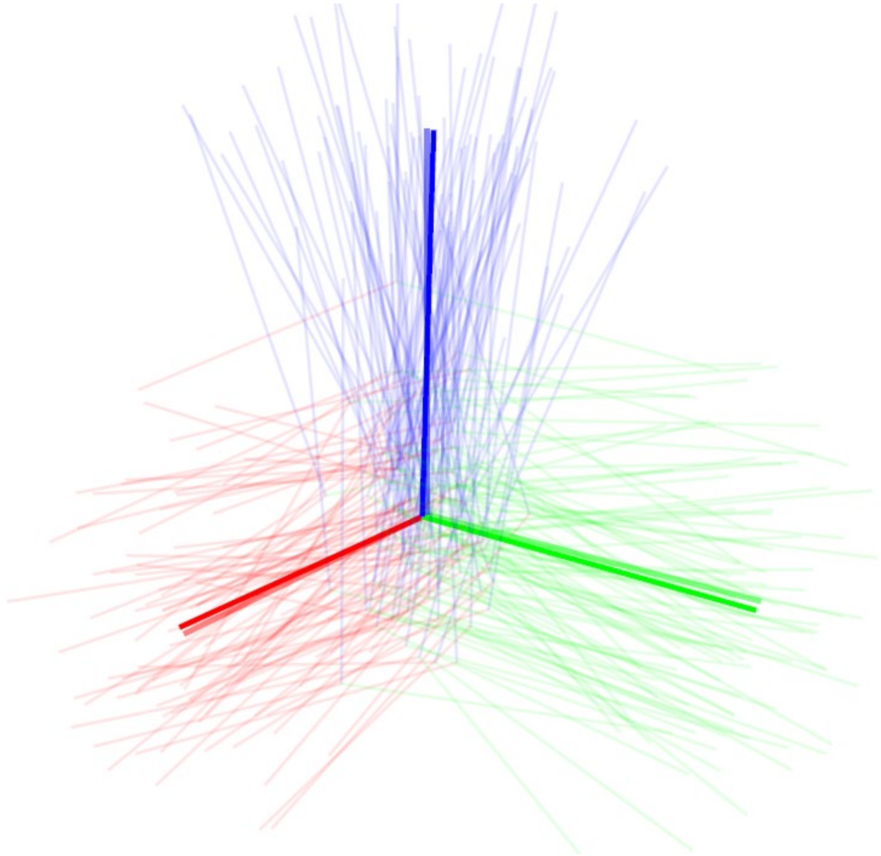
UiO : Department of Technology Systems
University of Oslo

