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Pose in 3D

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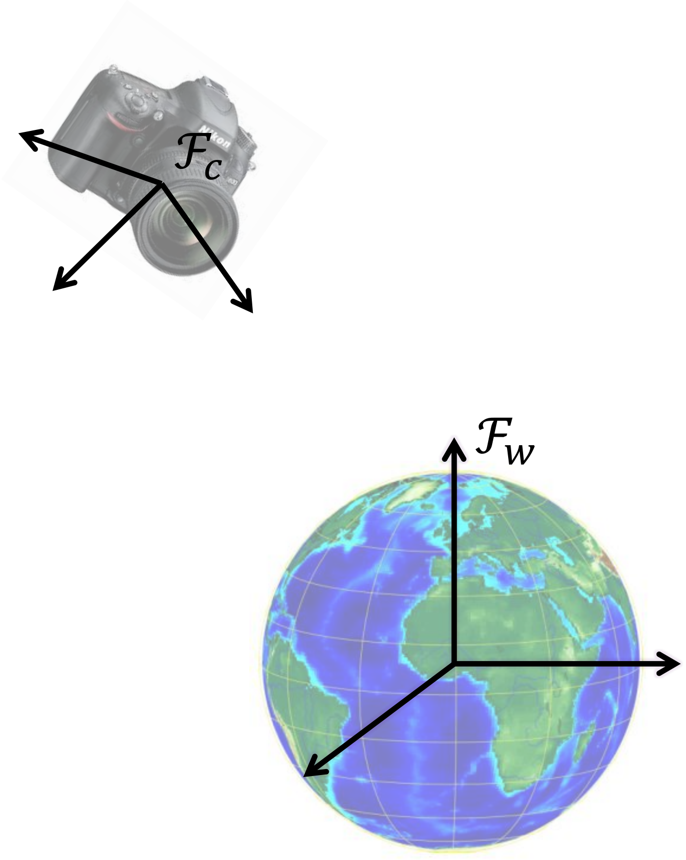
What is pose?

- A term describing the relationship between coordinate frames

- Pose = {Position, Orientation}



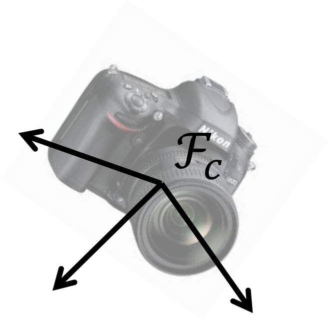
What is pose?

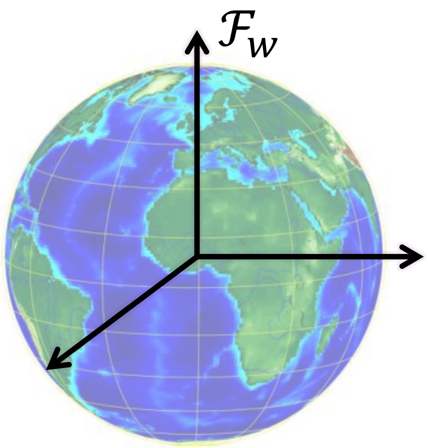
- A term describing the relationship between coordinate frames
- Pose = {Position, Orientation}

The pose of $\mathcal{F}_c$ relative to $\mathcal{F}_w$

$\Updownarrow$

How $\mathcal{F}_w$ should rotate and translate in order to coincide with $\mathcal{F}_c$



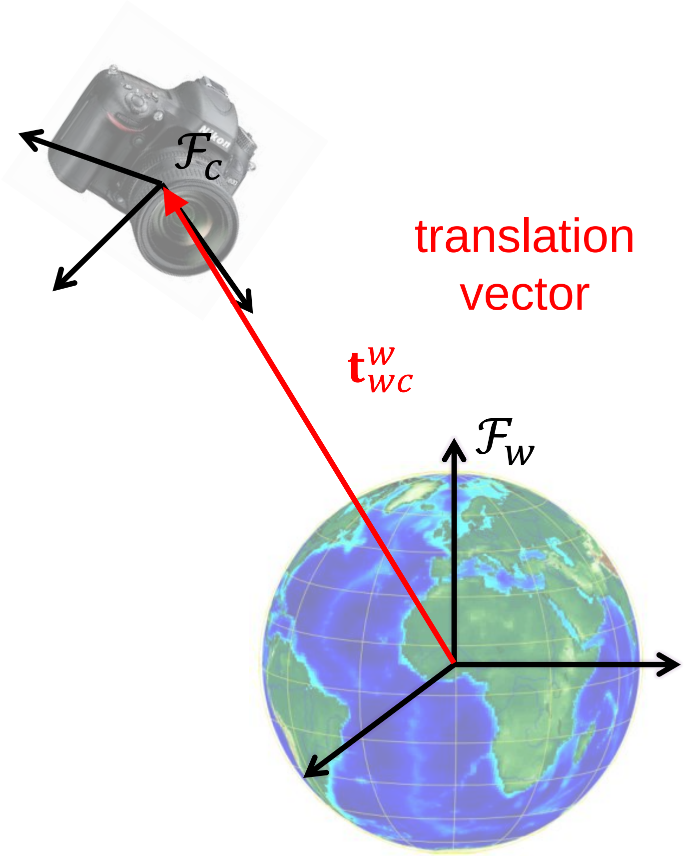


Pose

- The pose of the camera frame $\mathcal{F}_c$ with respect to the world frame $\mathcal{F}_w$ can be represented by the Euclidean transformation matrix

$$\mathbf{T}_{wc} = \begin{bmatrix} \mathbf{R}_{wc} & \mathbf{t}_{wc}^{w} \\ \mathbf{0}^T & 1 \end{bmatrix} \in SE(3)$$

where $\mathbf{R}_{wc} \in SO(3)$ is a rotation matrix and $\mathbf{t}_{wc}^{w} \in \mathbb{R}^3$ is a translation vector given in world coordinates



