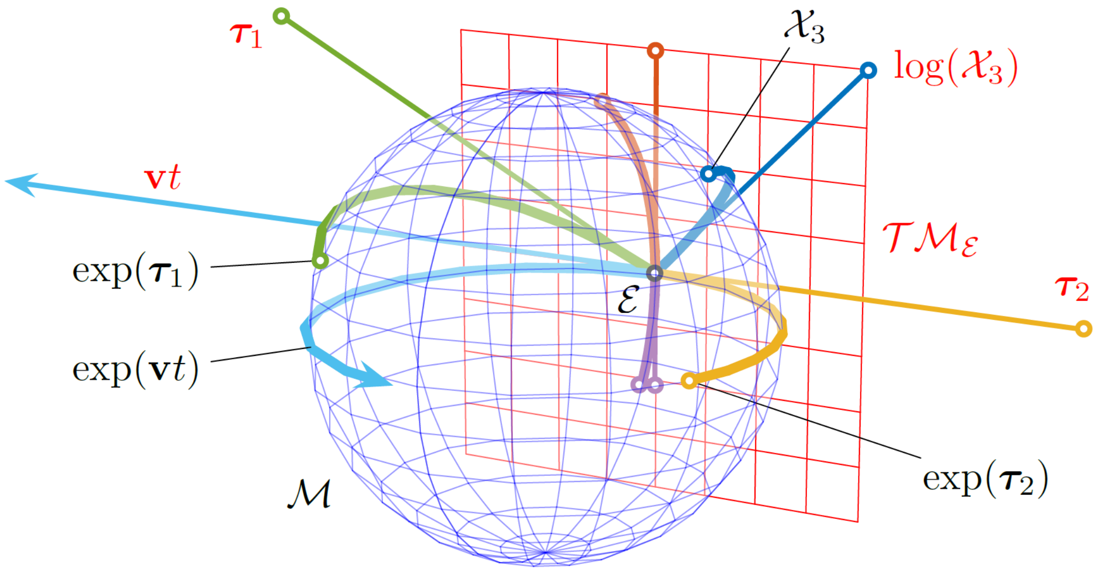
UiO : Department of Technology Systems
University of Oslo

Lecture 5.4
An introduction to Lie theory

Trym Vegard Haavardsholm



Orientations and poses are «special»

The **special orthogonal group in 3D** is the set of valid rotation matrices

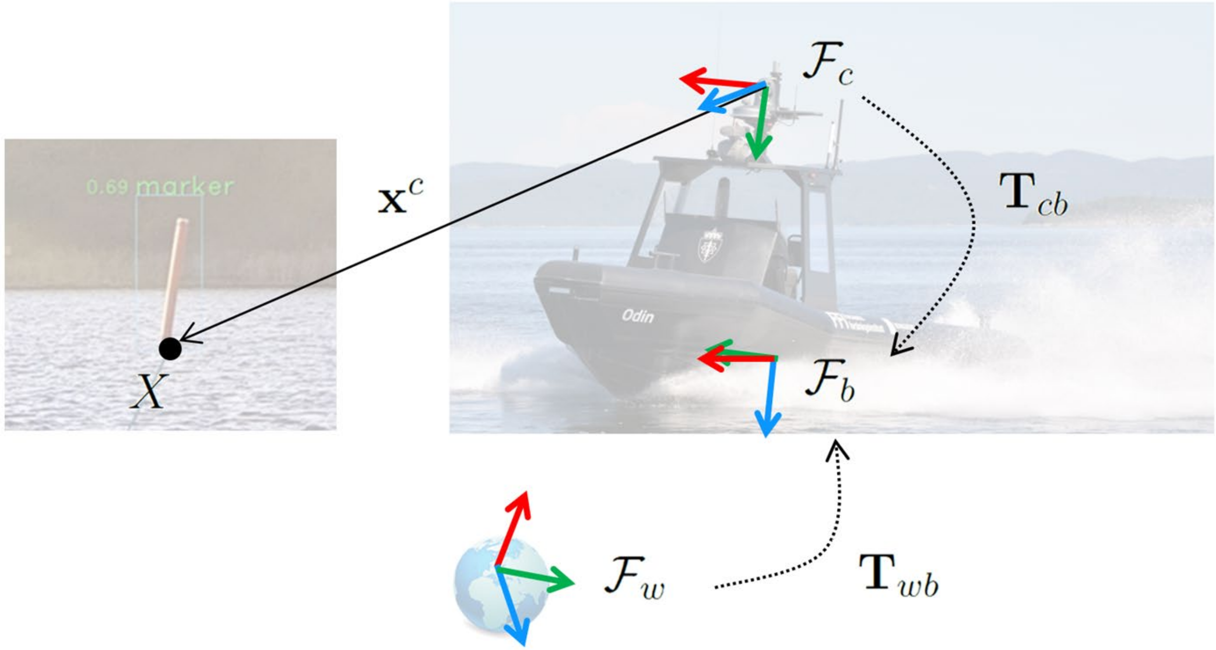
$$SO(3) = \left\{ \mathbf{R} \in \mathbb{R}^{3\times 3} \;\middle|\; \mathbf{R}\mathbf{R}^\top = \mathbf{I}, \det \mathbf{R} = 1 \right\}$$

The **special Euclidean group in 3D** is the set of valid Euclidean transformation matrices

$$SE(3) = \left\{ \mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} \;\middle|\; \mathbf{R} \in SO(3), \mathbf{t} \in \mathbb{R}^3 \right\}$$

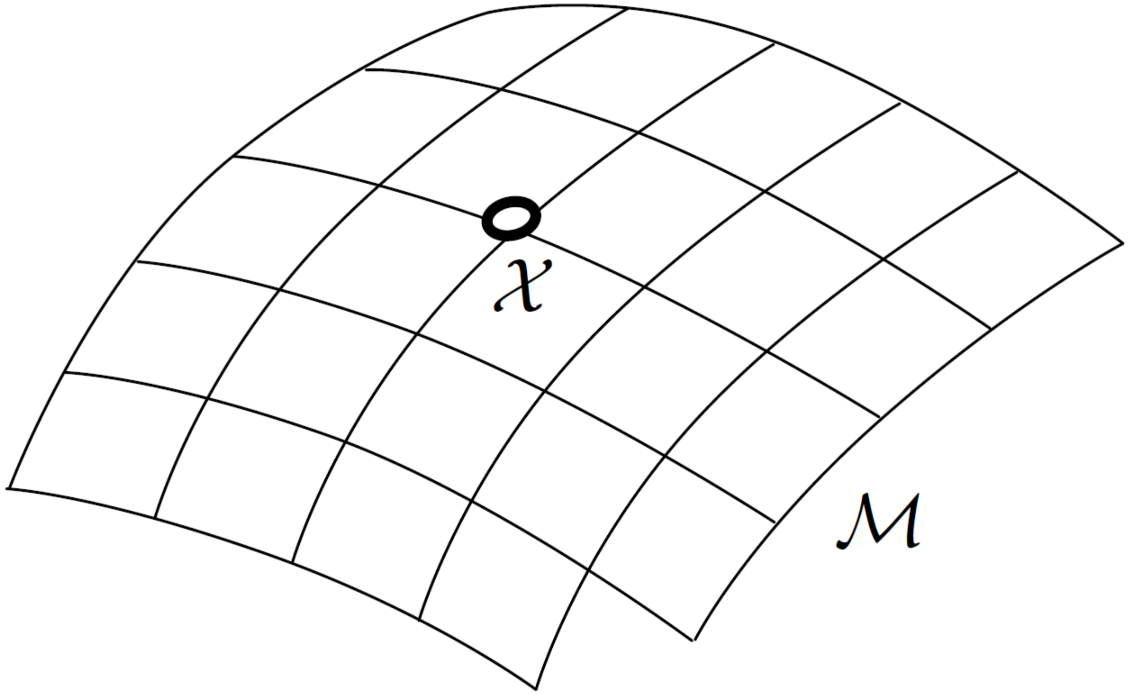
Example

```cpp
#include "Eigen/Eigen"
#include "sophus/so3.hpp"
#include "sophus/se3.hpp"
#include <iostream>

constexpr double pi = 3.14159265358979323846;

int main()
{
  Eigen::Matrix4d mat4_cb;
  mat4_cb <<
    0, 1, 0, 0,
    0, 0, 1, 2,
    1, 0, 0, 0,
    0, 0, 0, 1;

  const auto T_cb = Sophus::SE3d::fitToSE3(mat4_cb);

  std::cout << "T_cb = " << std::endl << T_cb.matrix() << std::endl;

}
```



Orientations and poses lie on manifolds

Orientations and poses lie on **manifolds** in higher-dimensional spaces, which are **not vector spaces**!