Lecture 5.5
Applying Lie theory in practice

Trym Vegard Haavardsholm



TEK5030

The exponential map

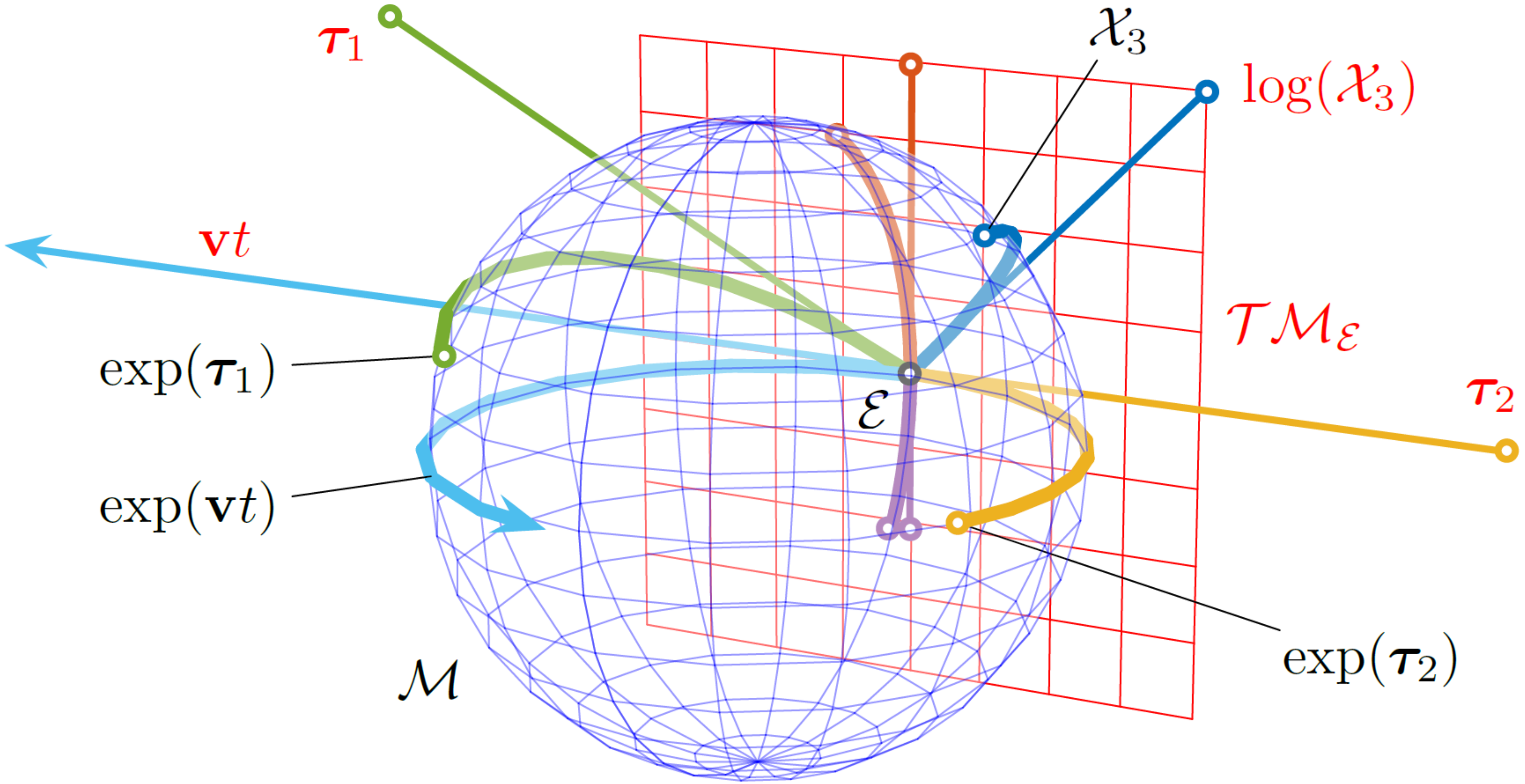


Image source: Solà, J., Deray, J., & Atchuthan, D. (n.d.). A micro Lie theory for state estimation in robotics

Plus and minus operators

It is convenient to express perturbations using plus and minus operators.

The **right plus and minus operators** are defined as:

$$\mathcal{Y} = \mathcal{X} \oplus {}^{\mathcal{X}}\boldsymbol{\tau} \triangleq \mathcal{X} \circ \mathrm{Exp}({}^{\mathcal{X}}\boldsymbol{\tau}) \ \in \mathcal{M}$$

$${}^{\mathcal{X}}\boldsymbol{\tau} = \mathcal{Y} \ominus \mathcal{X} \ \triangleq \mathrm{Log}(\mathcal{X}^{-1} \circ \mathcal{Y}) \in \mathcal{T}\mathcal{M}_{\mathcal{X}}$$

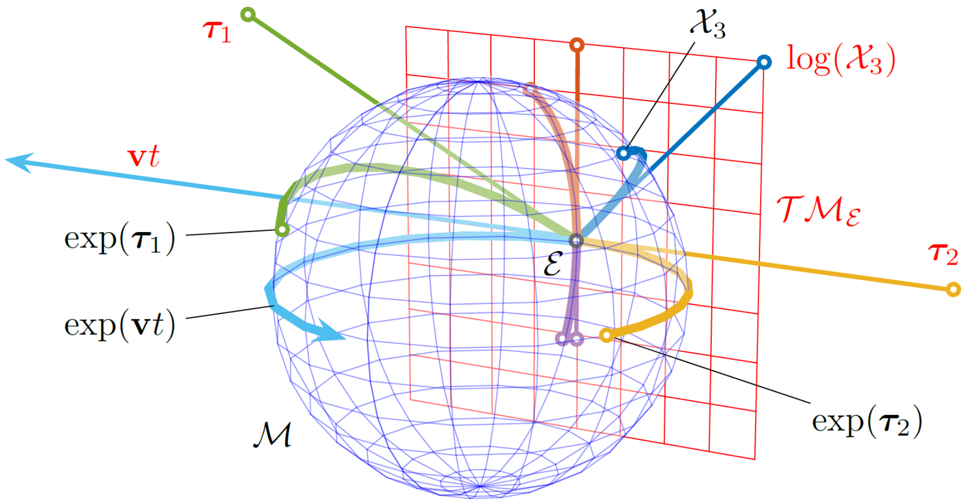


Image source: Solà, J., Deray, J., & Atchuthan, D. (n.d.). A micro Lie theory for state estimation in robotics (licensed under CC BY-NC-SA 4.0)

