UiO : Department of Technology Systems
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Basic projective geometry

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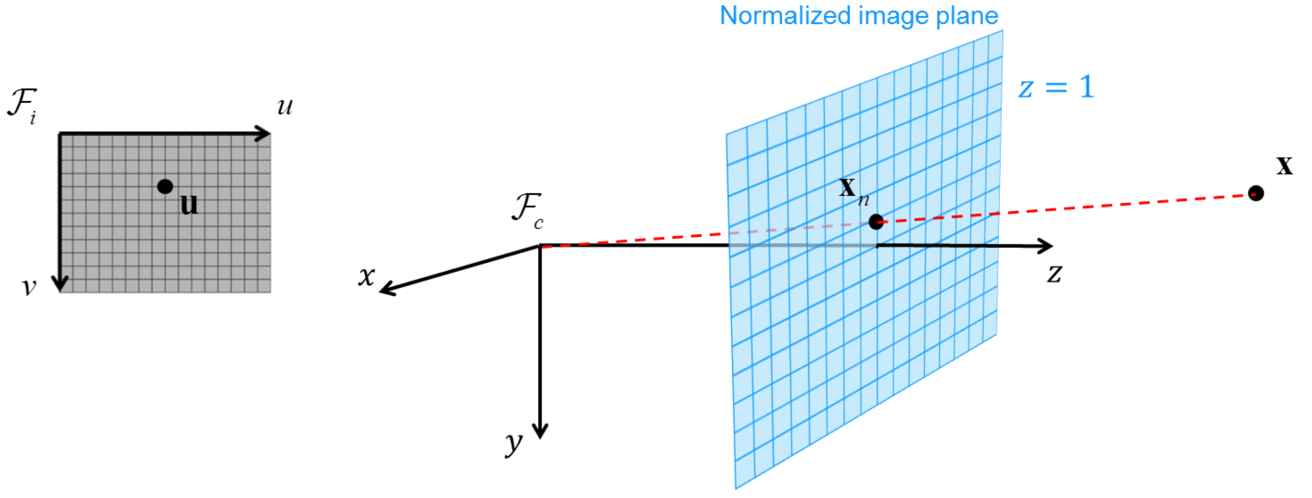
Motivation

- Projective geometry is an alternative to Euclidean geometry

- Many results, derivations and expressions in computer vision are easiest described in the projective framework
  - The perspective camera model



Projective representation versus Euclidean

$$\tilde{\mathbf{u}} = \begin{bmatrix} f_u & s & c_u \\ 0 & f_v & c_v \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tilde{\mathbf{x}}$$

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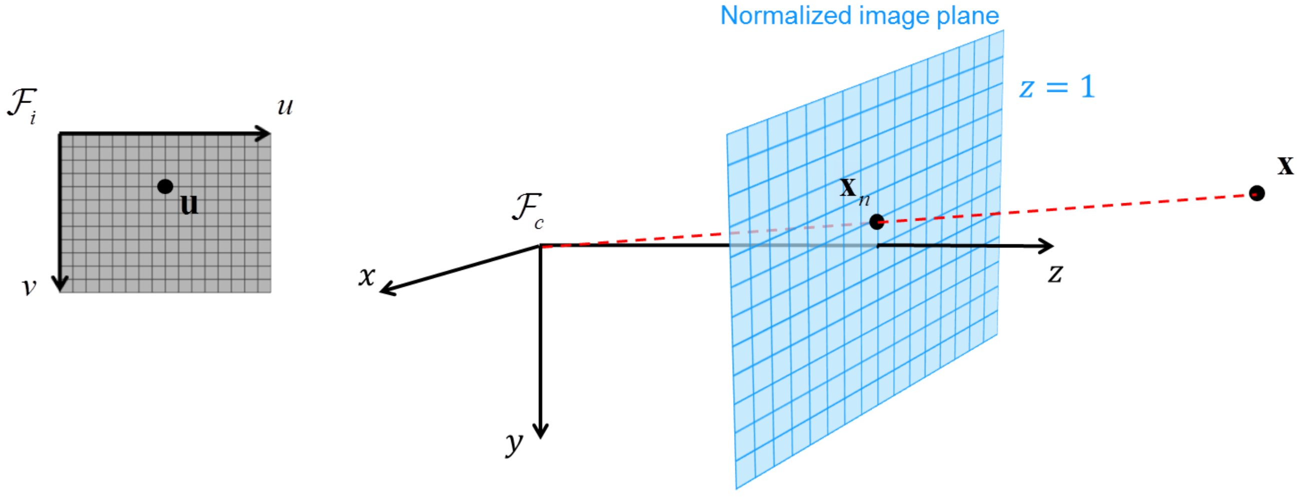

Projective representation versus **Euclidean**

$$\mathbf{u} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} f_u & s & c_u \\ 0 & f_v & c_v \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{z} \mathbf{x}$$

$$\mathbf{u} = \begin{bmatrix} f_u \frac{x}{z} + s \frac{y}{z} + c_u \\ f_v \frac{y}{z} + c_v \end{bmatrix}$$

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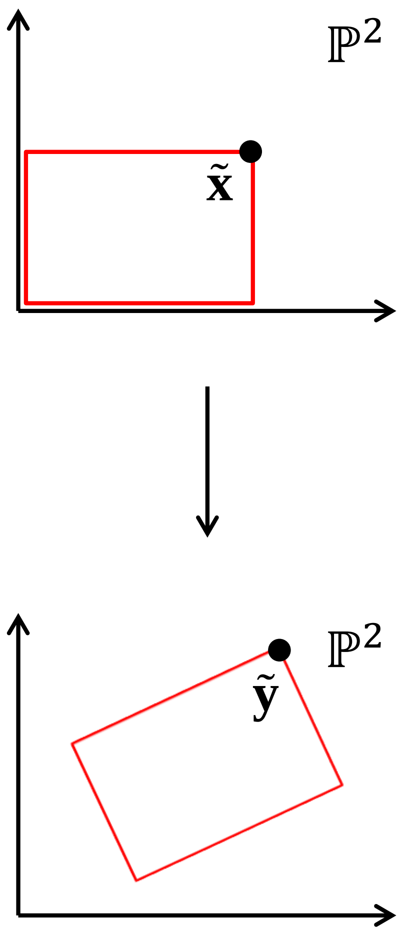
  - Transformations



$$\tilde{\mathbf{x}} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix}^{-1} \tilde{\mathbf{y}}$$

LINEAR

$$\tilde{\mathbf{y}} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix} \tilde{\mathbf{x}}$$

Projective representation versus Euclidean

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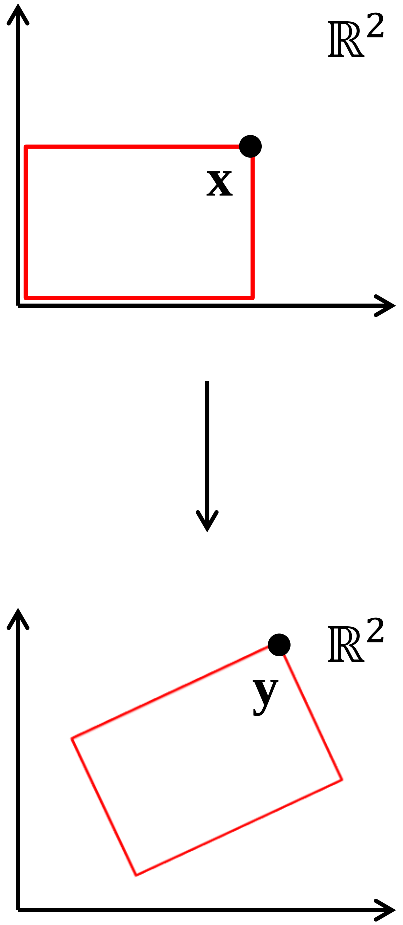


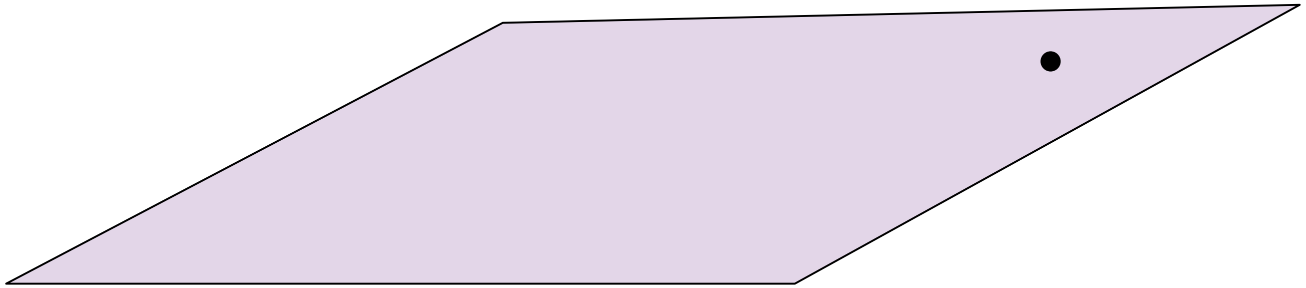
$$\mathbf{x} = \mathbf{R}^{-1}(\mathbf{y} - \mathbf{t})$$