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Orientations and poses lie on manifolds

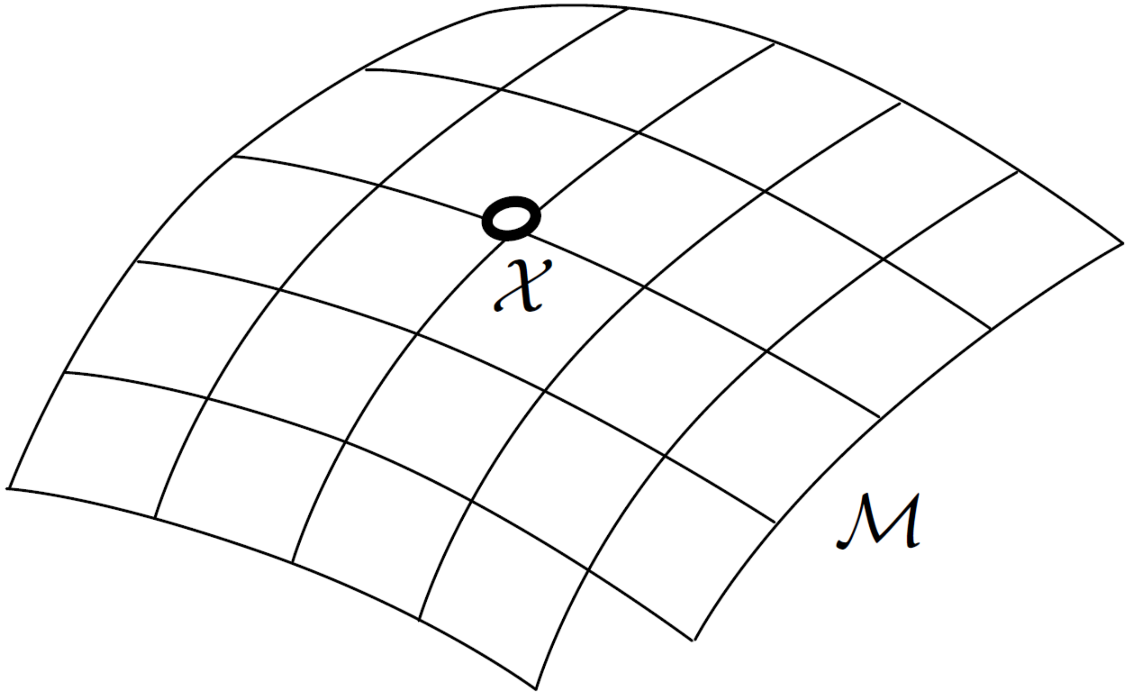
Orientations and poses lie on **manifolds** in higher-dimensional spaces, which are **not vector spaces**!

For example:

$$\mathbf{R} \in SO(3)$$

$$\delta\mathbf{R} \in \mathbb{R}^{3\times 3}$$

$$\mathbf{R} + \delta\mathbf{R} \notin SO(3)$$



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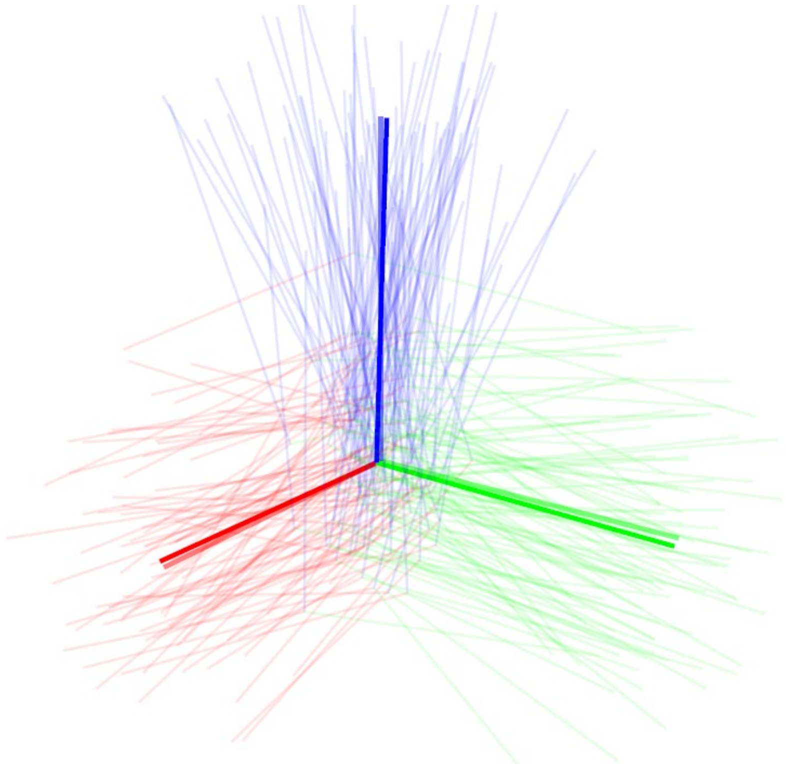
Orientations and poses lie on manifolds

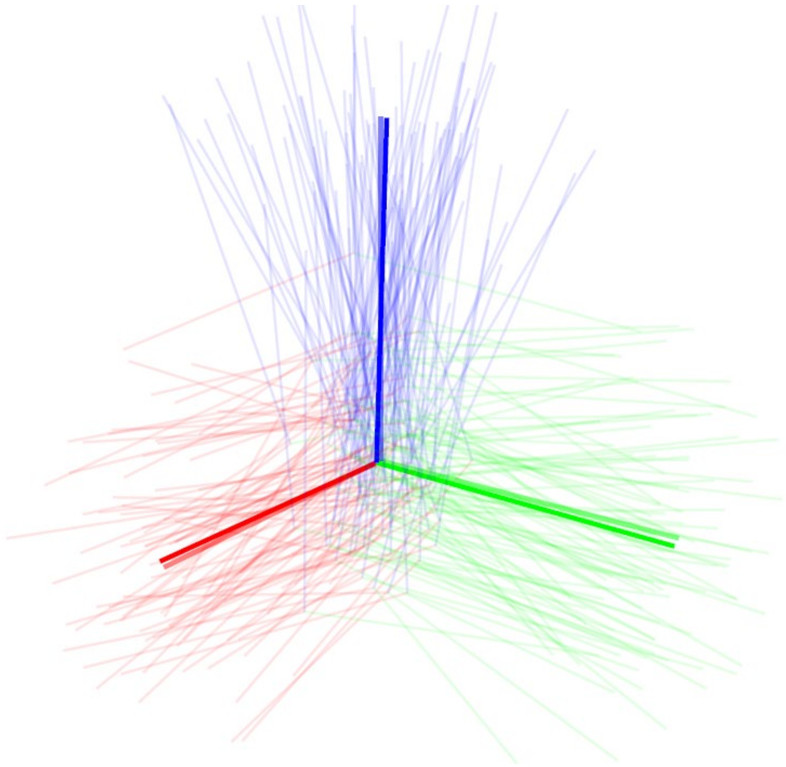
So how can we **compute the mean** of a set of poses?



Orientations and poses lie on manifolds

So how can we **compute the mean** of a set of poses?

Or **represent the probability distribution** of that set?

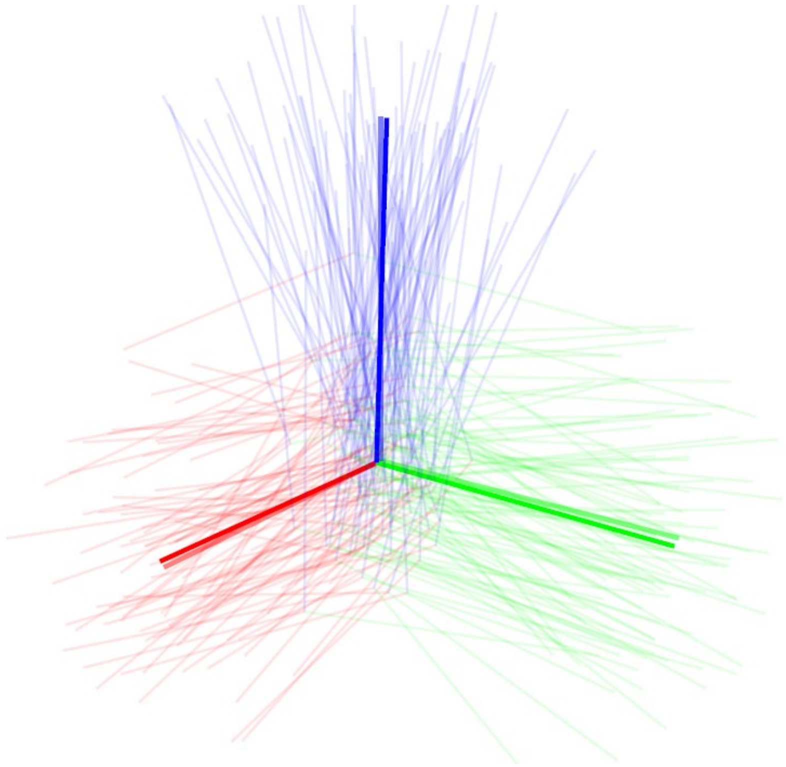


Orientations and poses lie on manifolds

So how can we **compute the mean** of a set of poses?