Plus and minus operators example: Interpolation

Vectors:

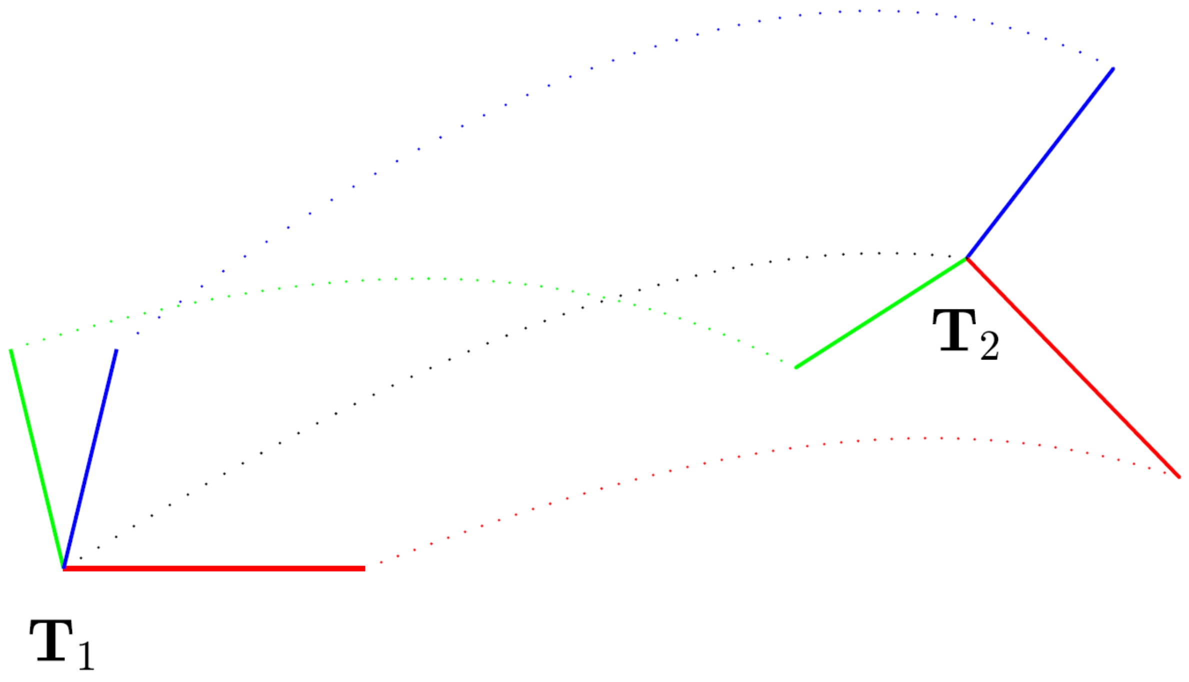
$$\mathbf{t} = \mathbf{t}_1 + \alpha(\mathbf{t}_2 - \mathbf{t}_1)$$

Group elements:

$$\begin{aligned}\mathcal{X} &= \mathcal{X}_1 \oplus \alpha(\mathcal{X}_2 \ominus \mathcal{X}_1) \\ &= \mathcal{X}_1 \circ \mathrm{Exp}(\alpha \, \mathrm{Log}(\mathcal{X}_1^{-1} \circ \mathcal{X}_2))\end{aligned}$$

Poses:

$$\mathbf{T} = \mathbf{T}_1 \circ \mathrm{Exp}(\alpha \, \mathrm{Log}(\mathbf{T}_1^{-1} \circ \mathbf{T}_2))$$



Uncertainty for Lie groups

We can represent a random variable on the manifold as a perturbation

$$\mathcal{X} = \bar{\mathcal{X}} \oplus \boldsymbol{\tau}, \quad \boldsymbol{\tau} = \mathcal{X} \ominus \bar{\mathcal{X}}$$

$$\boldsymbol{\tau} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_{\mathcal{X}})$$

$$\mathcal{X} \sim \mathcal{N}(\bar{\mathcal{X}}, \boldsymbol{\Sigma}_{\mathcal{X}})$$