NON LINEAR

$$\mathbf{y} = \mathbf{R}\mathbf{x} + \mathbf{t}$$

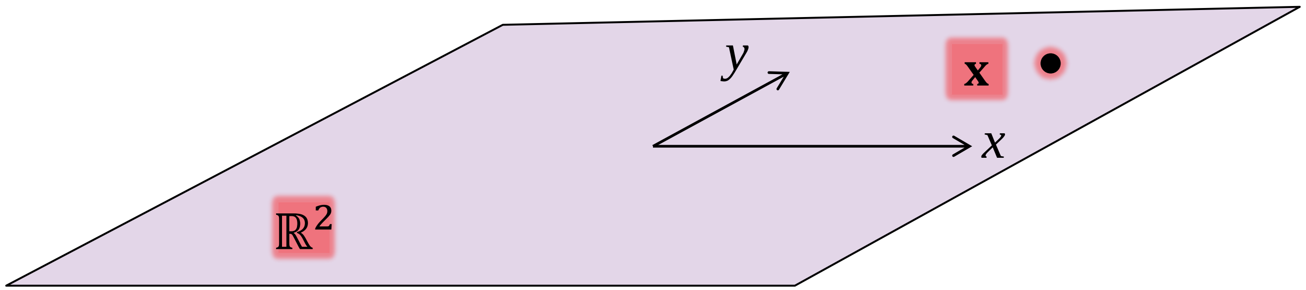
Projective representation versus **Euclidean**

Points in the projective plane $\mathbb{P}^2$



How to describe points in the plane?

Points in the projective plane $\mathbb{P}^2$



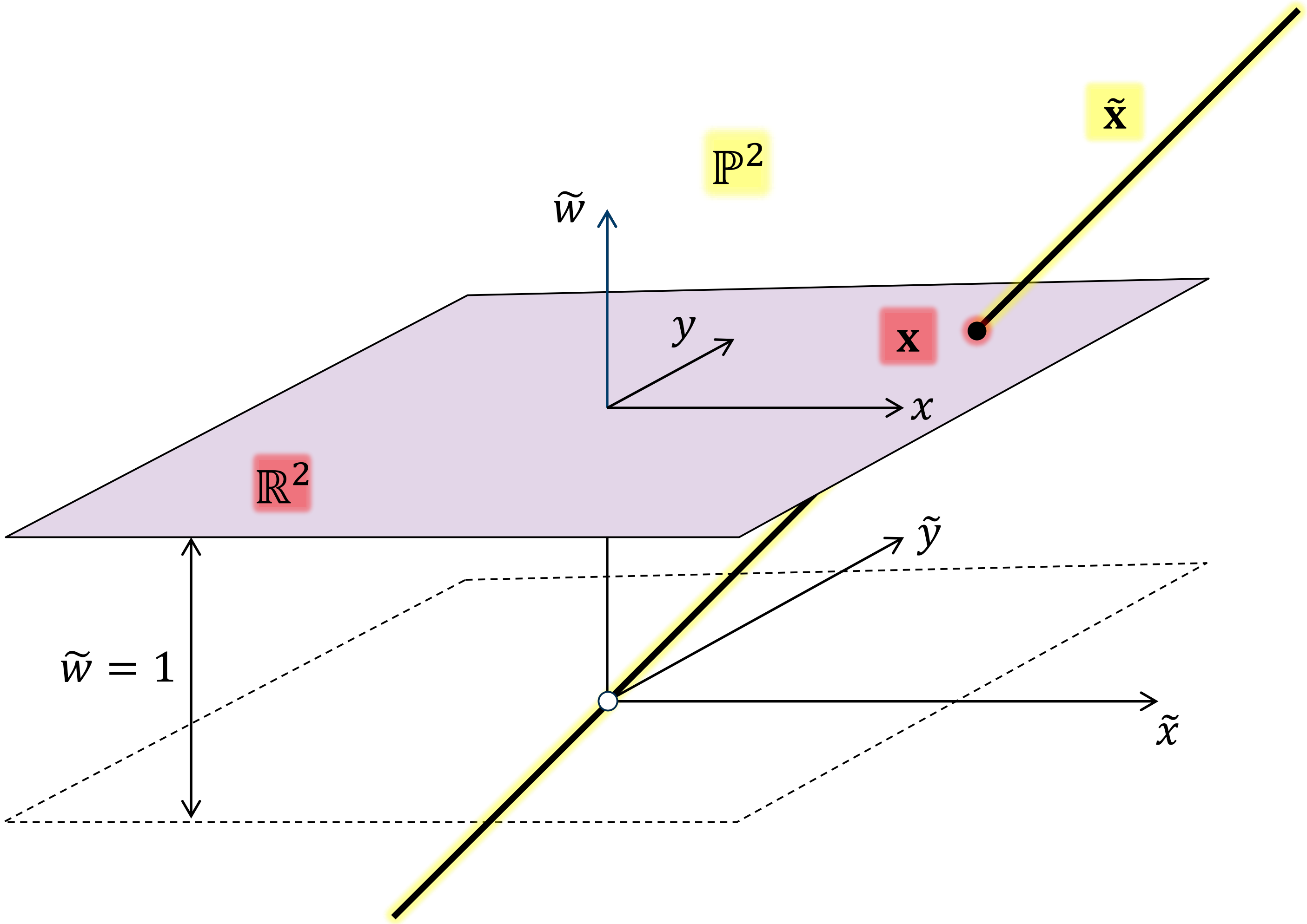
How to describe points in the plane?

Euclidean plane $\mathbb{R}^2$

- Choose a 2D coordinate frame
- Points have 2 unique coordinates

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$$

Points in the projective plane $\mathbb{P}^2$



How to describe points in the plane?

Euclidean plane $\mathbb{R}^2$

- Choose a 2D coordinate frame
- Points have 2 unique coordinates

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$$

Projective plane $\mathbb{P}^2$

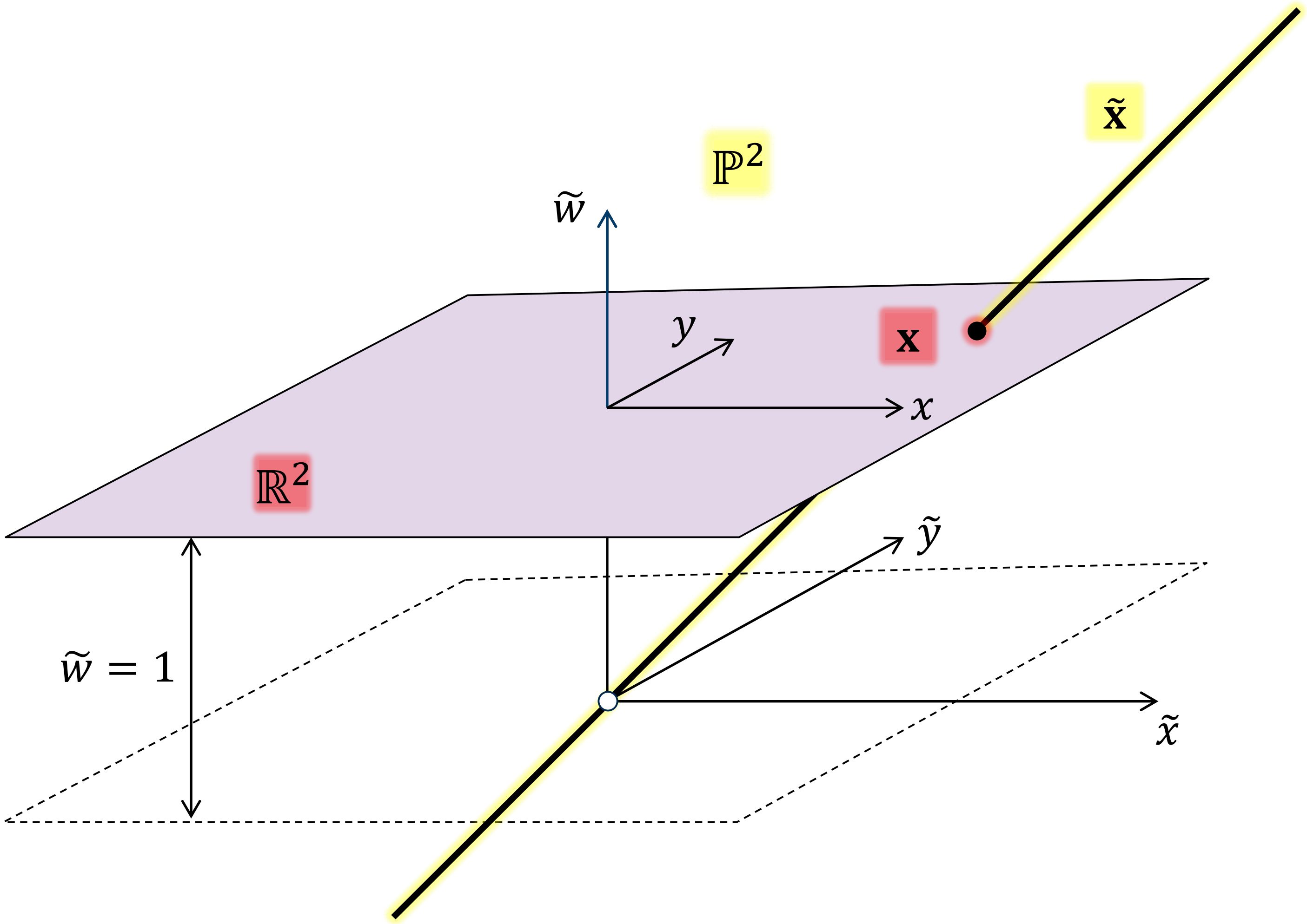
- Expand coordinate frame to 3D
- Points have 3 **homogeneous coordinates**