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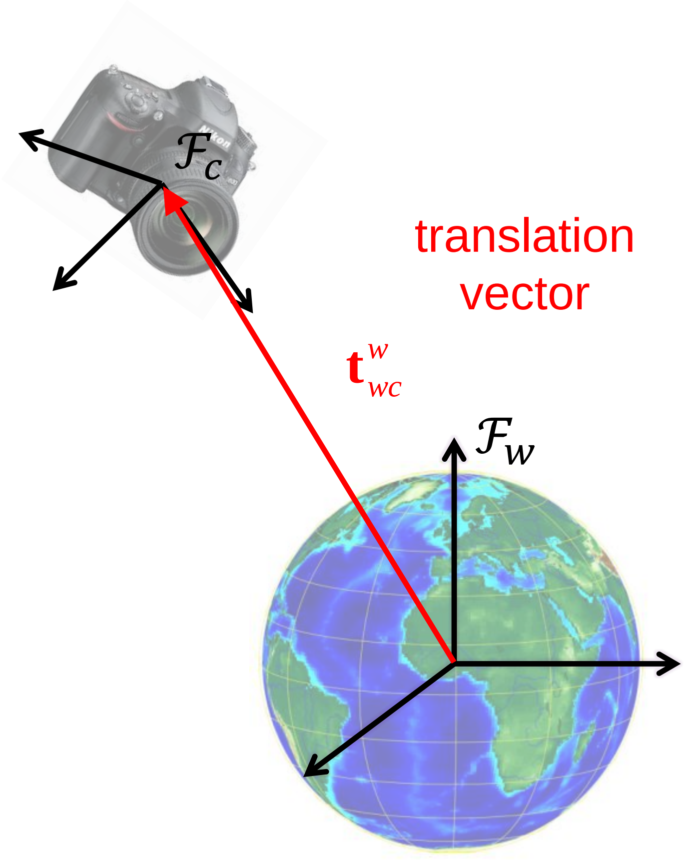
NOTATION

$\mathbf{T}_{ab}$ = The pose of $\mathcal{F}_b$ relative to $\mathcal{F}_a$

$\mathbf{R}_{ab}$ = The orientation of $\mathcal{F}_b$ relative to $\mathcal{F}_a$

$\mathbf{t}_{ab}^{c}$ = The translation of $\mathcal{F}_b$ relative to $\mathcal{F}_a$ given in $\mathcal{F}_c$ coordinates



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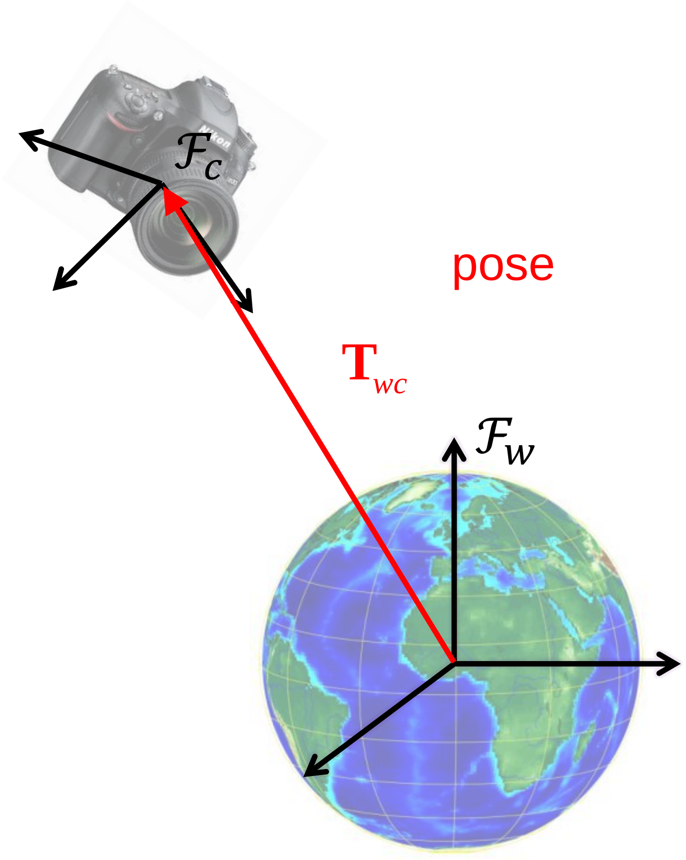
$$SE(3) = \left\{ \mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} \;\middle|\; \mathbf{R}\mathbf{R}^T = \mathbf{1}, \det \mathbf{R} = 1, \mathbf{t} \in \mathbb{R}^3 \right\}$$

- In illustrations we often represent the pose as an arrow similar to that of the translation vector



Pose – Inverse

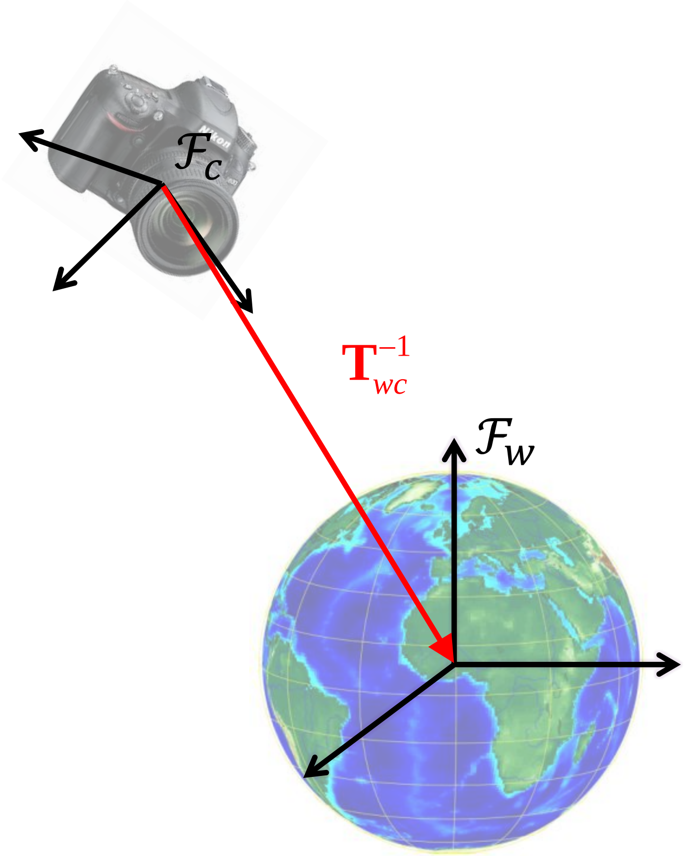
- The opposite pose, the pose of $\mathcal{F}_w$ with respect to $\mathcal{F}_c$, is given by the inverse

$$\mathbf{T}_{cw} = \mathbf{T}_{wc}^{-1}$$

- One can show that

$$\mathbf{T}_{cw} = \begin{bmatrix} \mathbf{R}_{wc} & \mathbf{t}_{wc}^{w} \\ \mathbf{0}^T & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{R}_{wc}^T & -\mathbf{R}_{wc}^T \mathbf{t}_{wc}^{w} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