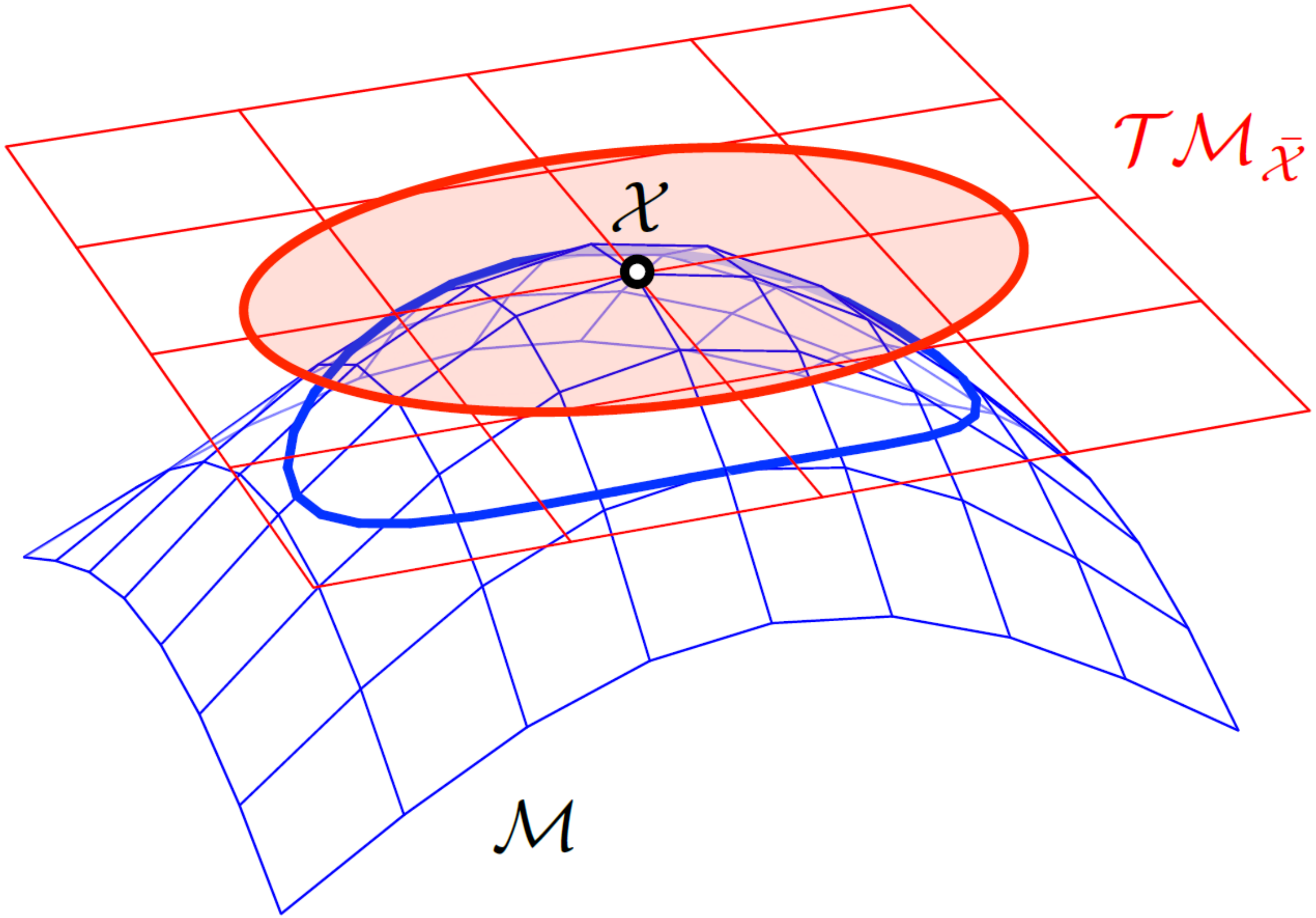


Image source: Solà, J., Deray, J., & Atchuthan, D. (n.d.). A micro Lie theory for state estimation in robotics

