Jacobians with vectors and poses

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$$\mathbf{J} = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \triangleq \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{bmatrix} \in \mathbb{R}^{n \times m}$$

The Jacobian

Given a multivariate function $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ with inputs $\mathbf{x} \in \mathbb{R}^m$ and outputs $f(\mathbf{x}) \in \mathbb{R}^n$ the **Jacobian matrix** stacks all **the partial derivatives** as:

$$\mathbf{J} = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \triangleq \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{bmatrix} \in \mathbb{R}^{n \times m}$$

The Jacobian

We will use the following convenient notation:

$$\mathbf{J} = \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \triangleq \lim_{\mathbf{h} \to 0} \frac{f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})}{\mathbf{h}} \in \mathbb{R}^{n \times m}$$

This lets us define a compact procedure for finding the Jacobian:

$$\lim_{\mathbf{h} \to 0} \frac{f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})}{\mathbf{h}} = \ldots = \lim_{\mathbf{h} \to 0} \frac{\mathbf{J}\mathbf{h}}{\mathbf{h}} \triangleq \frac{\partial (\mathbf{J}\mathbf{h})}{\partial \mathbf{h}} = \mathbf{J}$$

The Jacobian: Example

$$\lim_{\mathbf{h}\to 0} \frac{f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})}{\mathbf{h}} = \ldots = \lim_{\mathbf{h}\to 0} \frac{\mathbf{J}\mathbf{h}}{\mathbf{h}} \triangleq \frac{\partial(\mathbf{J}\mathbf{h})}{\partial \mathbf{h}} = \mathbf{J}$$

$$\begin{aligned} \mathbf{J}_{\mathbf{x}}^{\mathbf{A}\mathbf{x}} = \frac{\partial \mathbf{A}\mathbf{x}}{\partial \mathbf{x}} &= \lim_{\mathbf{h}\to 0} \frac{\mathbf{A}(\mathbf{x}+\mathbf{h}) - \mathbf{A}\mathbf{x}}{\mathbf{h}} \\ &= \lim_{\mathbf{h}\to 0} \frac{\mathbf{A}\mathbf{x} + \mathbf{A}\mathbf{h} - \mathbf{A}\mathbf{x}}{\mathbf{h}} \\ &= \lim_{\mathbf{h}\to 0} \frac{\mathbf{A}\mathbf{h}}{\mathbf{h}} \\ &= \mathbf{A}. \end{aligned}$$

The chain rule

For $\mathbf{y} = f(\mathbf{x})$ and $\mathbf{z} = g(\mathbf{y})$ we have $\mathbf{z} = g(f(\mathbf{x}))$
We can find the derivative of $\mathbf{z}$ with respect to $\mathbf{x}$ by applying the chain rule:

$$\frac{\partial \mathbf{z}}{\partial \mathbf{x}} = \frac{\partial \mathbf{z}}{\partial \mathbf{y}} \frac{\partial \mathbf{y}}{\partial \mathbf{x}}$$

We will use the corresponding notation:

$$\mathbf{J}_{\mathbf{x}}^{\mathbf{z}} = \mathbf{J}_{\mathbf{y}}^{\mathbf{z}} \mathbf{J}_{\mathbf{x}}^{\mathbf{y}}$$

First-order Taylor approximation

We can linearise the function $f(\mathbf{x})$ with the first-order Taylor approximation