- Hence $\mathbf{R}_{cw} = \mathbf{R}_{wc}^T$ and $\mathbf{t}_{cw}^c = -\mathbf{R}_{wc}^T \mathbf{t}_{wc}^w$



Pose – Action on points

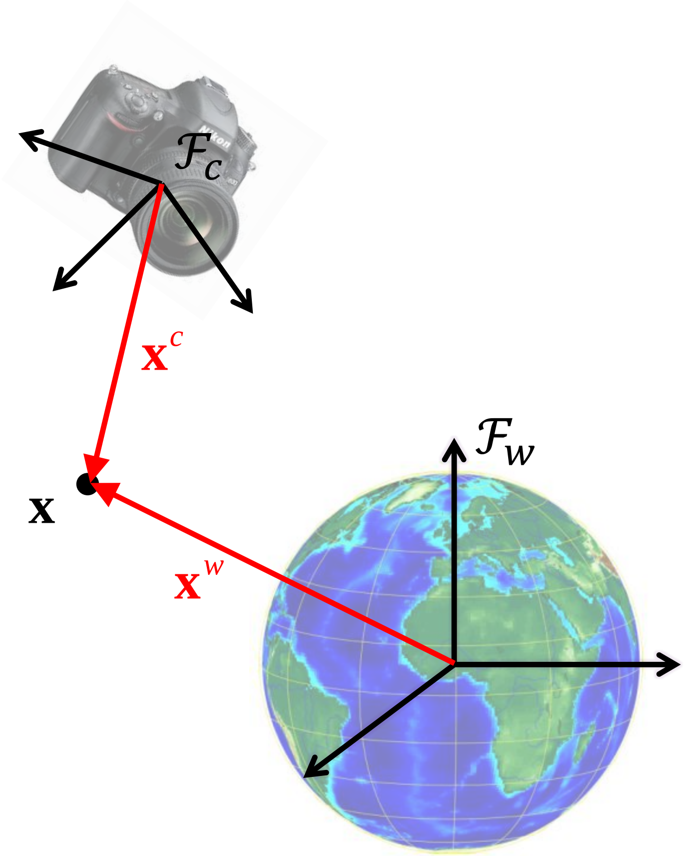
- The action of the pose $\mathbf{T}_{cw}$ on a point $\mathbf{x}$ is defined to be the transformation

$$\mathbf{x}^c = \mathbf{T}_{cw} \cdot \mathbf{x}^w$$

- For the matrix representation, this corresponds to the matrix product

$$\tilde{\mathbf{x}}^c = \mathbf{T}_{cw}\tilde{\mathbf{x}}^w$$

$$\mathbf{x}^c = \mathbf{R}_{cw}\mathbf{x}^w + \mathbf{t}_{cw}^c$$



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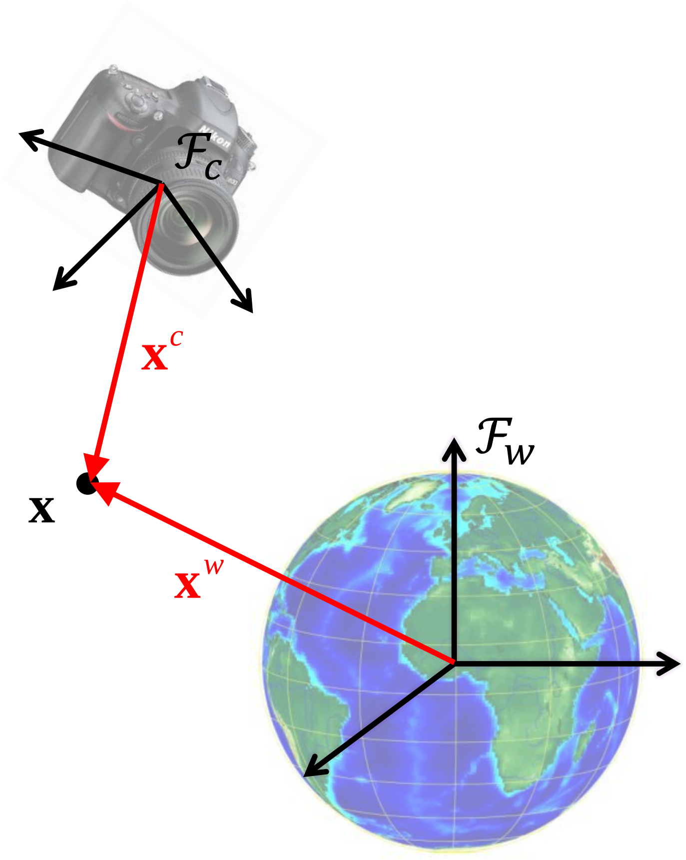