Uncertainty for Lie groups: Example

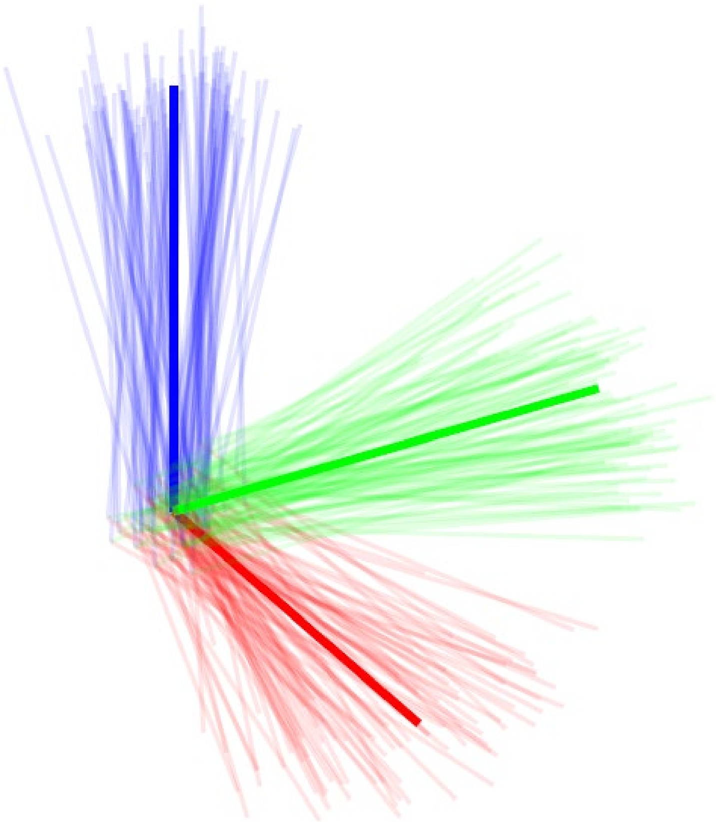
Drawing random poses from

$$\mathbf{T} \sim \mathcal{N}(\bar{\mathbf{T}}, \mathbf{\Sigma}_{\mathbf{T}})$$

by drawing random vectors:

$$\boldsymbol{\xi} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_{\mathbf{T}})$$

