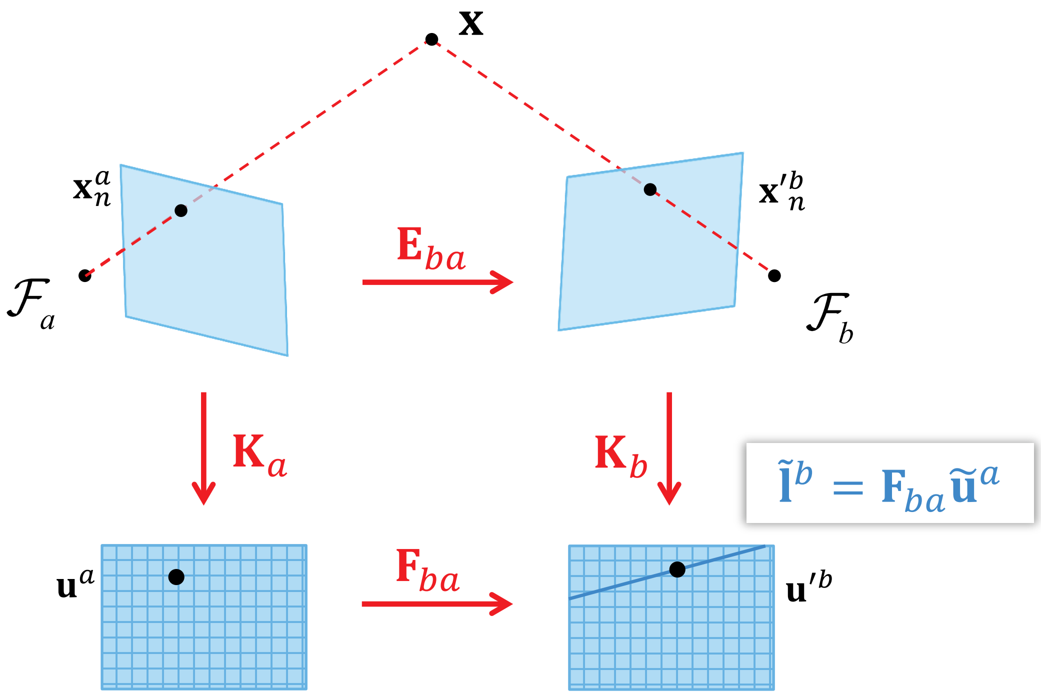
- The essential matrix $\mathbf{E}$ and the fundamental matrix $\mathbf{F}$ represent the epipolar geometry

$$\left(\tilde{\mathbf{x}}_n'^b\right)^T \mathbf{E}_{ba}\tilde{\mathbf{x}}_n^a = 0 \qquad \left(\tilde{\mathbf{u}}'^b\right)^T \mathbf{F}_{ba}\tilde{\mathbf{u}}^a = 0$$

- $\mathbf{E}$ and $\mathbf{F}$ can be estimated from correspondences
  - $\mathbf{F} \leftarrow$ RANSAC, 7-pt or 8-pt
  - $\mathbf{E} \leftarrow$ RANSAC, 5-pt

- $\mathbf{E}$ and $\mathbf{F}$ maps points to their corresponding epipolar lines

$$\tilde{\mathbf{l}}^b = \mathbf{F}_{ba}\tilde{\mathbf{u}}^a$$



$$\mathbf{F}_{ba} = \mathbf{K}_b^{-T}\mathbf{E}_{ba}\mathbf{K}_a^{-1}$$

$$\mathbf{F}_{ab} = \mathbf{K}_a^{-T}\mathbf{E}_{ab}\mathbf{K}_b^{-1}$$

Further reading

- Online book by Richard Szeliski – Computer Vision: Algorithms and Applications
  http://szeliski.org/Book/drafts/SzeliskiBook_20100903_draft.pdf
  – Chapter 7.2 covers two-frame structure from motion where the essential matrix and the fundamental matrix are central quantities in the discussion

- Online book by Timothy D. Barfoot – State Estimation for Robotics
  http://asrl.utias.utoronto.ca/~tdb/bib/barfoot_ser17.pdf
  – Chapter 6.4 covers the essential matrix and the fundamental matrix ++

- David Nistér, *An Efficient Solution to the Five-Point Relative Pose Problem,* 2004
  http://www.ee.oulu.fi/research/imag/courses/Sturm/nister04.pdf

- Richard I. Hartley, *In Defense of the Eight-Point Algorithm,* 1997
  https://www.cse.unr.edu/~bebis/CS485/Handouts/hartley.pdf