Note
The indexes are always pairwise equal

$$\tilde{\mathbf{x}}^{a} = \mathbf{T}_{ab}\tilde{\mathbf{x}}^{b}$$

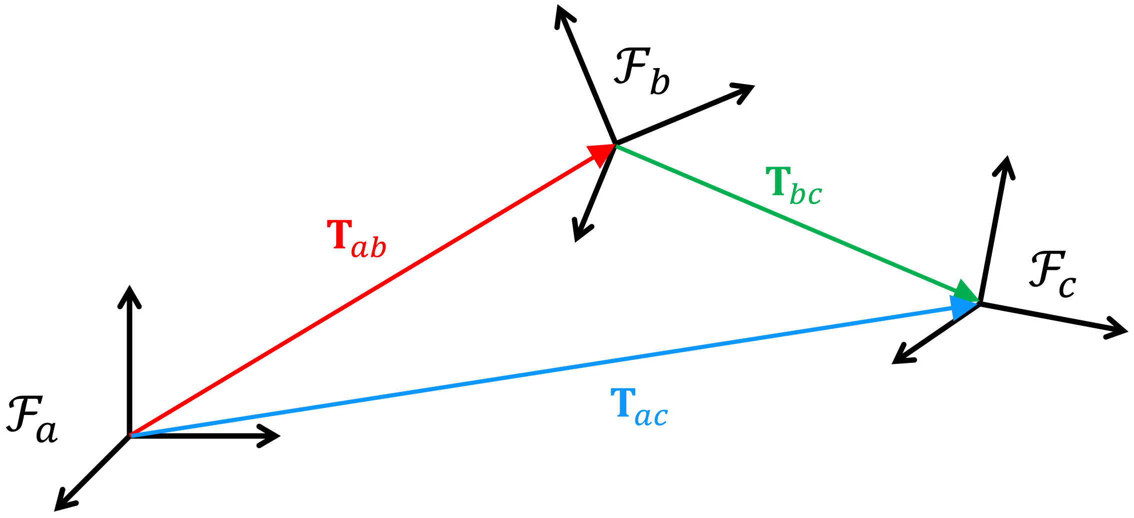
destination frame

source frame

Pose – Composition

We can chain together consecutive poses by compounding transformation matrices

$$\mathbf{T}_{ac} = \mathbf{T}_{ab}\mathbf{T}_{bc}$$



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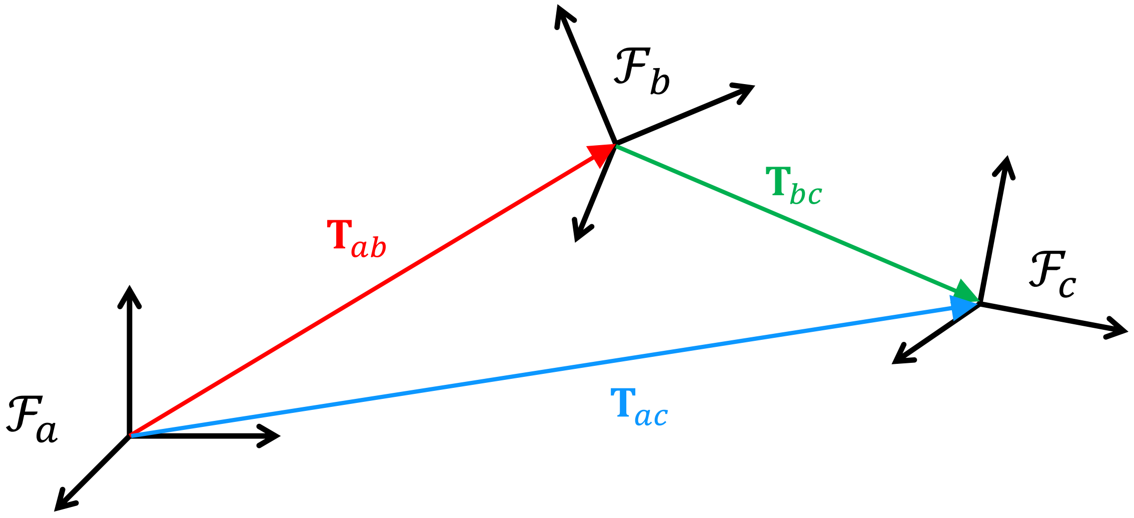


Note

The indexes are always pairwise equal

$$\tilde{\mathbf{x}}^{a} = \mathbf{T}_{ab}\mathbf{T}_{bc}\tilde{\mathbf{x}}^{c}$$

destination frame

intermediate frame

source frame

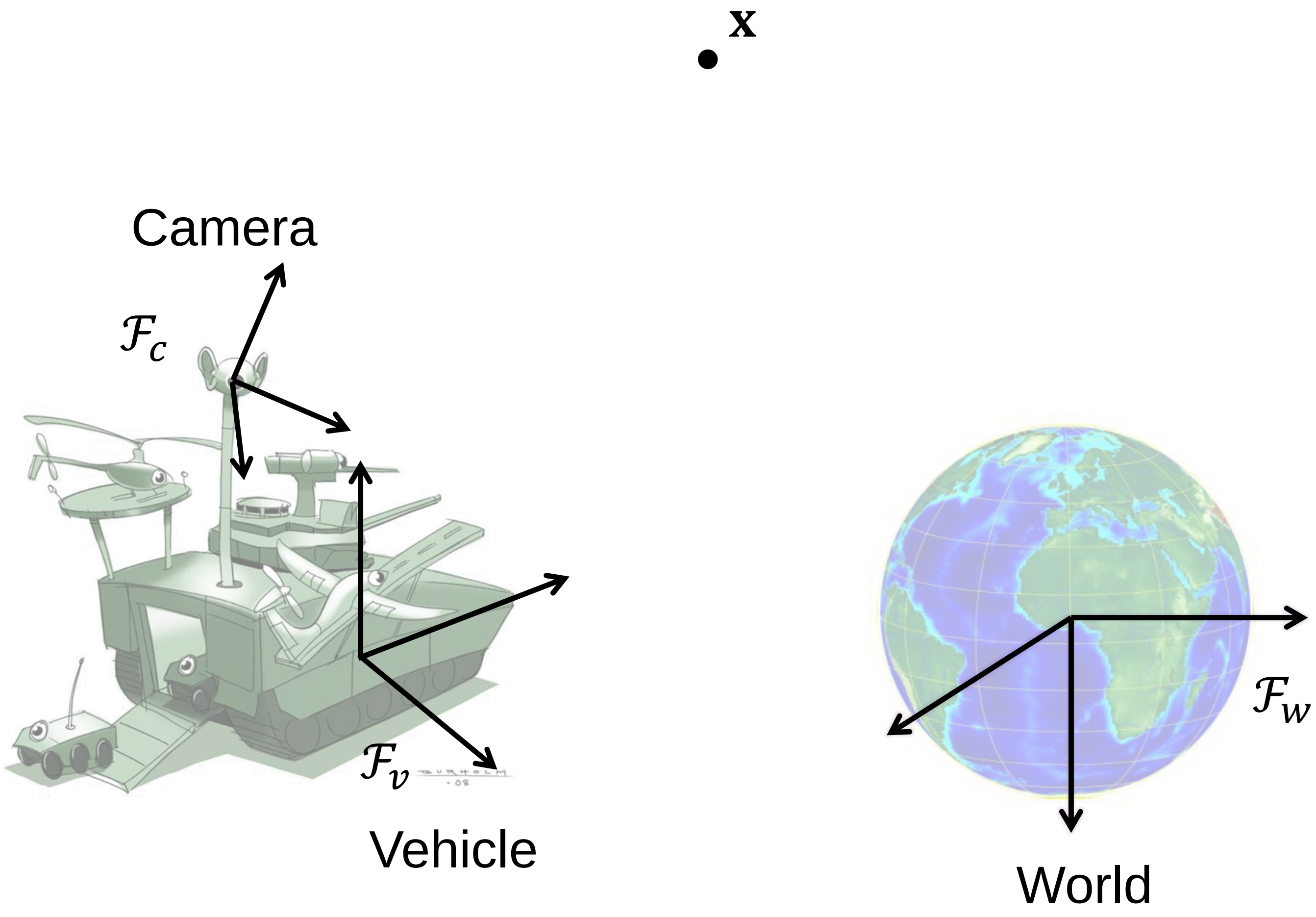
Example – Camera on a vehicle in the world