$$\tilde{\mathbf{x}} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix} \in \mathbb{P}^2$$

where

$$\tilde{\mathbf{x}} = \lambda \tilde{\mathbf{x}} \ \forall \ \lambda \in \mathbb{R} \backslash \{0\}$$

Points in the projective plane $\mathbb{P}^2$



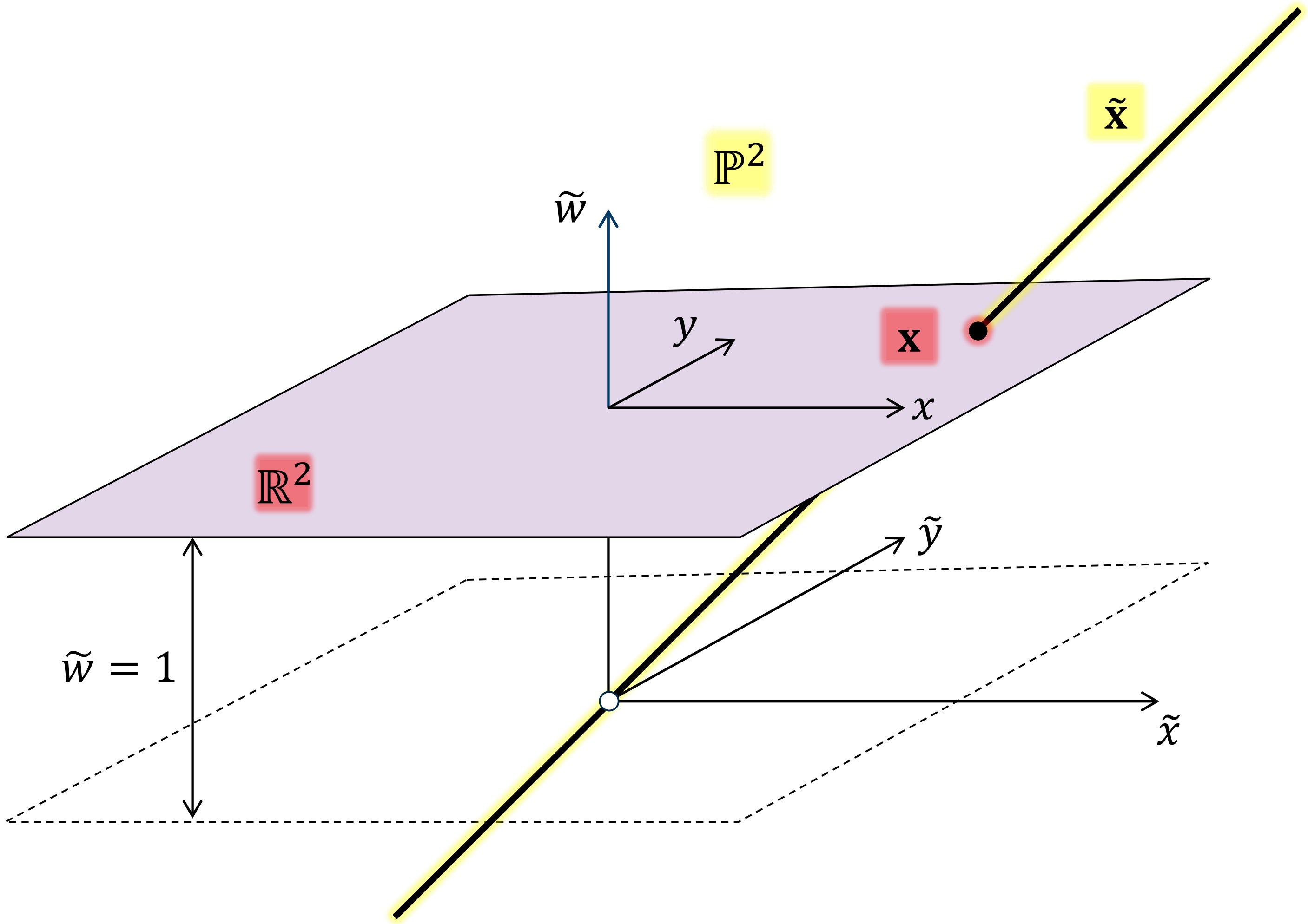
Observations

1. Any point $\mathbf{x} = [x, y]^T$ in the Euclidean plane has a corresponding homogeneous point $\tilde{\mathbf{x}} = [x, y, 1]^T$ in the projective plane

2. Homogeneous points of the form $[\tilde{x}, \tilde{y}, 0]^T$ does not have counterparts in the Euclidean plane

   They correspond to **points at infinity**

Points in the projective plane $\mathbb{P}^2$



Observations

3. When we work with geometrical problems in the plane, we can switch between the Euclidean representation and the projective representation

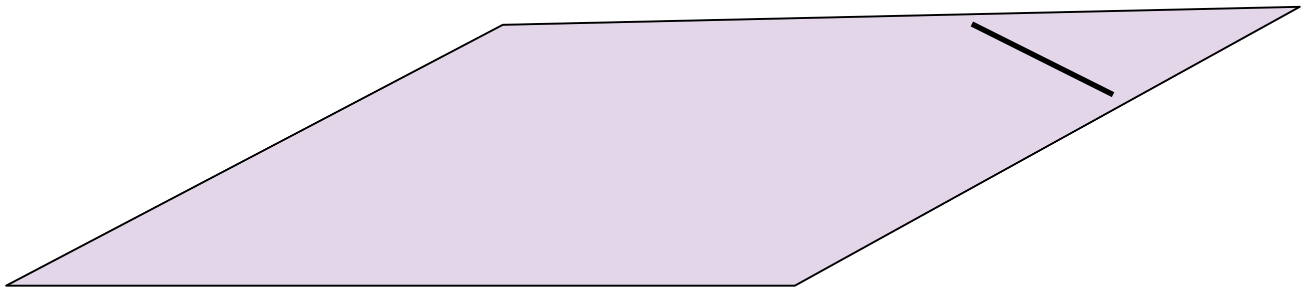
$$\mathbb{R}^2 \rightarrow \mathbb{P}^2$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbb{P}^2 \rightarrow \mathbb{R}^2$$

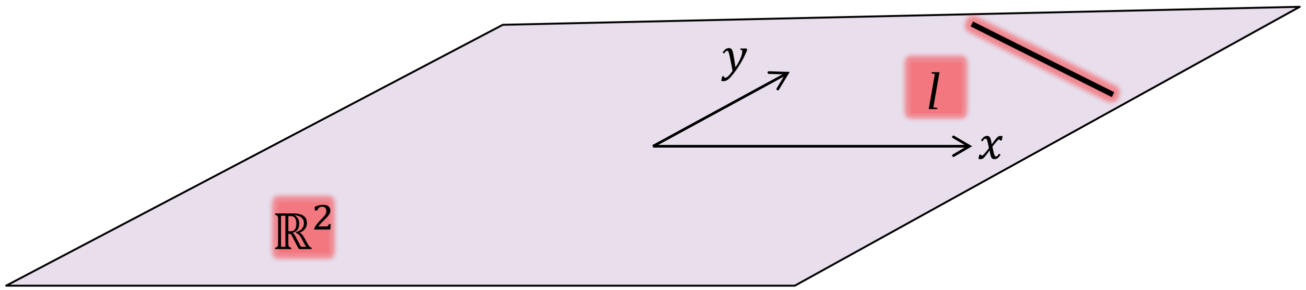
$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix} \mapsto \begin{bmatrix} \frac{\tilde{x}}{\tilde{w}} \\ \frac{\tilde{y}}{\tilde{w}} \end{bmatrix}$$

Lines in the projective plane $\mathbb{P}^2$



How to describe lines in the plane?