Or **represent the probability distribution** of that set?

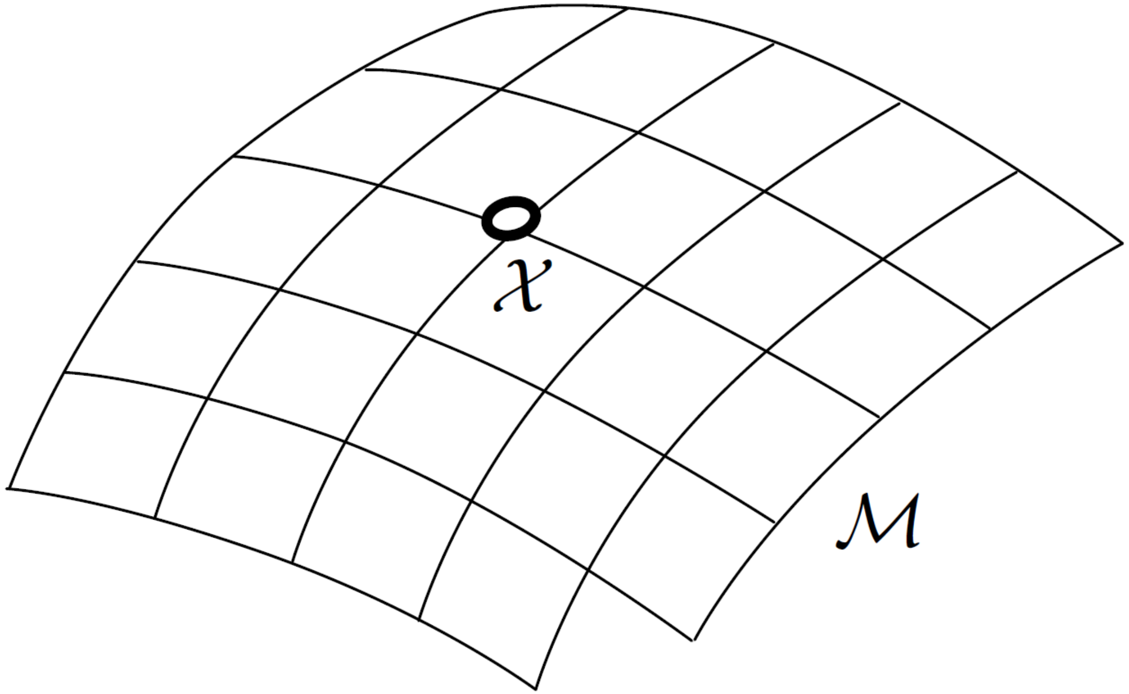
Or **compute the derivative**
of functions with orientations and poses?



$$\mathbf{J} = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \triangleq \lim_{\mathbf{h} \to 0} \frac{f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})}{\mathbf{h}} \in \mathbb{R}^{n \times m}$$

Lie theory lets us work on these manifolds

Orientations and poses are **matrix Lie groups**



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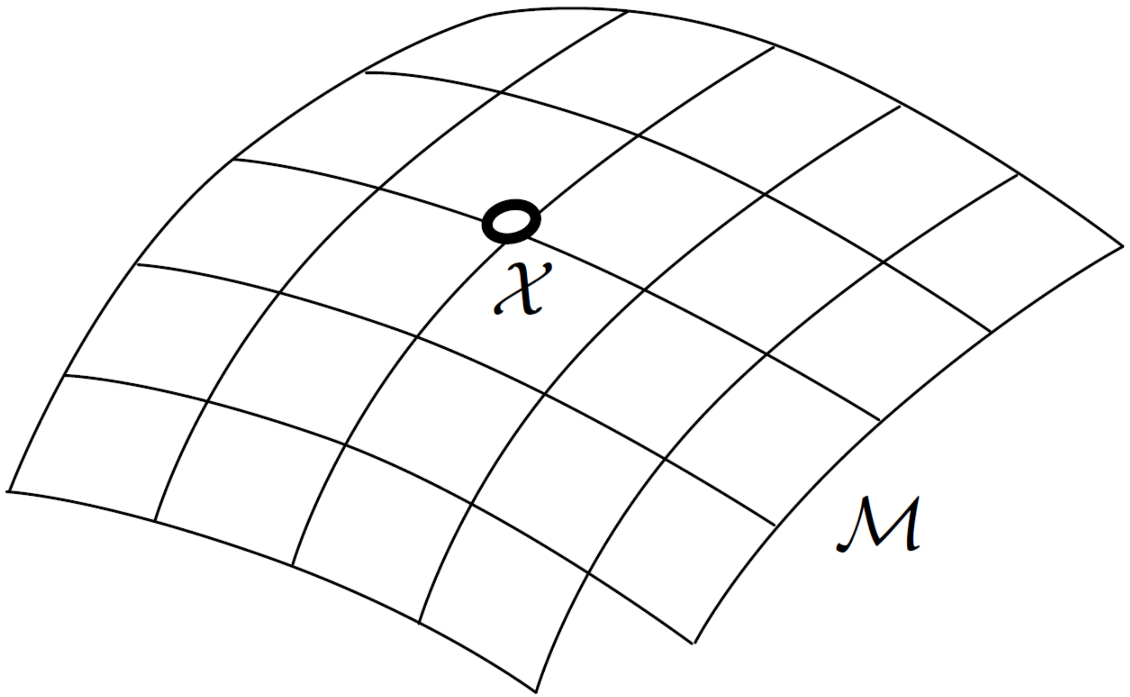
Lie theory lets us work on these manifolds

Orientations and poses are **matrix Lie groups**

A Lie group is a **group** on a **smooth manifold**



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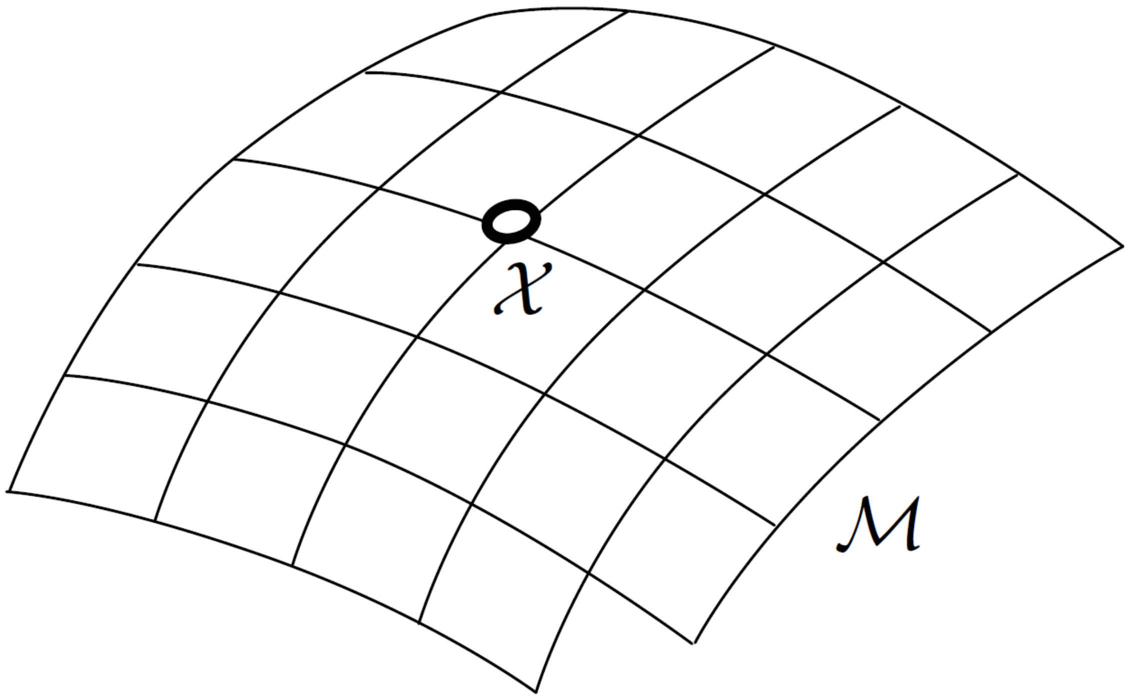
Lie theory lets us work on these manifolds

A Lie group is both:

- a smooth differential manifold
- a group $(\mathcal{G}, \circ)$ with set $\mathcal{G}$ and composition operation $\circ$ that satisfies the axioms:

$$\begin{aligned} \text{Closure under } \circ &: \quad \mathcal{X} \circ \mathcal{Y} \in \mathcal{G} \\ \text{Identity } \mathcal{E} &: \quad \mathcal{E} \circ \mathcal{X} = \mathcal{X} \circ \mathcal{E} = \mathcal{X} \\ \text{Inverse } \mathcal{X}^{-1} &: \quad \mathcal{X}^{-1} \circ \mathcal{X} = \mathcal{X} \circ \mathcal{X}^{-1} = \mathcal{E} \\ \text{Associativity} &: \quad (\mathcal{X} \circ \mathcal{Y}) \circ \mathcal{Z} = \mathcal{X} \circ (\mathcal{Y} \circ \mathcal{Z}) \end{aligned}$$