$$\mathbf{T} = \bar{\mathbf{T}} \oplus \boldsymbol{\xi}$$



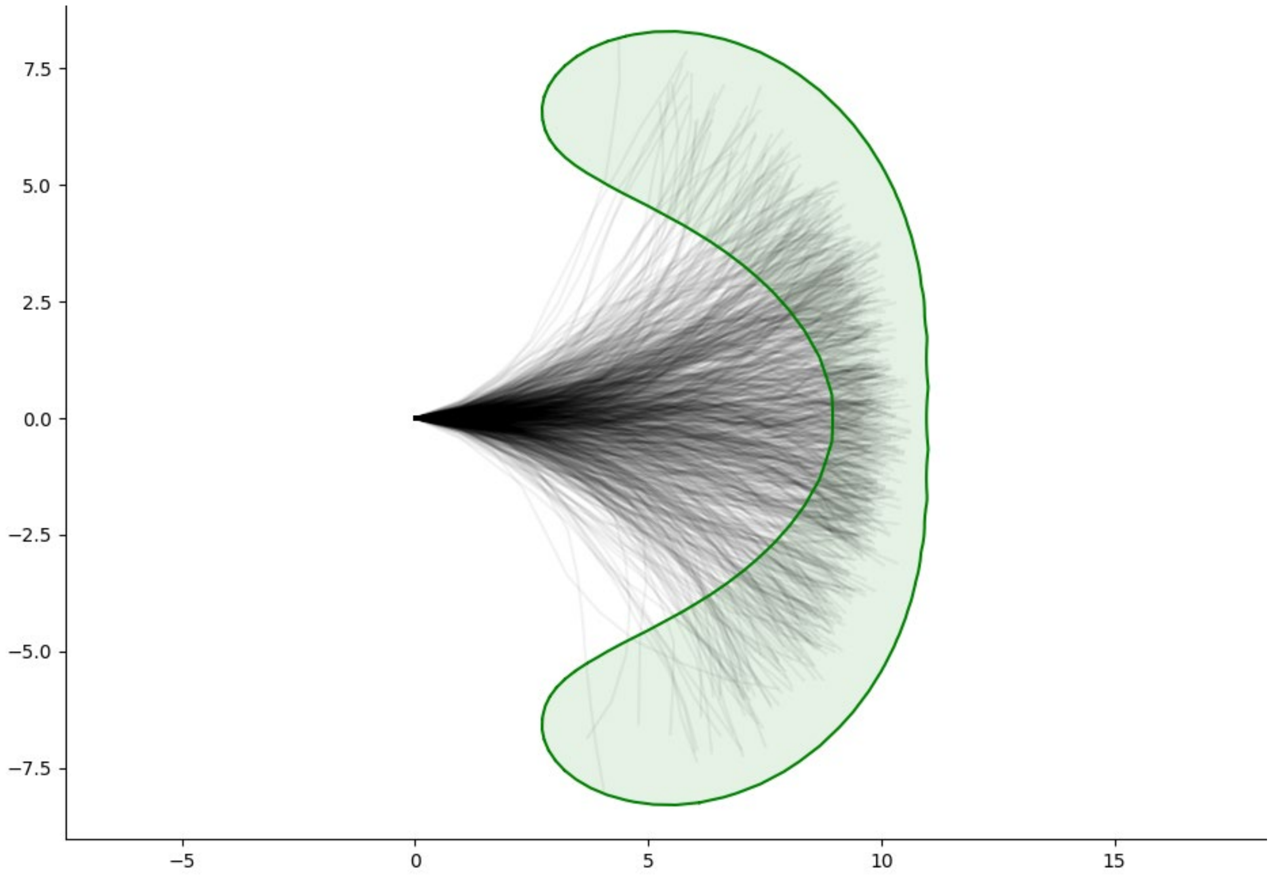
Uncertainty for Lie groups: Example

10 uncertain steps (pose compositions) of length 1 in the (local) *x*-direction

Black lines show randomly drawn paths

Green shape corresponds to the 95% error ellipsoid in the tangent space computed from the probabilistic model for the final pose, but transformed to the 2D-translation plane

The normal distribution in the tangent space models the real pose distribution very well



Working in the tangent space example: Mean element

Initialise for example with $\bar{\mathbf{T}}^0 \leftarrow \mathbf{T}_1$

for $t = 0, 1, \ldots, t^{max}$ **do**

Compute the mean tangent vector in the tangent space at $\bar{\mathbf{T}}^t$

$\bar{\boldsymbol{\xi}} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{T}_i \ominus \bar{\mathbf{T}}^t$

Update the hypothesis

$\bar{\mathbf{T}}^{t+1} \leftarrow \bar{\mathbf{T}}^t \oplus \bar{\boldsymbol{\xi}}$

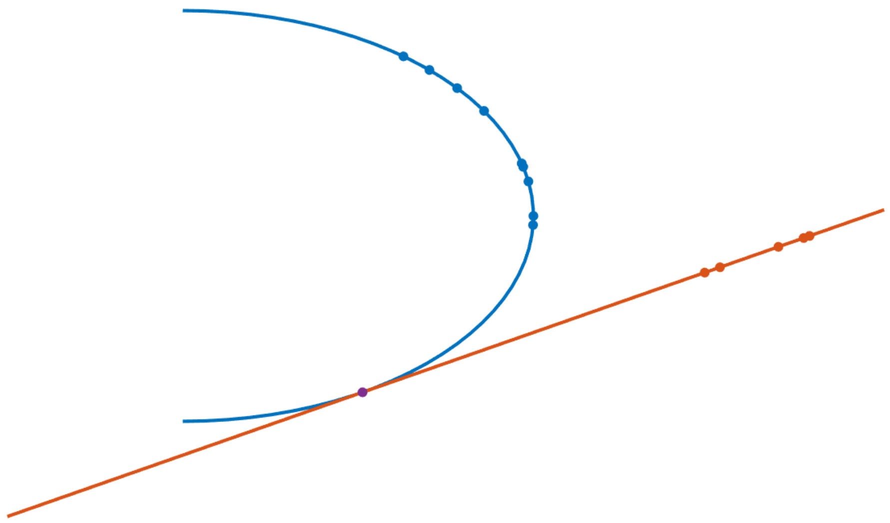
if $||\bar{\mathbf{T}}^t \ominus \bar{\mathbf{T}}^{t+1}||^2 < \epsilon$ **then**

$\bar{\mathbf{T}} \leftarrow \bar{\mathbf{T}}^{t+1}$

return

end

end



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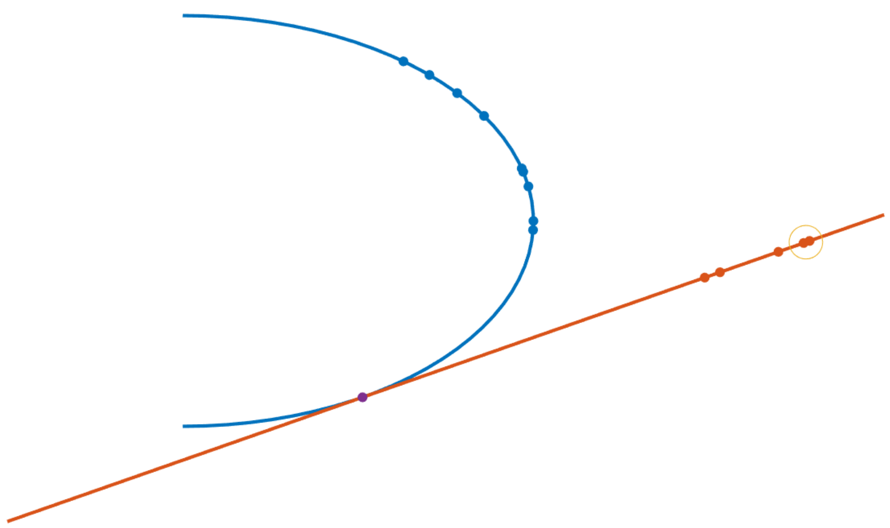