Lines in the projective plane $\mathbb{P}^2$



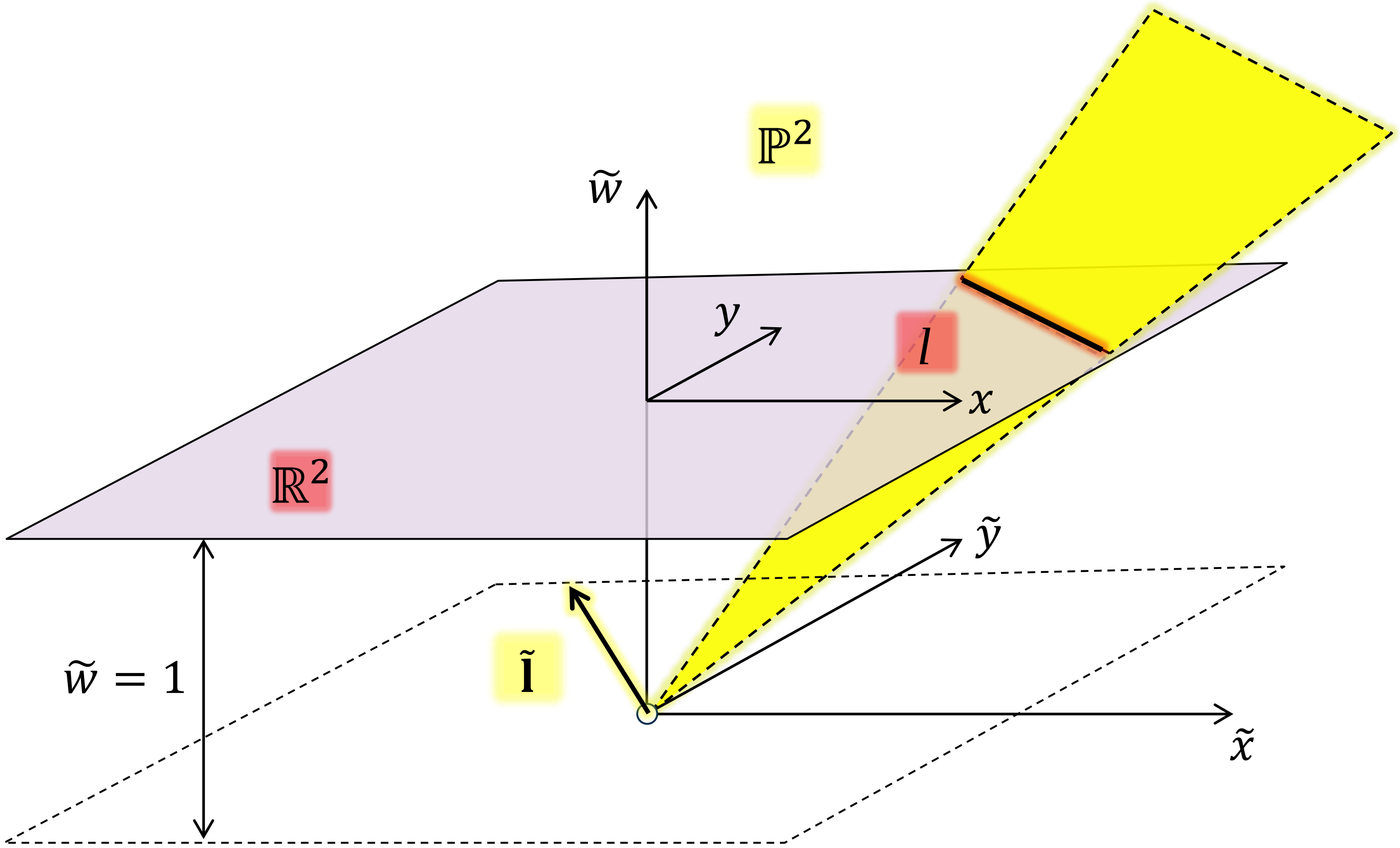
How to describe lines in the plane?

Euclidean plane $\mathbb{R}^2$

- 3 parameters $a, b, c \in \mathbb{R}$

$$l = \{(x, y) \mid ax + by + c = 0\}$$

Lines in the projective plane $\mathbb{P}^2$



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- 3 parameters $a, b, c \in \mathbb{R}$

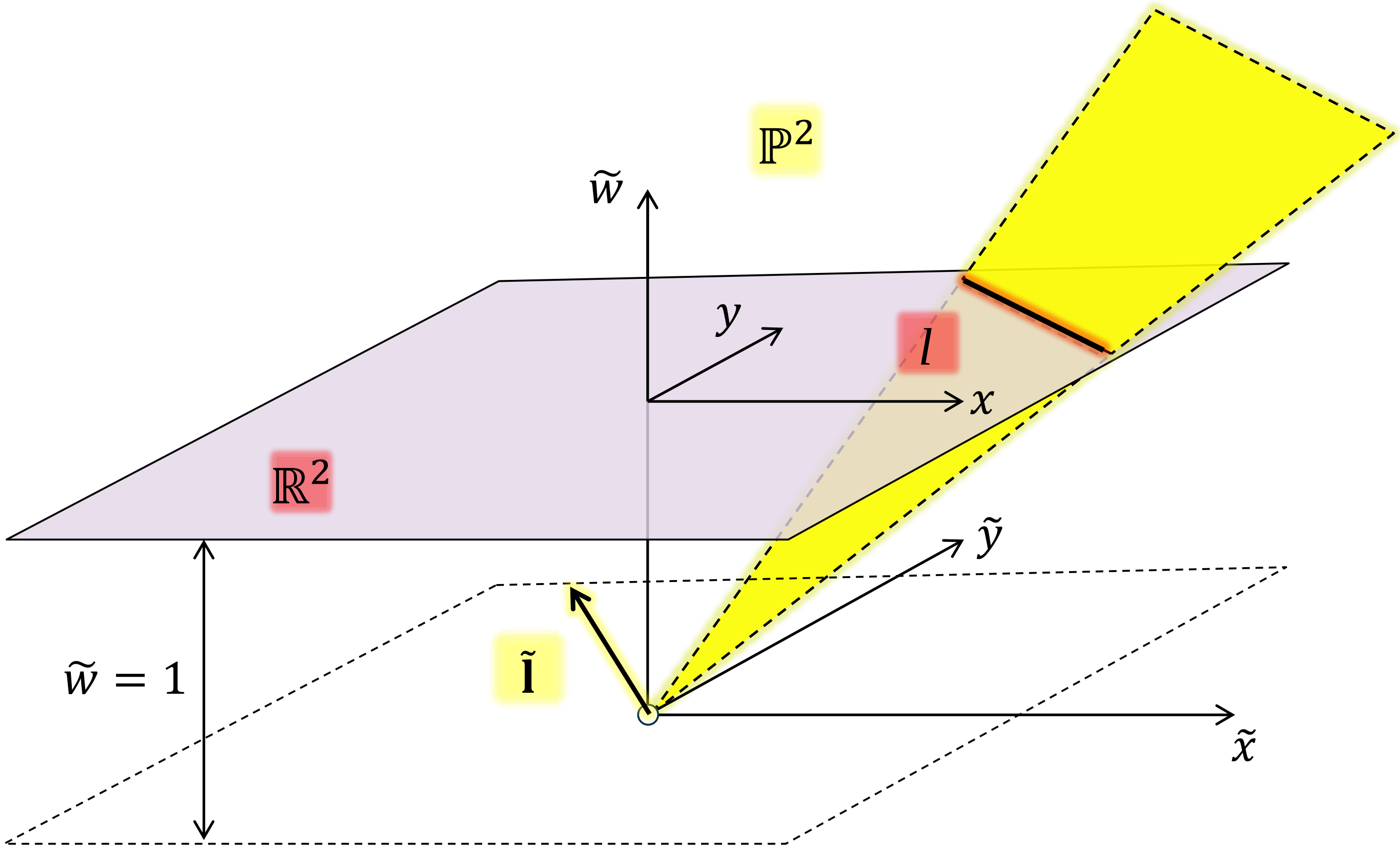
$$l = \{(x, y) \mid ax + by + c = 0\}$$

Projective plane $\mathbb{P}^2$

- Homogeneous vector $\tilde{\mathbf{l}} = [a, b, c]^T$

$$l = \left\{\tilde{\mathbf{x}} \in \mathbb{P}^2 \mid \tilde{\mathbf{l}}^T \tilde{\mathbf{x}} = 0\right\}$$