Action of $\mathcal{X} \in \mathcal{M}$ on $v \in \mathcal{V}$: $\mathcal{X} \cdot v$



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The *SO*(3) group

The **special orthogonal group in 3D** is the set of valid rotation matrices

$$SO(3) = \left\{ \mathbf{R} \in \mathbb{R}^{3\times 3} \;\middle|\; \mathbf{R}\mathbf{R}^\top = \mathbf{I}, \det \mathbf{R} = 1 \right\}$$

and is closed under matrix multiplication with identity $\mathbf{I}$.
Inversion is achieved with transposition

$$\mathbf{R}^{-1} = \mathbf{R}^\top$$

and composition with matrix multiplication

$$\mathbf{R}_a \circ \mathbf{R}_b = \mathbf{R}_a \mathbf{R}_b$$

The group action on vectors is given by the product

$$\mathbf{R} \cdot \mathbf{x} = \mathbf{R}\mathbf{x}$$

The *SE*(3) group

The **special Euclidean group in 3D** is the set of valid Euclidean transformation matrices

$$SE(3) = \left\{ \mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} \,\middle|\, \mathbf{R} \in SO(3), \mathbf{t} \in \mathbb{R}^3 \right\}$$

and is closed under matrix multiplication with identity $\mathbf{I}$.
Inversion is achieved with matrix inversion

$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{R}^\top & -\mathbf{R}^\top \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

and composition with matrix multiplication

$$\mathbf{T}_a \circ \mathbf{T}_b = \mathbf{T}_a \mathbf{T}_b = \begin{bmatrix} \mathbf{R}_a \mathbf{R}_b & \mathbf{R}_a \mathbf{t}_b + \mathbf{t}_a \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

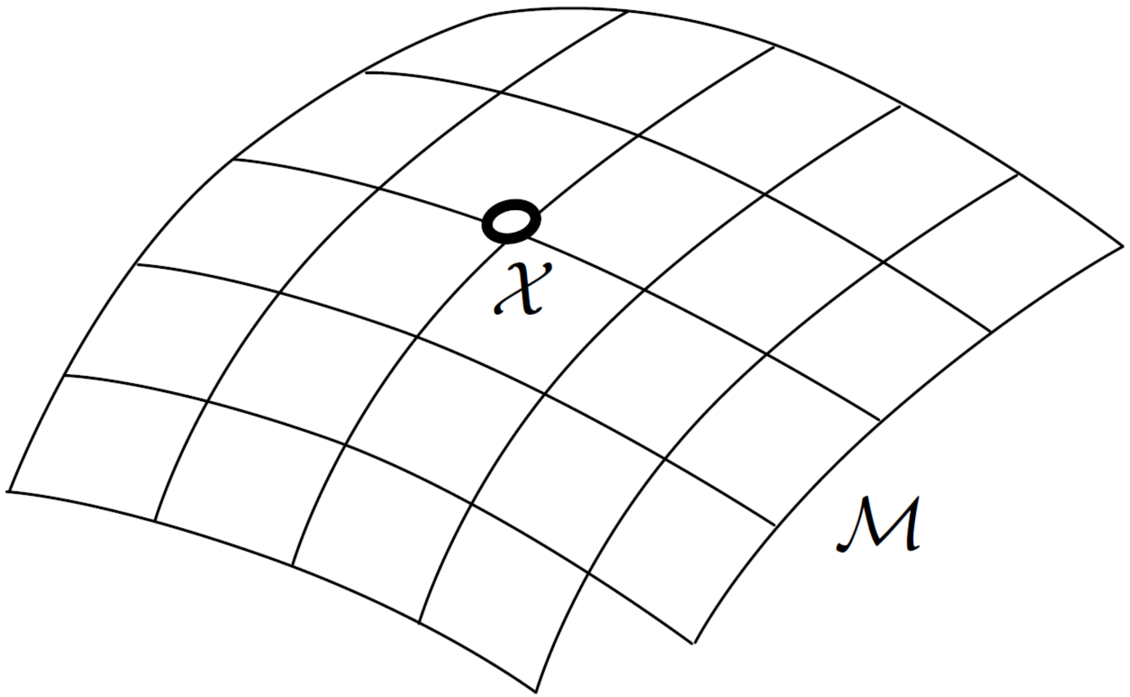
The group action on vectors is given by the product

$$\mathbf{T} \cdot \mathbf{x} = \mathbf{T}\tilde{\mathbf{x}} = \mathbf{R}\mathbf{x} + \mathbf{t}.$$

Lie theory lets us work on these manifolds

Orientations and poses are **matrix Lie groups**

A Lie group is a **group** on a **smooth manifold**



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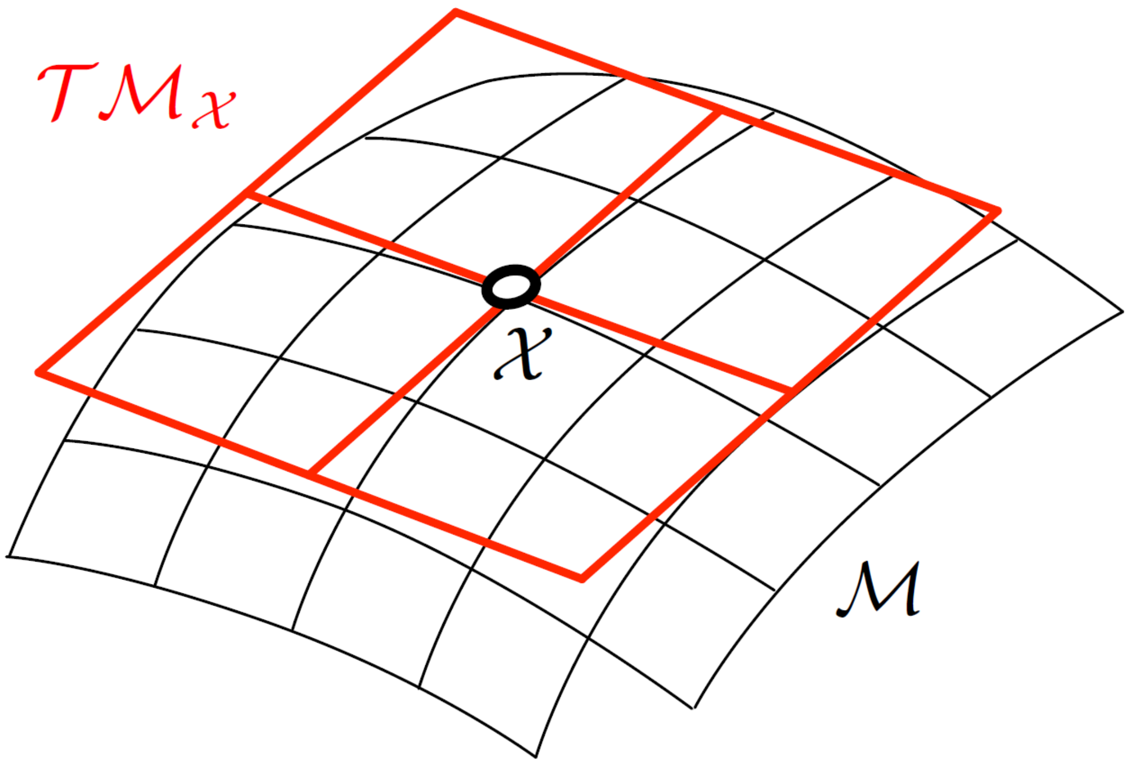
Lie theory lets us work on these manifolds

Orientations and poses are **matrix Lie groups**

A Lie group is a **group** on a **smooth manifold**

Lie theory describes the **tangent space** around elements of a Lie group, and defines **exact mappings** between the tangent space and the manifold