A point $\mathbf{x}$ has a known position relative to a camera mounted on a vehicle

The vehicle has a known pose relative to the world

The camera has a known pose relative to the vehicle

Find expressions for $\mathbf{x}^v$ and $\mathbf{x}^w$



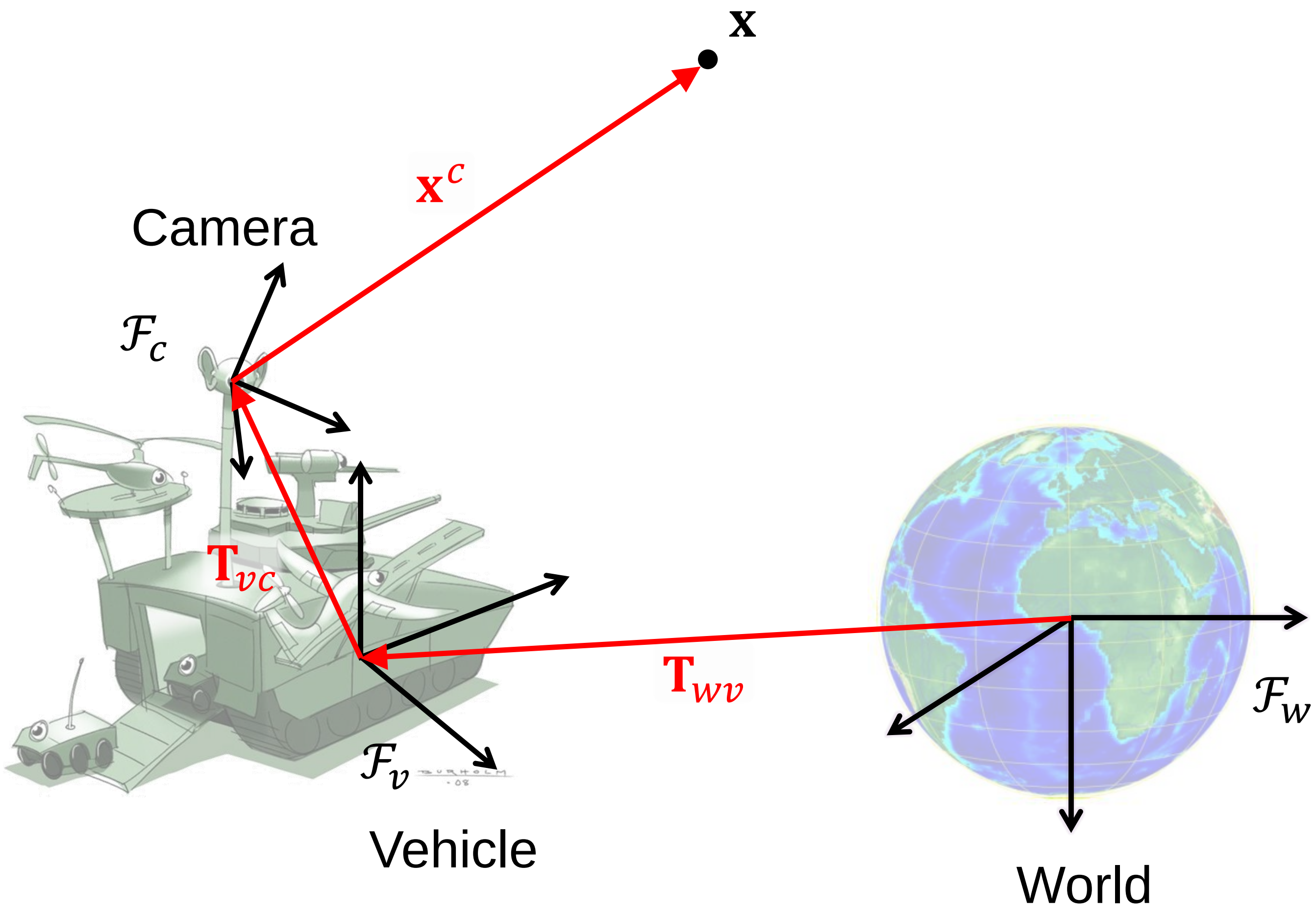
Example – Camera on a vehicle in the world

A point $\mathbf{x}$ has a known position relative to a camera mounted on a vehicle $\mathbf{x}^c$