Lines in the projective plane $\mathbb{P}^2$



Observations

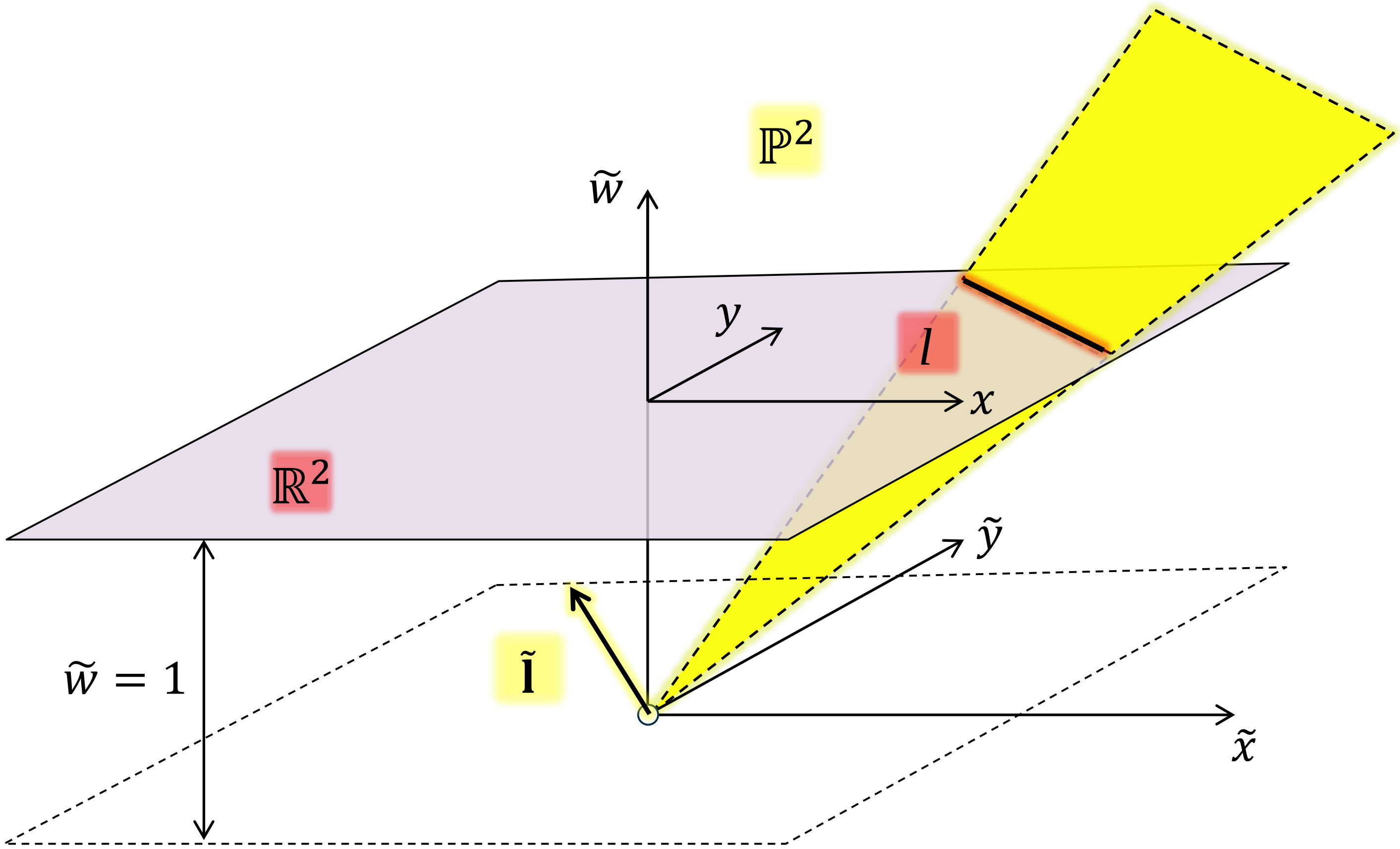
1. Points and lines in the projective plane have the same representation, we say that points and lines are dual objects in $\mathbb{P}^2$

2. All lines in the Euclidean plane have a corresponding line in the projective plane

3. The line $\tilde{\mathbf{l}} = [0,0,1]^T$ in the projective plane does not have an Euclidean counterpart

   This line consists entirely of ideal points, and is know as *the line at infinity*

Lines in the projective plane $\mathbb{P}^2$



Properties of lines in the projective plane

1. In the projective plane, all lines intersect, parallel lines intersect at infinity

   Two lines $\tilde{\mathbf{l}}_1$ and $\tilde{\mathbf{l}}_2$ intersect in the point
   $$\tilde{\mathbf{x}} = \tilde{\mathbf{l}}_1 \times \tilde{\mathbf{l}}_2$$

2. The line passing through points $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$ is given by
   $$\tilde{\mathbf{l}} = \tilde{\mathbf{x}}_1 \times \tilde{\mathbf{x}}_2$$

Example 1

$$\mathbf{x}_1 = (2, 4)$$
$$\mathbf{x}_2 = (5, 13)$$

Determine the line passing through the two points $\mathbf{x}_1$ and $\mathbf{x}_2$

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Homogeneous representation of the points

$$\tilde{\mathbf{x}}_1 = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \in \mathbb{P}^2 \qquad \tilde{\mathbf{x}}_2 = \begin{bmatrix} 5 \\ 13 \\ 1 \end{bmatrix} \in \mathbb{P}^2$$

Homogeneous representation of line

$$\tilde{\mathbf{l}} = \tilde{\mathbf{x}}_1 \times \tilde{\mathbf{x}}_2 = [\tilde{\mathbf{x}}_1]_\times \tilde{\mathbf{x}}_2 = \begin{bmatrix} 0 & -1 & 4 \\ 1 & 0 & -2 \\ -4 & 2 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 13 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

Equation of the line

$$-3x + y + 2 = 0 \quad \Leftrightarrow \quad y = 3x - 2$$

Matrix representation of the cross product

$$\mathbf{u} \times \mathbf{v} \;\mapsto\; [\mathbf{u}]_\times \mathbf{v}$$

where

$$[\mathbf{u}]_\times \stackrel{\text{def}}{=} \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

Example 2

$$y = x - 2$$
$$y = -2x + 3$$

At which point does these two lines intersect?

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$$y = x - 2$$
$$y = -2x + 3$$

At which point does these two lines intersect?

$$y = x - 2 \quad \mapsto \quad \tilde{\mathbf{l}}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \in \mathbb{P}^2 \qquad\qquad y = -2x + 3 \quad \mapsto \quad \tilde{\mathbf{l}}_2 = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix} \in \mathbb{P}^2$$

Point of intersection

$$\tilde{\mathbf{x}} = \tilde{\mathbf{l}}_1 \times \tilde{\mathbf{l}}_2 = [\tilde{\mathbf{l}}_1]_\times \tilde{\mathbf{l}}_2 = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ -3 \end{bmatrix} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} {}^{-5}/_{-3} \\ {}^{1}/_{-3} \end{bmatrix} \approx \begin{bmatrix} 1.67 \\ -0.33 \end{bmatrix}$$

Example 3

$$y = x - 2$$
$$y = x + 3$$

At which point does these two lines intersect?

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Euclidean geometry
Parallel lines never intersect!

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Point of intersection

$$\tilde{\mathbf{x}} = \tilde{\mathbf{l}}_1 \times \tilde{\mathbf{l}}_2 = [\tilde{\mathbf{l}}_1]_\times \tilde{\mathbf{l}}_2 = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \\ 0 \end{bmatrix}$$

Projective geometry
All lines intersect!

Parallel lines intersect at infinity

Example 4

Cameras can observe points that are "infinitely" far away

$$\tilde{\mathbf{u}} = \begin{bmatrix} f_u & s & c_u \\ 0 & f_v & c_v \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ \color{red}{0} \end{bmatrix}$$

In images of planar surfaces we can see how the surface converges towards a line

Any two parallel lines in the plane will appear to intersect on this line

Image: Flicker.com (Melita)

Example 4

Cameras can observe points that are "infinitely" far away

$$\tilde{\mathbf{u}} = \begin{bmatrix} f_u & s & c_u \\ 0 & f_v & c_v \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$$