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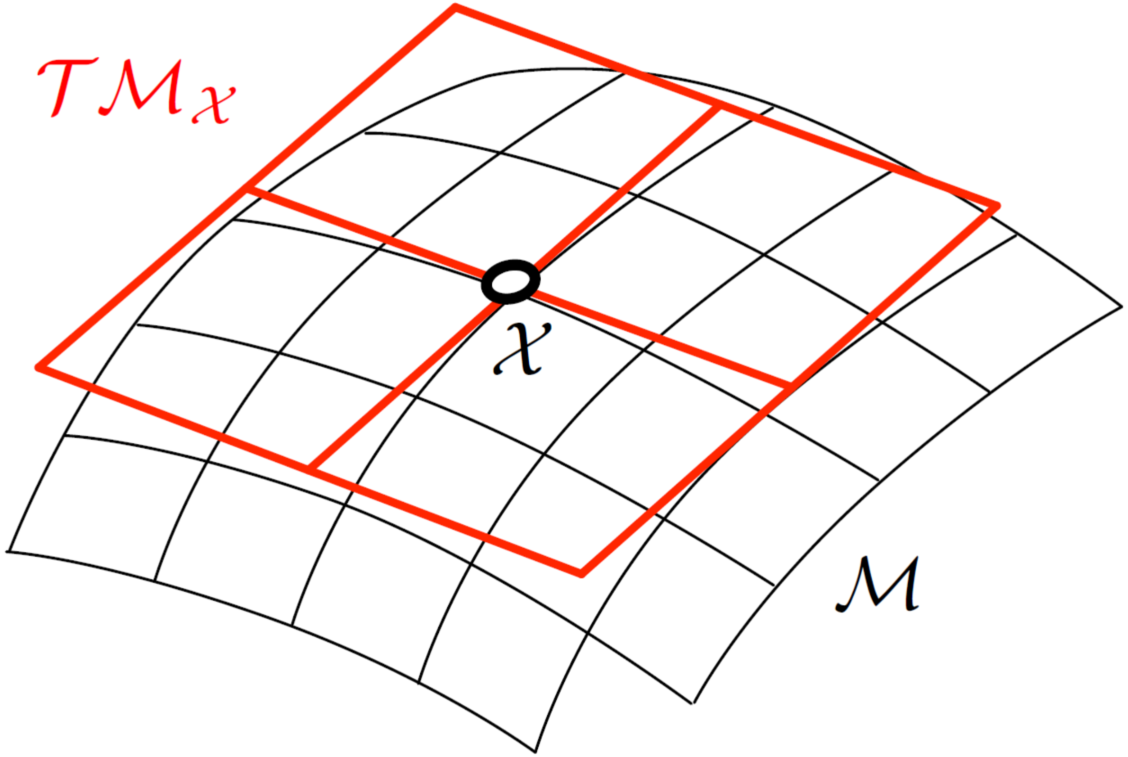
Lie theory lets us work on these manifolds

Orientations and poses are **matrix Lie groups**

A Lie group is a **group** on a **smooth manifold**

Lie theory describes the **tangent space** around elements of a Lie group, and defines **exact mappings** between the tangent space and the manifold

The tangent space is a **vector space** with the same dimension as the number of degrees of freedom of the group transformations



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Lie algebra

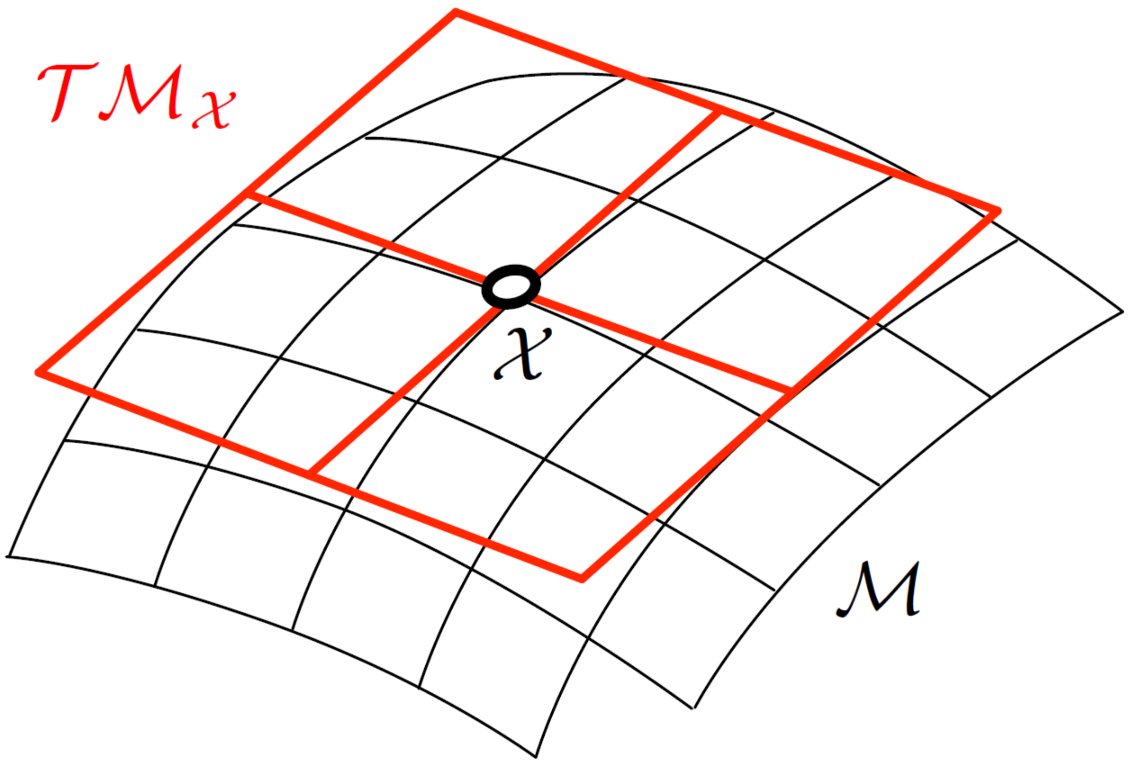
The tangent space at the identity $\mathcal{TM}_\mathcal{E}$ is called the Lie algebra of $\mathcal{M}$:

$$\text{Lie algebra} \quad : \qquad \mathfrak{m} \triangleq \mathcal{TM}_\mathcal{E}$$

The Lie algebra is a vector space
with elements $\boldsymbol{\tau}^\wedge \in \mathfrak{m}$
which can be **identified** with vectors $\boldsymbol{\tau} \in \mathbb{R}^m$
through the linear maps

$$\text{Hat: } (\cdot)^\wedge : \mathbb{R}^m \to \mathfrak{m}; \qquad \boldsymbol{\tau}^\wedge = \sum_{i=1}^{m} \tau_i \mathbf{E}_i$$

$$\text{Vee: } (\cdot)^\vee : \mathfrak{m} \to \mathbb{R}^m; \qquad \boldsymbol{\tau} = (\boldsymbol{\tau}^\wedge)^\vee = \sum_{i=1}^{m} \tau_i \mathbf{e}_i$$



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The Lie algebra of *SO*(3)

The Lie algebra of *SO*(3) is given by

$$\mathfrak{so}(3) = \left\{ \boldsymbol{\theta}^\wedge = [\boldsymbol{\theta}]_\times \in \mathbb{R}^{3\times 3} \;\middle|\; \boldsymbol{\theta} \in \mathbb{R}^3 \right\}$$

where the tangent space vector $\boldsymbol{\theta} \triangleq \theta \mathbf{u}$ corresponds to the rotation on angle-axis form.
The Lie algebra can be decomposed into

$$\boldsymbol{\theta}^\wedge = [\boldsymbol{\theta}]_\times = \theta_1 \mathbf{E}_1 + \theta_2 \mathbf{E}_2 + \theta_3 \mathbf{E}_3$$

$$\mathbf{E}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \; \mathbf{E}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \; \mathbf{E}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The Lie algebra of *SO*(3)

$$[\mathbf{u}]_\times = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}_\times \triangleq \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

The Lie algebra of *SO*(3) is given by

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The Lie algebra of *SE*(3)

The Lie algebra of *SE*(3) is given by

$$\mathfrak{se}(3) = \left\{ \boldsymbol{\xi}^\wedge = \begin{bmatrix} [\boldsymbol{\theta}]_\times & \boldsymbol{\rho} \\ \mathbf{0}^\top & 0 \end{bmatrix} \in \mathbb{R}^{4\times4} \;\middle|\; \boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{bmatrix} \in \mathbb{R}^6 \right\}$$