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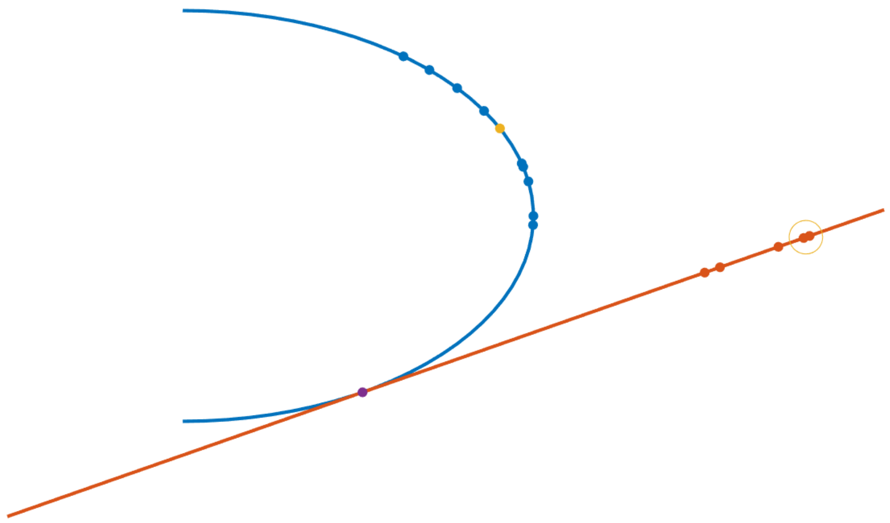

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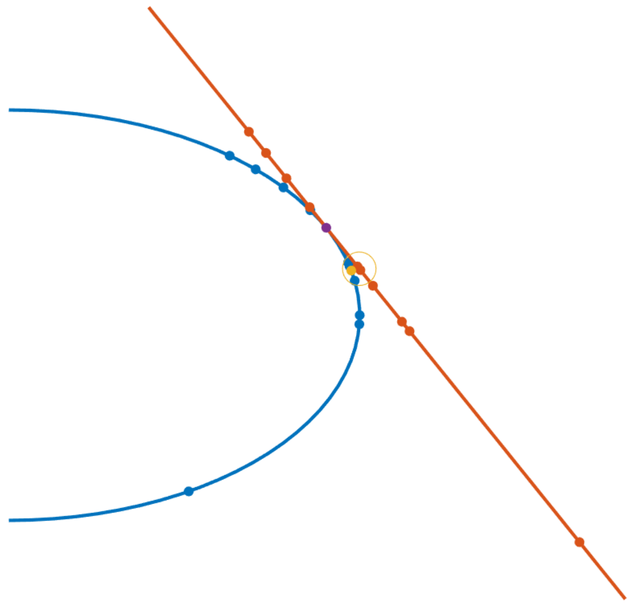

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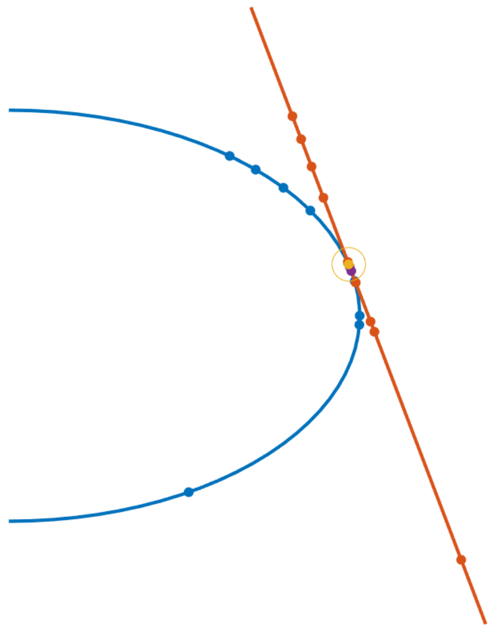