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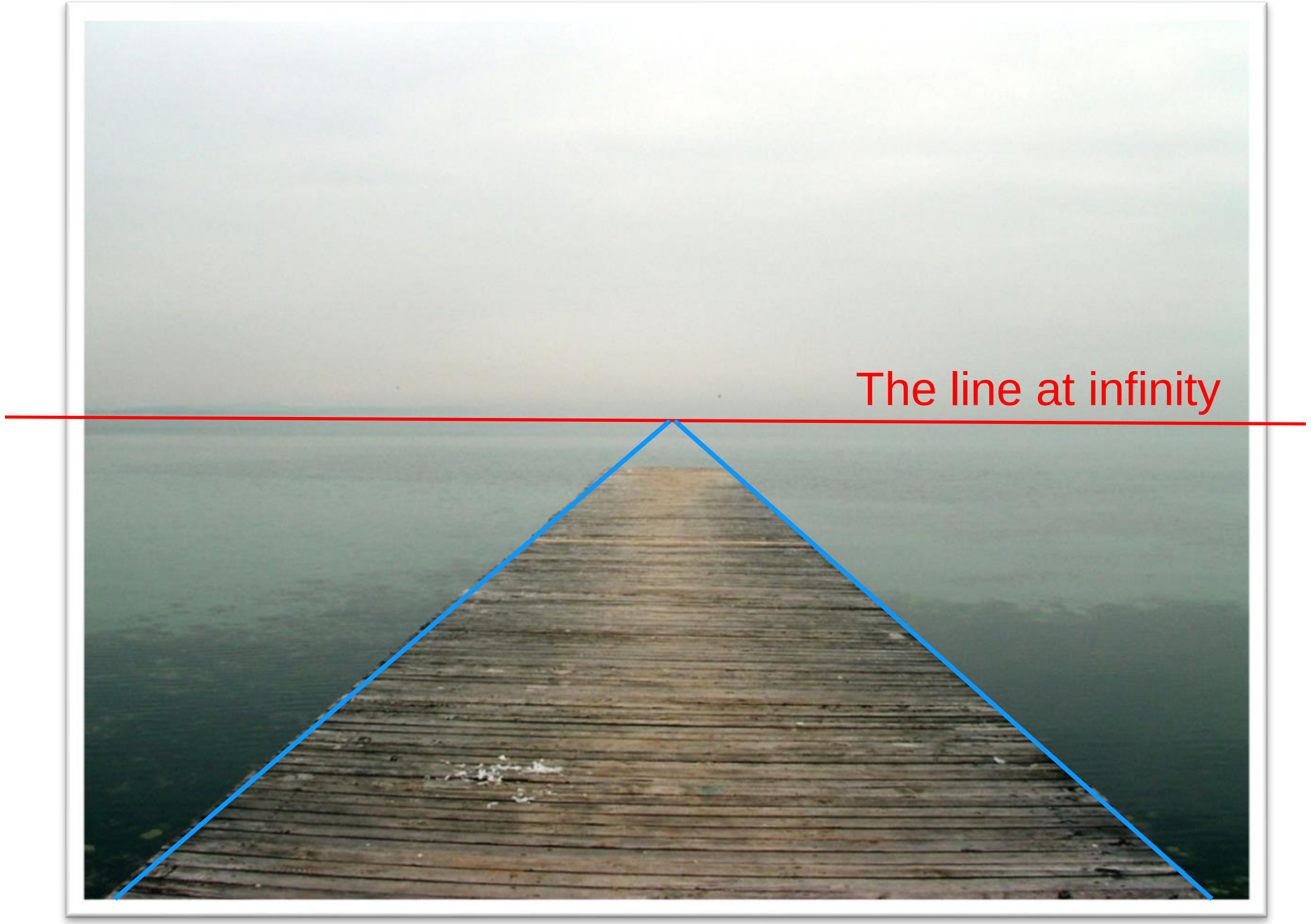
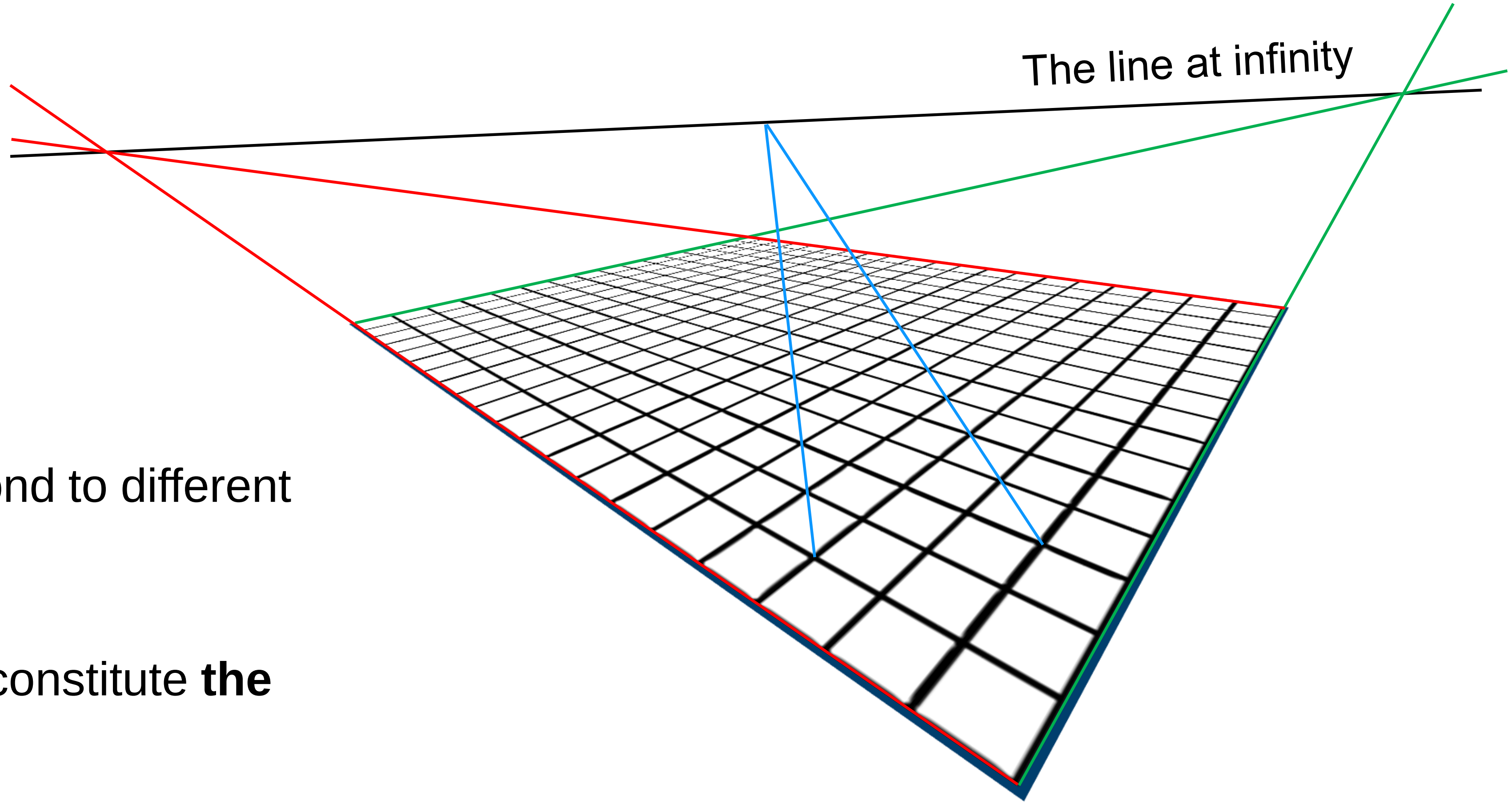


Image: Flicker.com (Melita)

Example 4



Different directions correspond to different points at infinity

The set of all infinite points constitute **the line at infinity**

Linear transformations of the projective plane $\mathbb{P}^2$

- A linear transformation of $\mathbb{P}^2$ can be represented by a invertible homogeneous 3×3 matrix

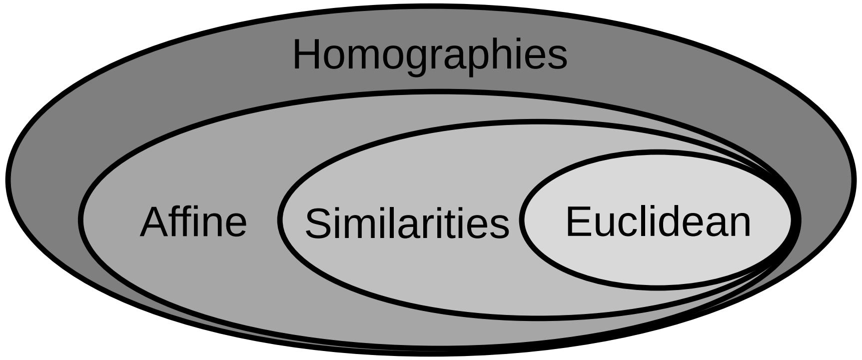
$$\begin{aligned} H: \quad \mathbb{P}^2 &\rightarrow \mathbb{P}^2 \\ \tilde{\mathbf{x}} &\mapsto \mathbf{H}\tilde{\mathbf{x}} \end{aligned}$$

  where

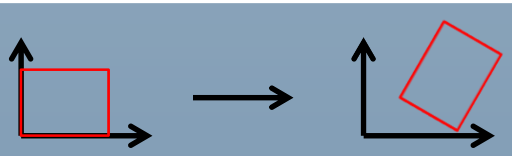
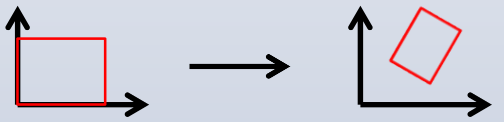
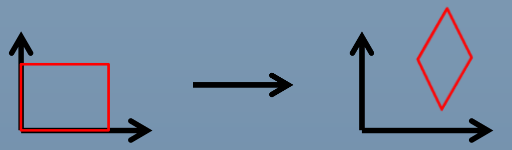
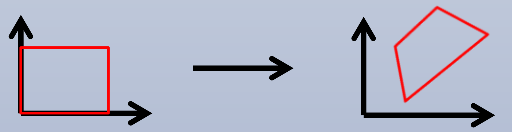
$$\mathbf{H} = \lambda\mathbf{H} \quad \forall \quad \lambda \in \mathbb{R} \backslash \{0\}$$