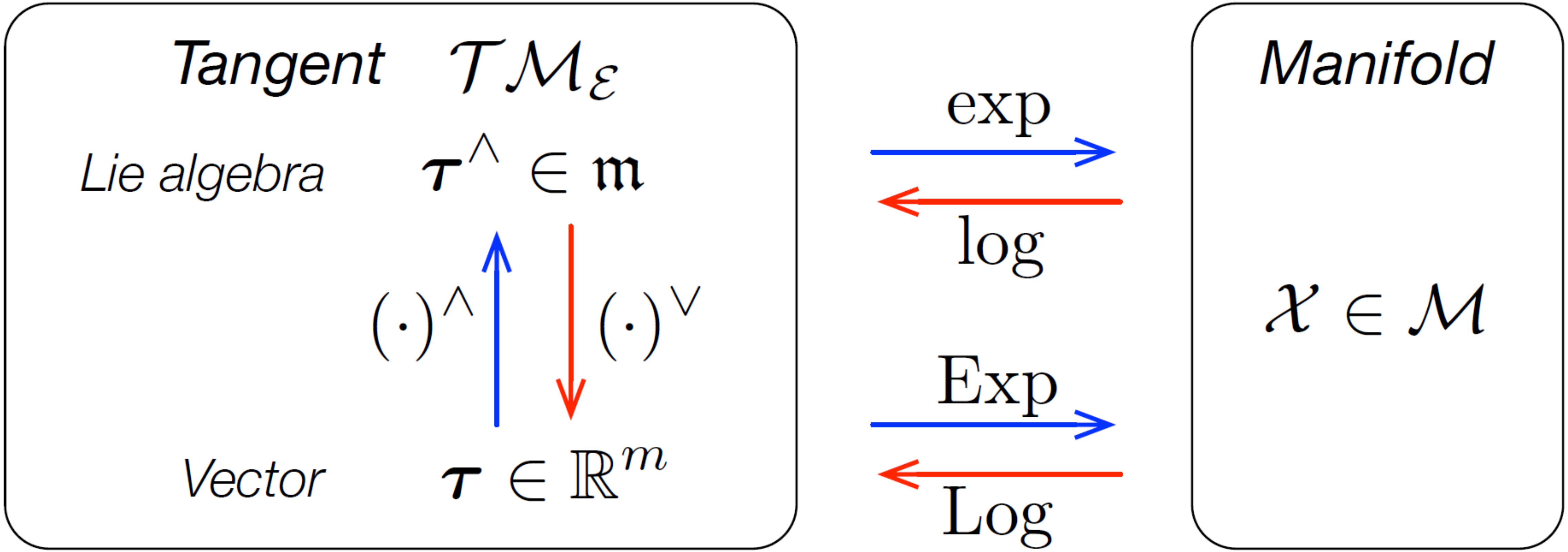
where the vectors $\boldsymbol{\rho}, \boldsymbol{\theta} \in \mathbb{R}^3$ correspond to the translational and rotational parts, respectively.
The Lie algebra can be decomposed into

$$\boldsymbol{\xi}^\wedge = \xi_1\mathbf{E}_1 + \xi_2\mathbf{E}_2 + \xi_3\mathbf{E}_3 + \xi_4\mathbf{E}_4 + \xi_5\mathbf{E}_5 + \xi_6\mathbf{E}_6$$

$$\mathbf{E}_1 = \begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix},\quad \mathbf{E}_2 = \begin{bmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{bmatrix},\quad \mathbf{E}_3 = \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{bmatrix},$$

$$\mathbf{E}_4 = \begin{bmatrix} 0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0 \end{bmatrix}, \mathbf{E}_5 = \begin{bmatrix} 0&0&1&0\\0&0&0&0\\-1&0&0&0\\0&0&0&0 \end{bmatrix}, \mathbf{E}_6 = \begin{bmatrix} 0&-1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}$$

The exponential map



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The exponential map

The **exponential map** transfers elements of the Lie algebra to elements of the group:

$$\exp : \mathfrak{m} \to \mathcal{M}; \qquad \mathcal{X} = \exp(\boldsymbol{\tau}^\wedge)$$

The inverse operation is **the logarithmic map:**

$$\log : \mathcal{M} \to \mathfrak{m}; \qquad \boldsymbol{\tau}^\wedge = \log(\mathcal{X})$$

The **capitalised exponential and logarithmic maps** are convenient compositions that work directly on the vector elements:

$$\begin{aligned} \mathrm{Exp} : \mathbb{R}^m \to \mathcal{M}; \qquad \mathcal{X} &= \mathrm{Exp}(\boldsymbol{\tau}) \triangleq \exp(\boldsymbol{\tau}^\wedge) \\ \mathrm{Log} : \mathcal{M} \to \mathbb{R}^m; \qquad \boldsymbol{\tau} &= \mathrm{Log}(\mathcal{X}) \triangleq \log(\mathcal{X})^\vee \end{aligned}$$

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The inverse operation is **the logarithmic map:**

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The matrix exponential:

$$\exp(\mathbf{X}) = \sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{X}^k$$

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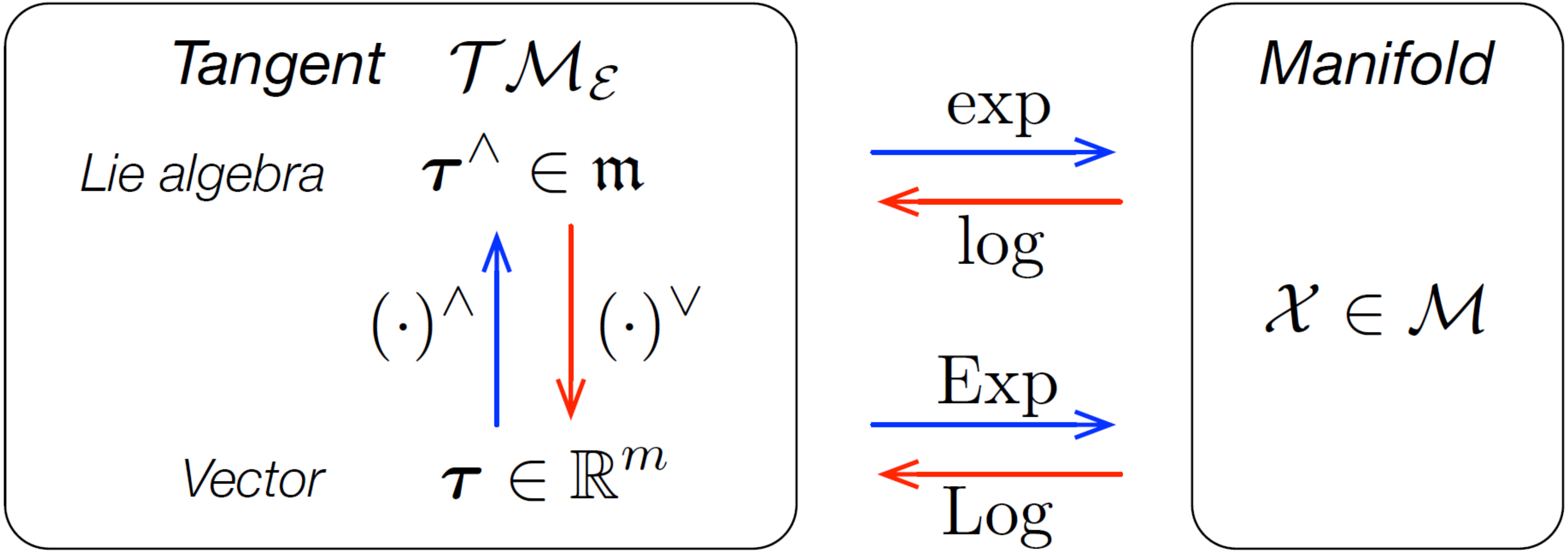
The exponential map



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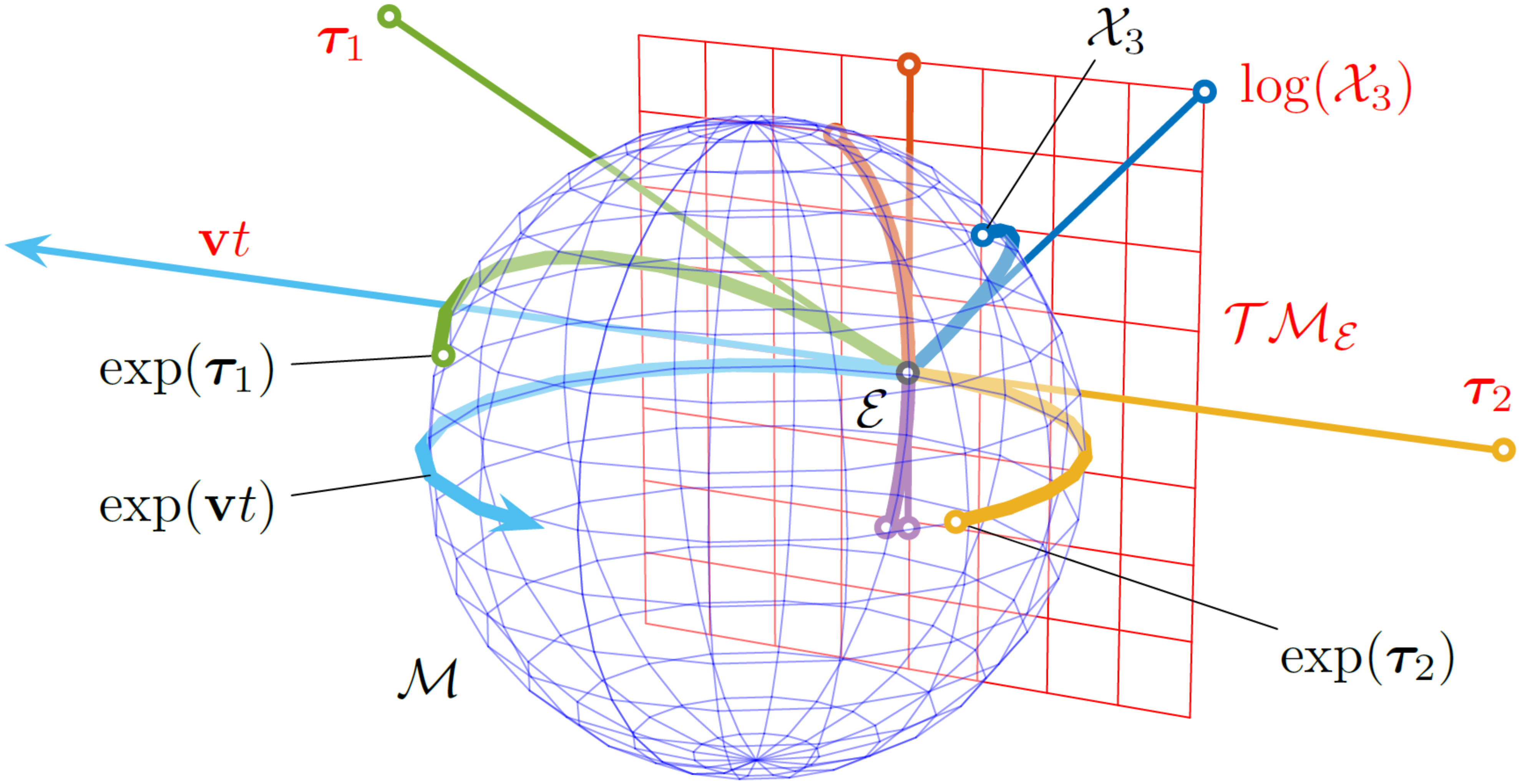