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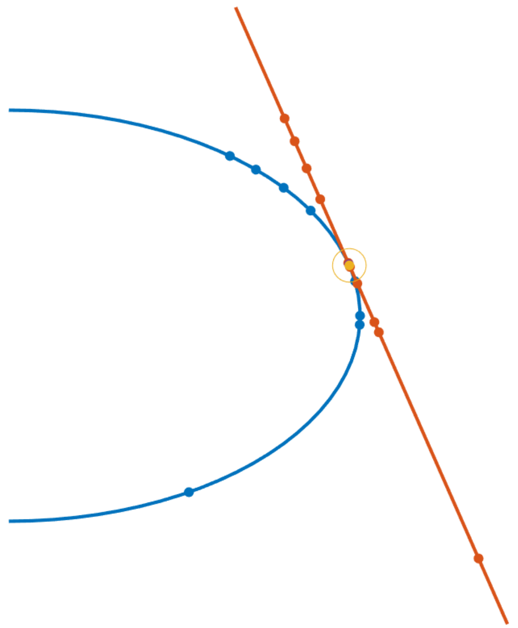

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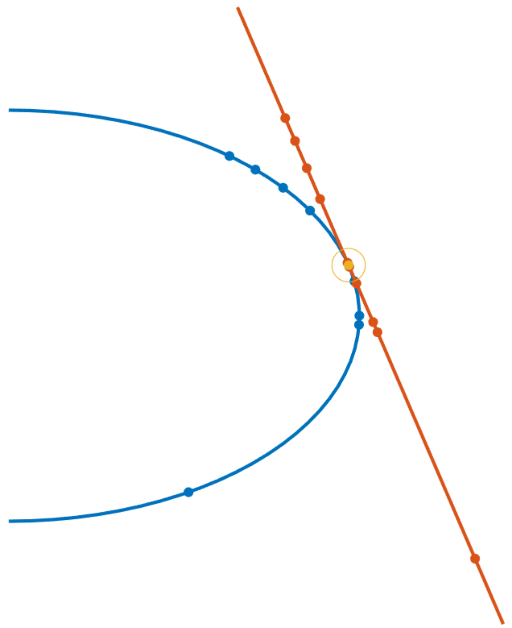

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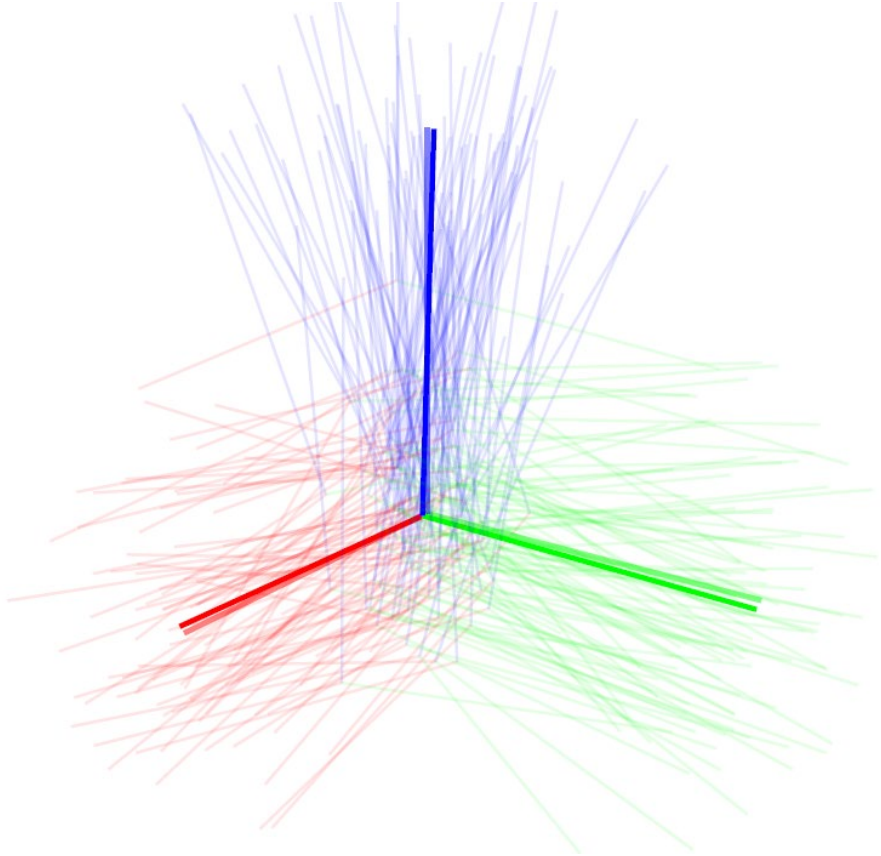

Resources

Learn more:

- The compendium

- Solà, J., Deray, J., & Atchuthan, D. (n.d.). A micro Lie theory for state estimation in robotics

Using Lie theory in practice:

- My python library pylie: https://github.com/tussedrotten/pylie

- The C++ library Sophus: https://github.com/strasdat/Sophus