- Important groups of linear projective transformations

- Each group is closed under
  - Matrix multiplication
  - Matrix inverse



Linear transformations of the projective plane $\mathbb{P}^2$

Transformation	Matrix	#DoF	Preserves	Visualization
Euclidean	$\begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$	3	Lengths **+ all below**	
Similarity	$\begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \quad s \in \mathbb{R}$	4	Angles **+ all below**	
Affine	$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}$	6	Parallelism, line at infinity **+ all below**	
Homography	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$	8	Straight lines	

Linear transformations of the projective plane $\mathbb{P}^2$

- Several image operations correspond to a linear projective transformation
  - Rotation
  - Translation
  - Resizing

$$\mathbf{H} = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$$

$$\begin{aligned} s &= 1 \\ \mathbf{R} &\neq \mathbf{I} \\ \mathbf{t} &= \mathbf{0} \end{aligned}$$

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- Several image operations correspond to a linear projective transformation
  - Rotation
  - Translation
  - Resizing

$$\mathbf{H} = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$$

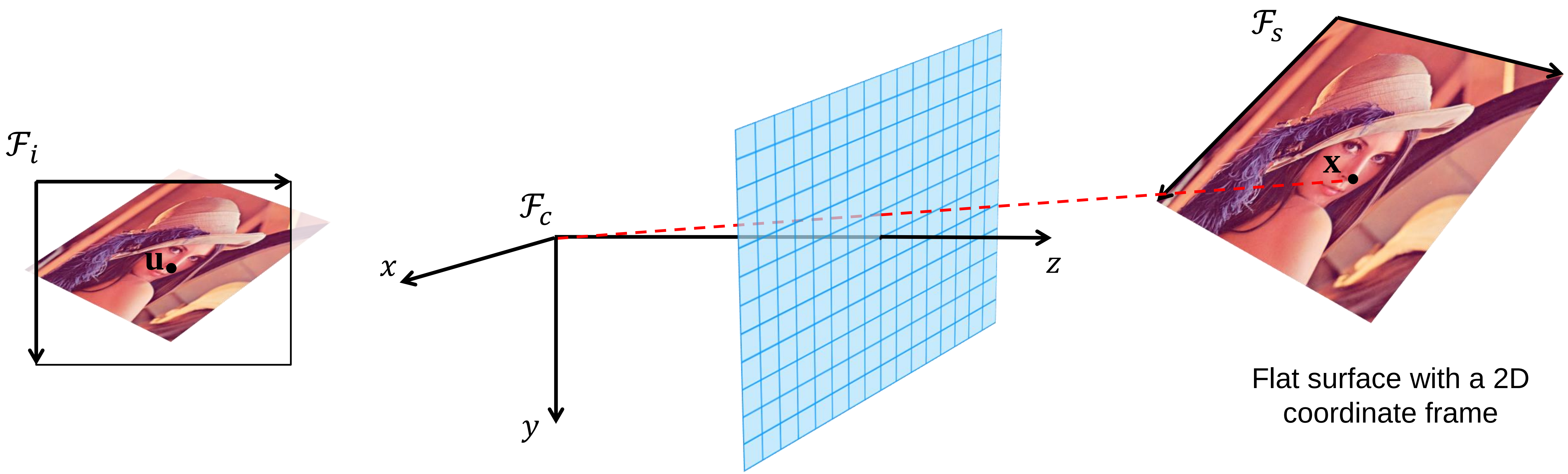
$$\longrightarrow$$

$$\begin{aligned} s &< 1 \\ \mathbf{R} &= \mathbf{I} \\ \mathbf{t} &= \mathbf{0} \end{aligned}$$