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The exponential map for *SO*(3)

The tangent space vector $\boldsymbol{\theta} = \theta\mathbf{u}$ corresponds to the angle-axis representation, and the Exp map is simply the Rodrigues' rotation formula:

$$\mathbf{R} = \mathrm{Exp}(\boldsymbol{\theta}) \triangleq \mathbf{I} + \sin\theta[\mathbf{u}]_\times + (1 - \cos\theta)[\mathbf{u}]_\times^2$$

The Log map is given by

$$\boldsymbol{\theta} = \mathrm{Log}(\mathbf{R}) \triangleq \frac{\theta}{2\sin\theta}(\mathbf{R} - \mathbf{R}^\top)^\vee$$

$$\theta = \arccos\left(\frac{\mathrm{tr}(\mathbf{R}) - 1}{2}\right)$$

When θ is small, the following approximation holds:

$$\mathbf{R} = \mathrm{Exp}(\boldsymbol{\theta}) \approx \mathbf{I} + \boldsymbol{\theta}^\wedge$$

The exponential map for *SE*(3)

The Exp map is given by:

$$\mathbf{T} = \mathrm{Exp}(\boldsymbol{\xi}) \triangleq \begin{bmatrix} \mathrm{Exp}(\boldsymbol{\theta}) & \mathbf{V}(\boldsymbol{\theta})\boldsymbol{\rho} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

$$\mathbf{V}(\boldsymbol{\theta}) = \mathbf{I} + \frac{1 - \cos\theta}{\theta}[\mathbf{u}]_\times + \frac{\theta - \sin\theta}{\theta}[\mathbf{u}]_\times^2$$

The Log map is given by:

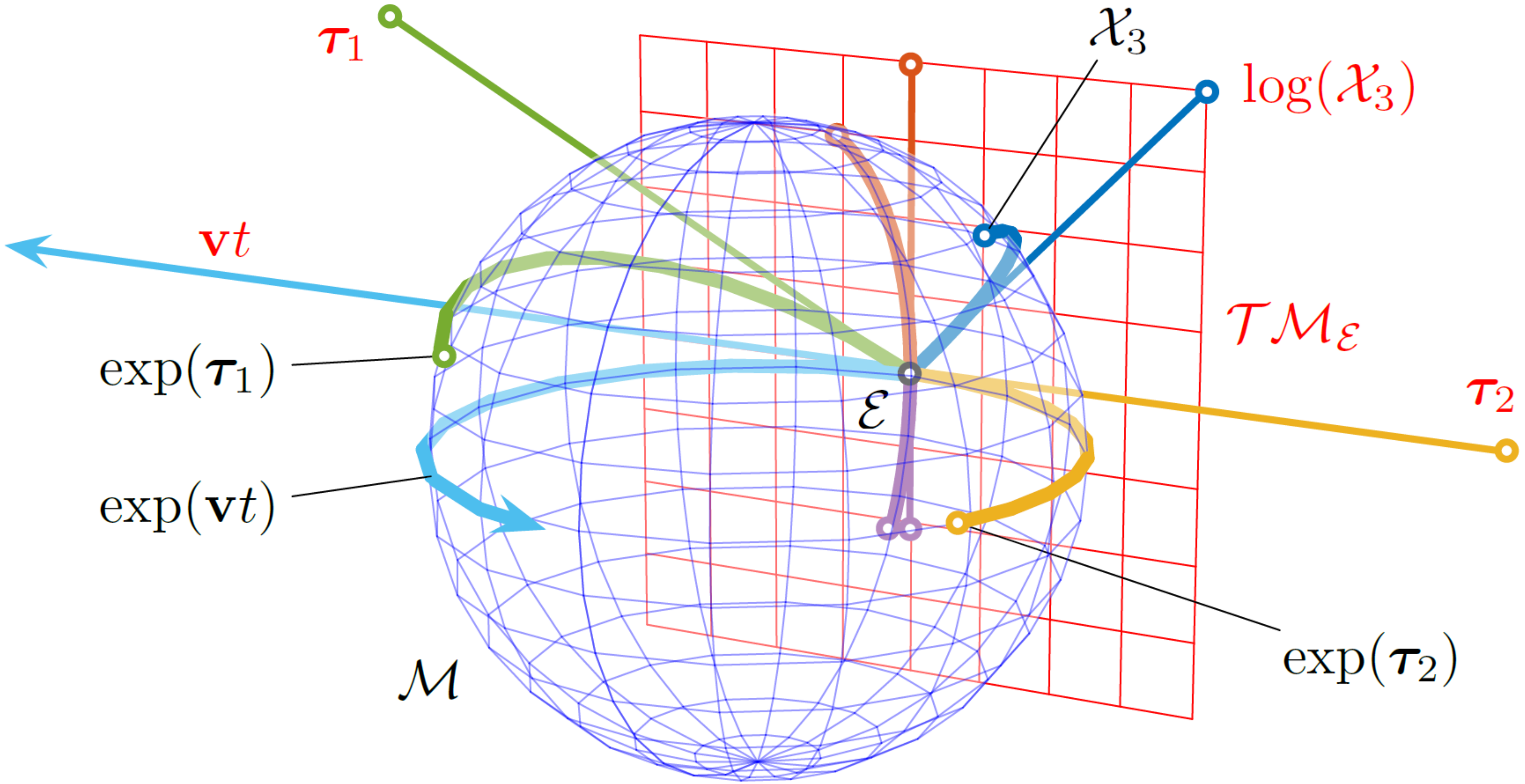
$$\boldsymbol{\xi} = \mathrm{Log}(\mathbf{T}) \triangleq \begin{bmatrix} \mathbf{V}^{-1}(\boldsymbol{\theta})\mathbf{t} \\ \boldsymbol{\theta} \end{bmatrix} \qquad \boldsymbol{\theta} = \mathrm{Log}(\mathbf{R})$$

$$\mathbf{V}^{-1}(\boldsymbol{\theta}) = \mathbf{I} - \frac{\theta}{2}[\mathbf{u}]_\times + \left(1 - \frac{\theta\sin\theta}{2(1 - \cos\theta)}\right)[\mathbf{u}]_\times^2$$

When θ is small, the following approximation holds:

$$\mathbf{T} = \mathrm{Exp}(\boldsymbol{\xi}) \approx \mathbf{I} + \boldsymbol{\xi}^\wedge$$

The exponential map



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Right and left perturbations

We can perform **perturbations** on the manifold expressed as tangent space vectors by combining one $\mathrm{Exp}/\mathrm{Log}$ operation with one composition.