The vehicle has a known pose relative to the world $\mathbf{T}_{wv}$

The camera has a known pose relative to the vehicle $\mathbf{T}_{vc}$

Find expressions for $\mathbf{x}^v$ and $\mathbf{x}^w$



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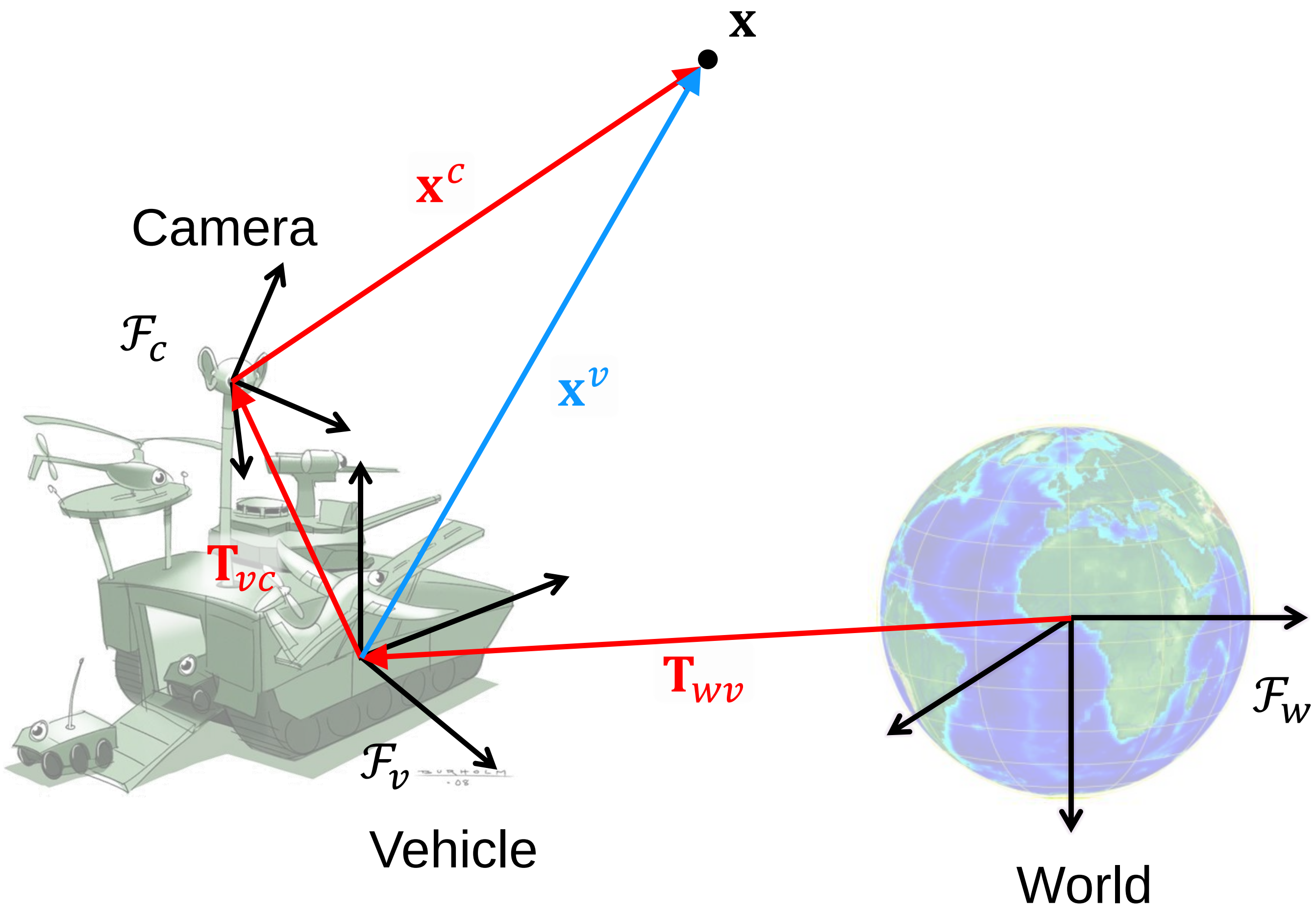
$$\mathbf{x}^v = \mathbf{T}_{vc} \cdot \mathbf{x}^c$$