Linear transformations of the projective plane $\mathbb{P}^2$

- Perspective imaging of a flat surface can be described by a homography

$$\mathbf{H}\tilde{\mathbf{x}}^s = \tilde{\mathbf{u}}^i$$

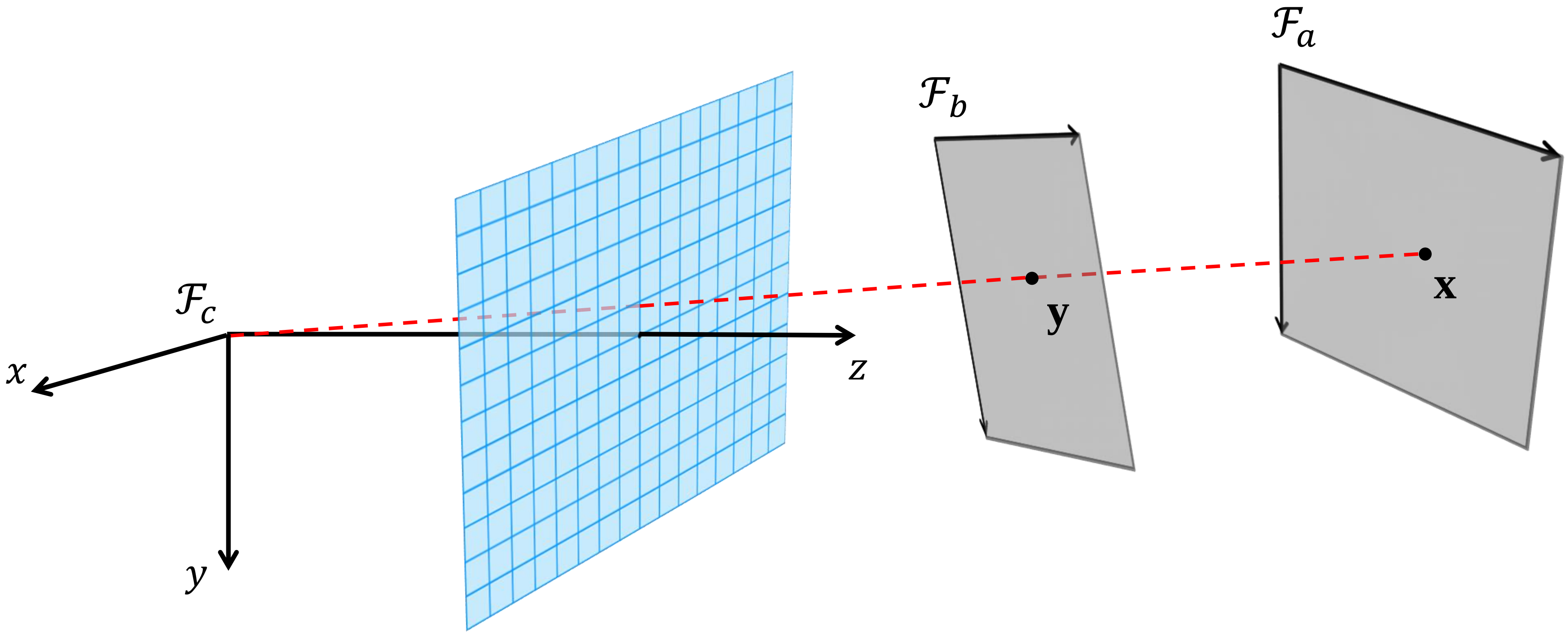


Flat surface with a 2D coordinate frame

Linear transformations of the projective plane $\mathbb{P}^2$

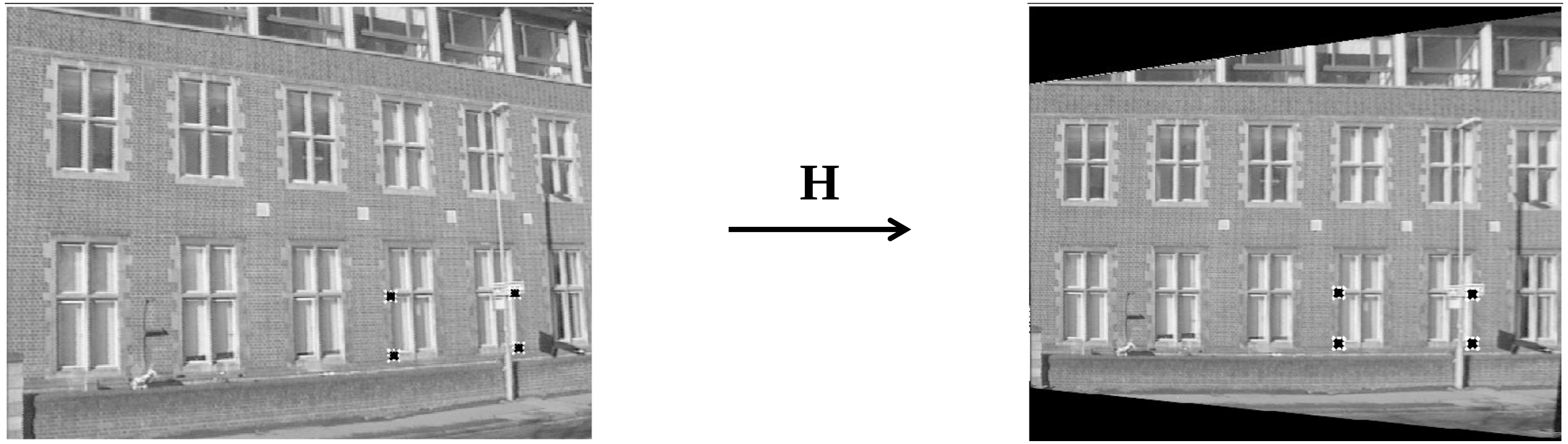
- The central projection between two planes corresponds to a homography

$$\mathbf{H}\tilde{\mathbf{x}}^a = \tilde{\mathbf{y}}^b$$



Linear transformations of the projective plane $\mathbb{P}^2$

- For images of a flat surface, a homography can be used to «change» the camera position



http://www.robots.ox.ac.uk/~vgg/hzbook.html http://www.robots.ox.ac.uk/~vgg/hzbook.html

The projective space $\mathbb{P}^3$

- The relationship between the Euclidean space $\mathbb{R}^3$ and the projective space $\mathbb{P}^3$ is much like the relationship between $\mathbb{R}^2$ and $\mathbb{P}^2$
  - We represent points in homogeneous coordinates

$$\tilde{\mathbf{x}} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{w} \end{bmatrix} = \lambda \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{w} \end{bmatrix} \quad \forall \quad \lambda \in \mathbb{R} \backslash \{0\}$$

  - Points at infinity have $\tilde{w} = 0$
  - We can transform between $\mathbb{R}^3$ and $\mathbb{P}^3$

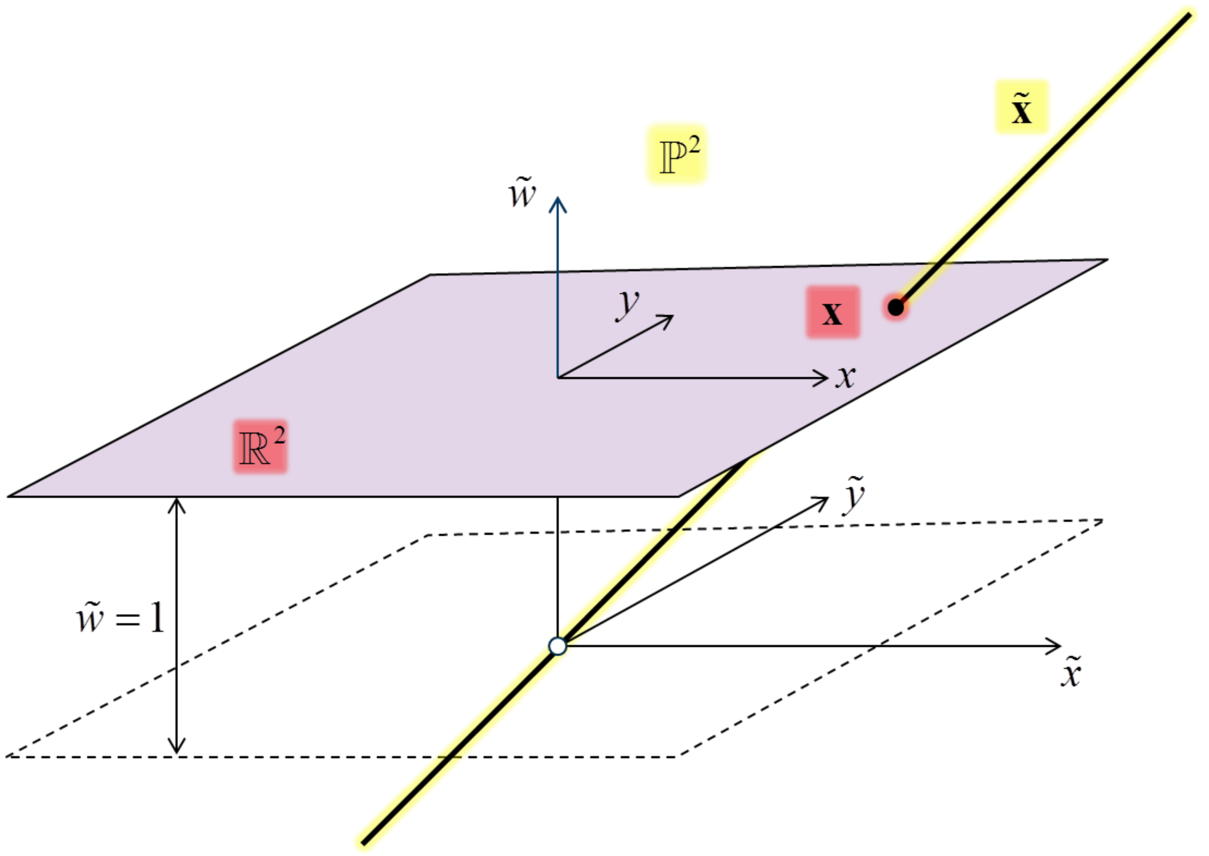
$$\begin{aligned} \mathbb{R}^3 &\to \mathbb{P}^3 \\ \mathbf{x} &\mapsto \tilde{\mathbf{x}} \\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} &\mapsto \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \end{aligned} \qquad \begin{aligned} \mathbb{P}^3 &\to \mathbb{R}^3 \\ \tilde{\mathbf{x}} &\mapsto \mathbf{x} \\ \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{w} \end{bmatrix} &\mapsto \begin{bmatrix} \tilde{x}/\tilde{w} \\ \tilde{y}/\tilde{w} \\ \tilde{z}/\tilde{w} \end{bmatrix} \end{aligned}$$

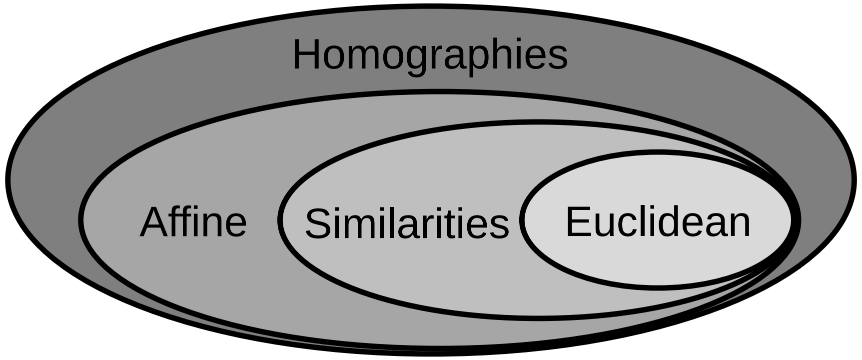
Linear transformations of the projective space $\mathbb{P}^3$

Transformation of $\mathbb{P}^3$	Matrix	#DoF	Preserves
Euclidean	$\begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$	6	Volumes, volume ratios, lengths + all below
Similarity	$\begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \quad s \in \mathbb{R}$	7	Angles + all below
Affine	$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$	12	Parallelism of planes, The plane at infinity + all below
Homography	$\begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix}$	15	Intersection and tangency of surfaces in contact, straight lines

Summary

- Projective plane $\mathbb{P}^2$ and space $\mathbb{P}^3$
  - Alternative representation of points
  - Homogeneous coordinates
  - Can swap between $\mathbb{R}^n$ and $\mathbb{P}^n$
- Linear projective transformations
  - Homogeneous matrices
  - Several matrix groups





$$\mathbb{R}^2 \rightarrow \mathbb{P}^2 \qquad \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbb{P}^2 \rightarrow \mathbb{R}^2 \qquad \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix} \mapsto \begin{bmatrix} \frac{\tilde{x}}{\tilde{w}} \\ \frac{\tilde{y}}{\tilde{w}} \end{bmatrix}$$

Supplementary material

Recommended

- *Richard Szeliski: Computer Vision: Algorithms and Applications 2nd ed*
  - Chapter 2 is about “image formation” and covers some projective geometry, focusing on transformations, in section 2.1.1-2.1.4