Right perturbations are performed in the **local frame**:

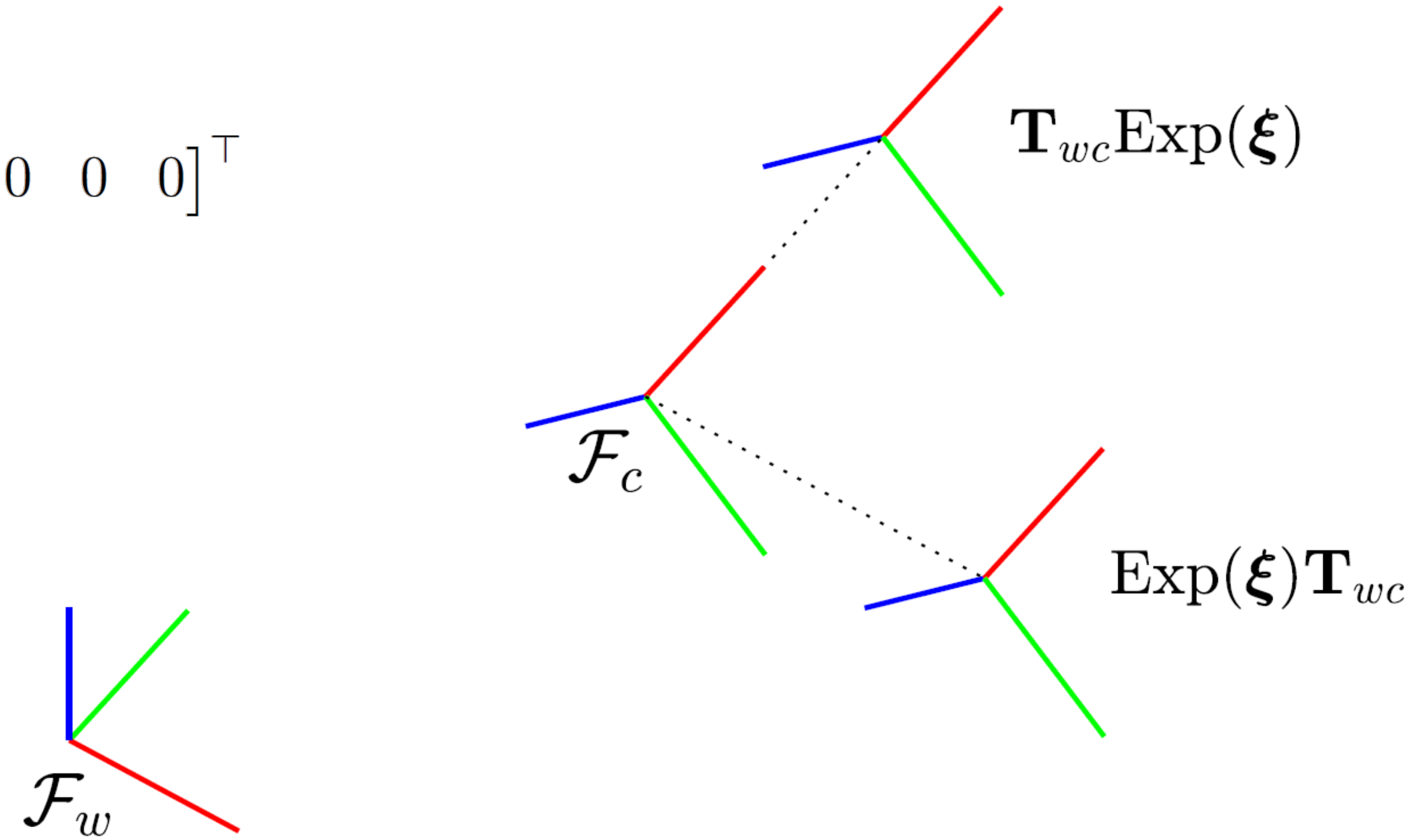
$$\mathcal{X} \circ \mathrm{Exp}({}^{\mathcal{X}}\boldsymbol{\tau})$$

Left perturbations are performed in the **global frame**:

$$\mathrm{Exp}({}^{\mathcal{E}}\boldsymbol{\tau}) \circ \mathcal{X}$$

Right and left perturbations example

$$\boldsymbol{\xi} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^\top$$



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Right perturbations are performed in the **local frame**:

$$\mathcal{X} \circ \mathrm{Exp}({}^{\mathcal{X}}\boldsymbol{\tau})$$

Left perturbations are performed in the **global frame**:

$$\mathrm{Exp}({}^{\mathcal{E}}\boldsymbol{\tau}) \circ \mathcal{X}$$

We will in the following consider **right** perturbations

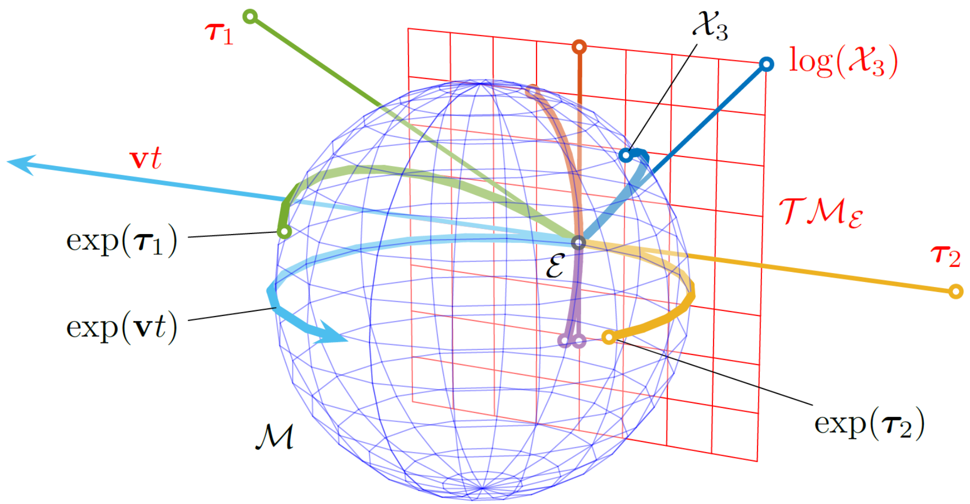
Plus and minus operators

It is convenient to express perturbations using plus and minus operators.

The **right plus and minus operators** are defined as:

$$\mathcal{Y} = \mathcal{X} \oplus {}^{\mathcal{X}}\boldsymbol{\tau} \triangleq \mathcal{X} \circ \operatorname{Exp}({}^{\mathcal{X}}\boldsymbol{\tau}) \in \mathcal{M}$$

$${}^{\mathcal{X}}\boldsymbol{\tau} = \mathcal{Y} \ominus \mathcal{X} \triangleq \operatorname{Log}(\mathcal{X}^{-1} \circ \mathcal{Y}) \in \mathcal{TM}_{\mathcal{X}}$$



Resources

Learn more:

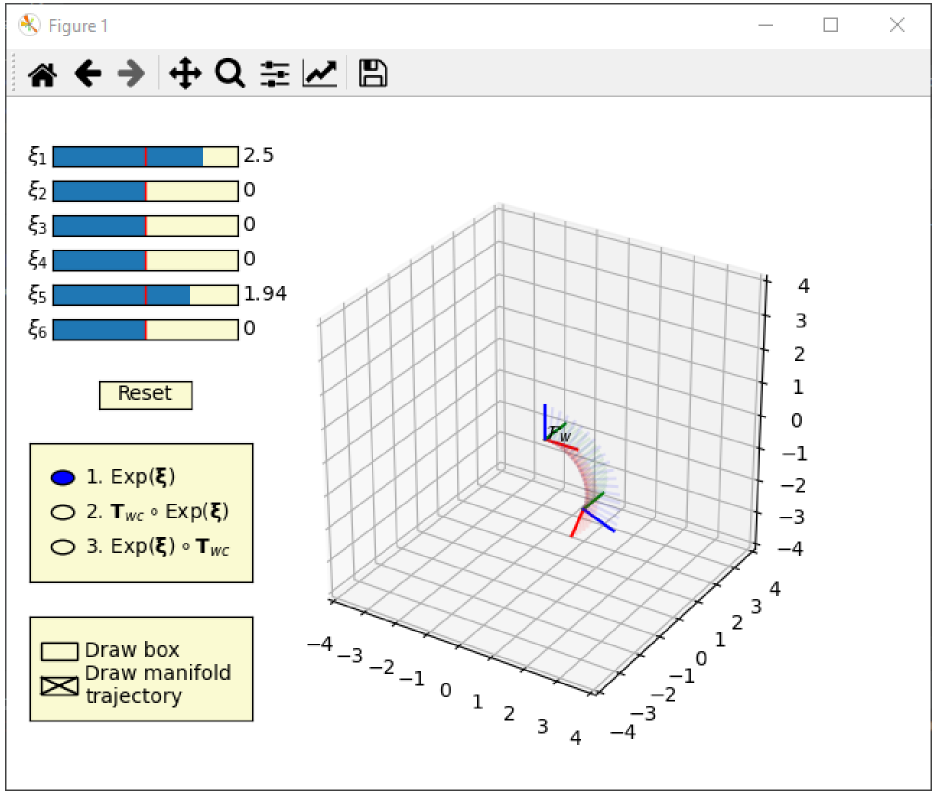
- The compendium

- Solà, J., Deray, J., & Atchuthan, D. (n.d.).
  A micro Lie theory for state estimation in robotics

Using Lie theory in practice:

- My python library pylie:
  https://github.com/tussedrotten/pylie

- The C++ library Sophus:
  https://github.com/strasdat/Sophus



Next lecture