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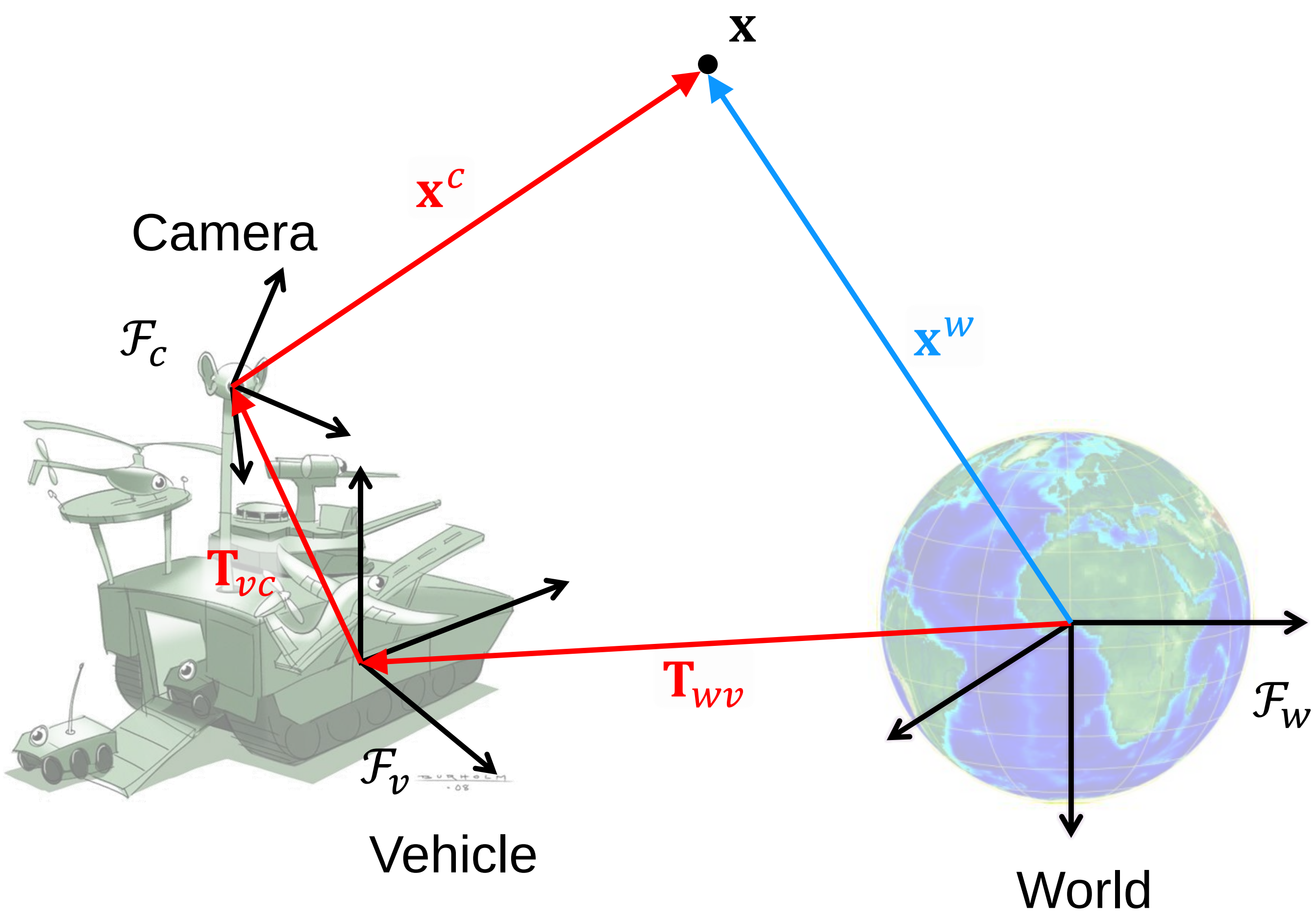
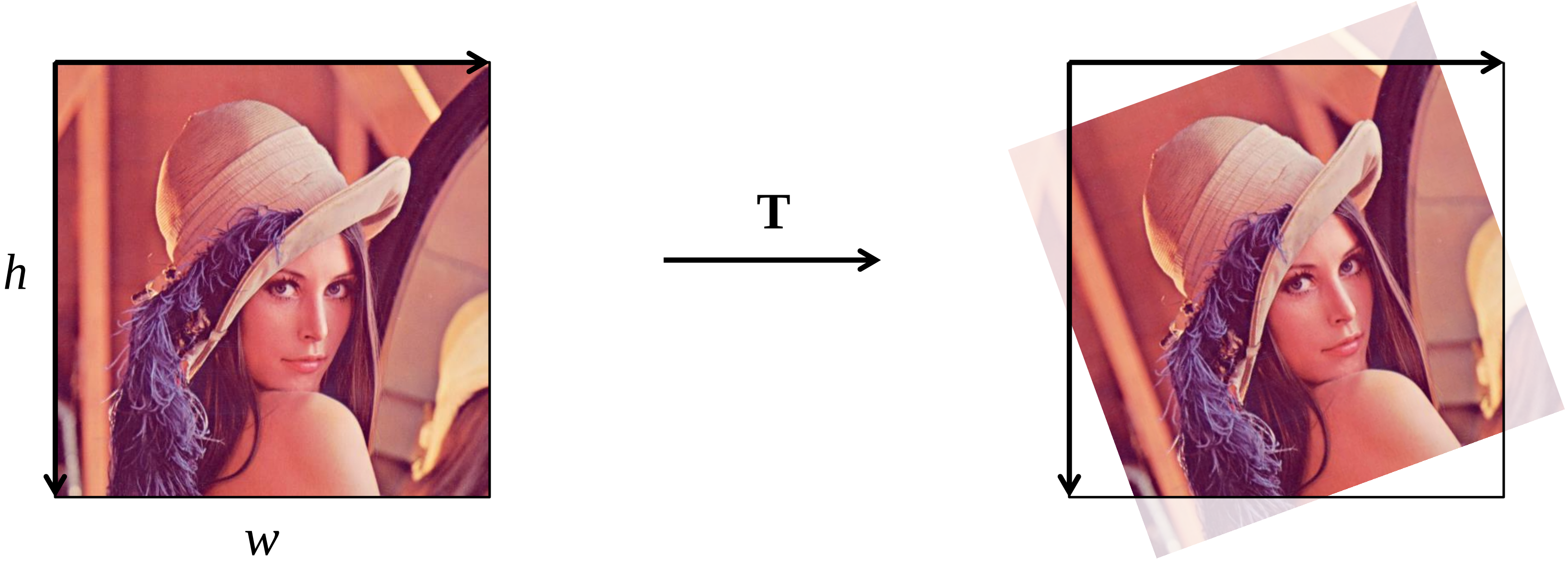
Find expressions for $\mathbf{x}^v$ and $\mathbf{x}^w$

$$\mathbf{x}^v = \mathbf{T}_{vc} \cdot \mathbf{x}^c$$

$$\mathbf{x}^w = \mathbf{T}_{wv}\mathbf{T}_{vc} \cdot \mathbf{x}^c$$



Example – Image rotation about center



Projective transformation

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & w/2 \\ 0 & 1 & h/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -w/2 \\ 0 & 1 & -h/2 \\ 0 & 0 & 1 \end{bmatrix}$$

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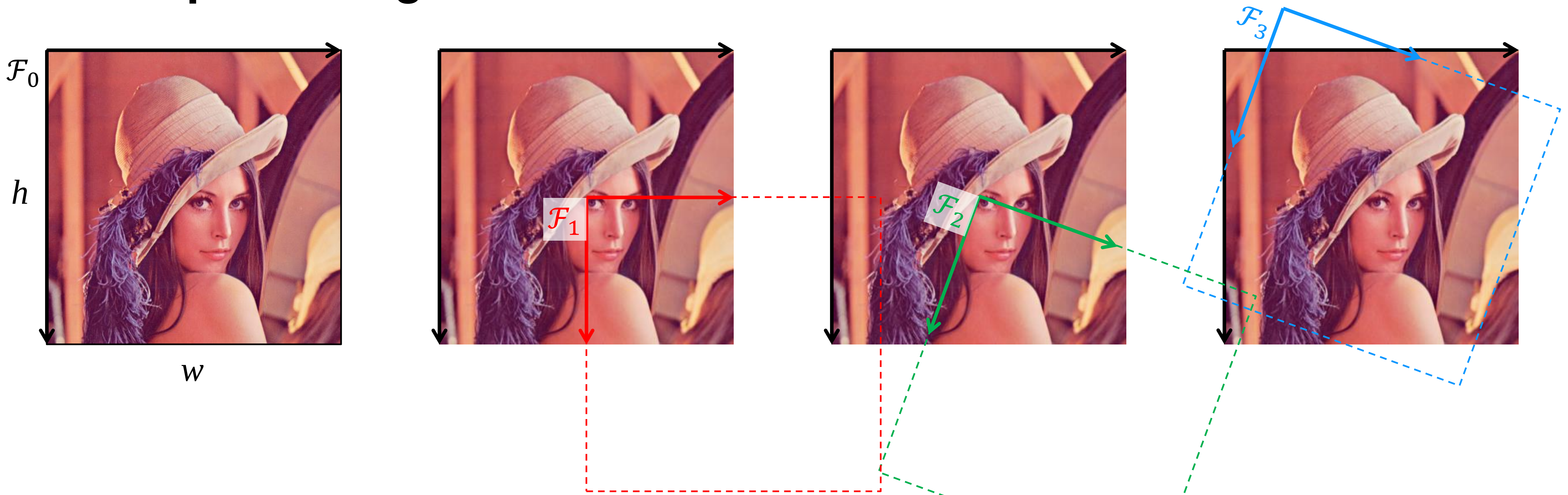


Pose of $\mathcal{F}_0$ relative to $\mathcal{F}_3$

$$\mathbf{T}_{30} = \begin{bmatrix} 1 & 0 & w/2 \\ 0 & 1 & h/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -w/2 \\ 0 & 1 & -h/2 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{T}_{32}\mathbf{T}_{21}\mathbf{T}_{10}$$

Pose – Other representations

- Several representation of rotation in 3D
  - Orthonormal rotation matrix $\mathbf{R}$
  - Euler angles $(\theta_1, \theta_2, \theta_3)$
  - Axis angle $\boldsymbol{\phi} = \phi \mathbf{v}$
  - Unit quaternions $q = q_0 + q_1 i + q_2 j + q_3 k$

- Several representations of pose in 3D
  - Transformation matrix $\mathbf{T}_{ab} \in SE(3)$
  - Pair of rotation matrix and translation vector $(\mathbf{R}_{ab}, \mathbf{t}_{ab})$
  - Euler angles and translation vector $(\theta_1, \theta_2, \theta_3, \mathbf{t}_{ab})$
  - Axis angle and translation vector $(\boldsymbol{\phi}, \mathbf{t}_{ab})$
  - Unit quaternion and translation vector $(q, \mathbf{t}_{ab})$

Summary

- Pose = {Position, Orientation}

- Representation

$$\mathbf{T}_{ab} = \begin{bmatrix} \mathbf{R}_{ab} & \mathbf{t}_{ab}^{a} \\ \mathbf{0} & 1 \end{bmatrix} \in SE(3)$$

- Properties
  - Composition

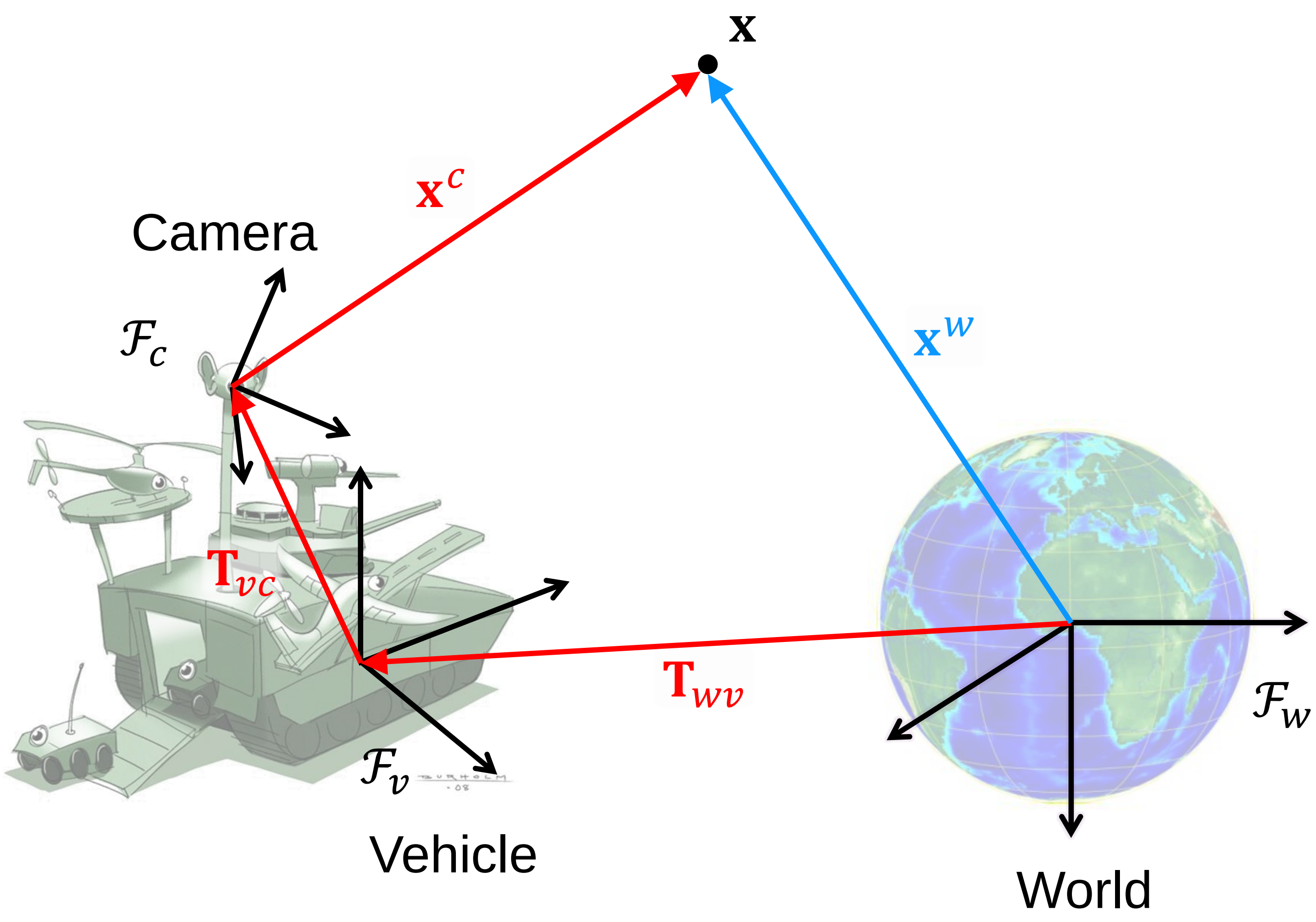
$$\mathbf{T}_{ab}\mathbf{T}_{bc} = \mathbf{T}_{ac}$$

  - Inverse

$$\mathbf{T}_{ab}^{-1} = \mathbf{T}_{ba}$$

  - Action on points

$$\mathbf{T}_{ab}\tilde{\mathbf{x}}^{b} = \tilde{\mathbf{x}}^{a}$$



Supplementary material

Recommended

- *Richard Szeliski: Computer Vision: Algorithms and Applications 2nd ed*
  - Szeliski does not focus directly on pose representation, but covers the topic indirectly several places e.g. in section 2.1 Geometric primitives and transformations

- *T. V. Haavardsholm: A Handbook In Visual SLAM*
  - Chapter 2 “3D geometry”, in particular section 2.3 “Representing pose”