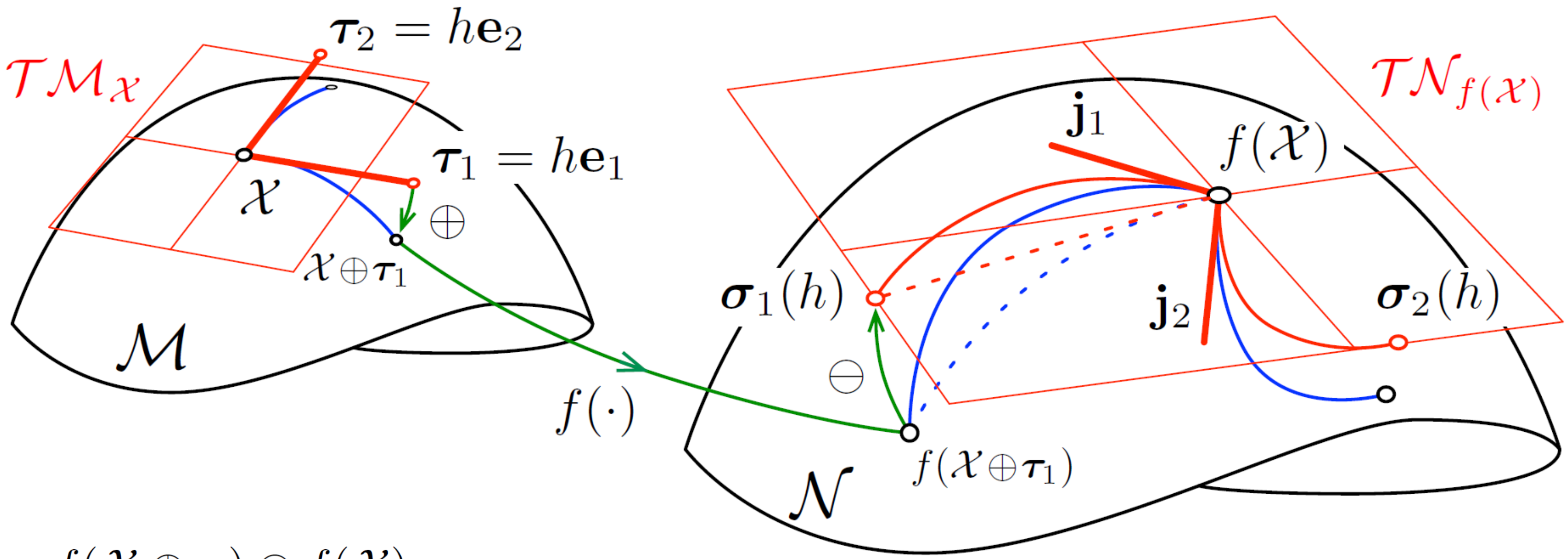
$$f(\mathbf{x}+\mathbf{h}) \xrightarrow[\mathbf{h}\rightarrow 0]{} f(\mathbf{x}) + \mathbf{J}_{\mathbf{x}}^{f(\mathbf{x})}\mathbf{h}$$

Derivatives on Lie groups

We can express the derivative of functions acting on Lie groups similarly using the right plus and minus operators:

$$\mathbf{J} = \frac{{}^{\mathcal{X}}\partial f(\mathcal{X})}{\partial \mathcal{X}} \triangleq \lim_{\boldsymbol{\tau} \to 0} \frac{f(\mathcal{X} \oplus \boldsymbol{\tau}) \ominus f(\mathcal{X})}{\boldsymbol{\tau}} \in \mathbb{R}^{n \times m}$$
$$= \lim_{\boldsymbol{\tau} \to 0} \frac{\operatorname{Log}\left(f(\mathcal{X})^{-1} \circ f(\mathcal{X} \circ \operatorname{Exp}(\boldsymbol{\tau}))\right)}{\boldsymbol{\tau}}$$

Derivatives on Lie groups



$$\mathbf{J} = \frac{{}^{\mathcal{X}}\partial f(\mathcal{X})}{\partial \mathcal{X}} \triangleq \lim_{\boldsymbol{\tau} \to 0} \frac{f(\mathcal{X} \oplus \boldsymbol{\tau}) \ominus f(\mathcal{X})}{\boldsymbol{\tau}} \in \mathbb{R}^{n \times m}$$

$$= \lim_{\boldsymbol{\tau} \to 0} \frac{\mathrm{Log}\left(f(\mathcal{X})^{-1} \circ f(\mathcal{X} \circ \mathrm{Exp}(\boldsymbol{\tau}))\right)}{\boldsymbol{\tau}}$$

Derivatives on Lie groups: Example

$$
\begin{aligned}
\mathbf{J}_{\mathcal{Y}}^{\mathcal{X}\circ\mathcal{Y}} &= \lim_{\boldsymbol{\tau}\to 0} \frac{(\mathcal{X}\circ(\mathcal{Y}\oplus\boldsymbol{\tau}))\ominus(\mathcal{X}\circ\mathcal{Y})}{\boldsymbol{\tau}} \\
&= \lim_{\boldsymbol{\tau}\to 0} \frac{\operatorname{Log}\left((\mathcal{X}\circ\mathcal{Y})^{-1}\circ(\mathcal{X}\circ(\mathcal{Y}\circ\operatorname{Exp}(\boldsymbol{\tau})))\right)}{\boldsymbol{\tau}} \\
&= \lim_{\boldsymbol{\tau}\to 0} \frac{\operatorname{Log}\left((\mathcal{X}\circ\mathcal{Y})^{-1}(\mathcal{X}\circ\mathcal{Y})\circ\operatorname{Exp}(\boldsymbol{\tau})\right)}{\boldsymbol{\tau}} \\
&= \lim_{\boldsymbol{\tau}\to 0} \frac{\boldsymbol{\tau}}{\boldsymbol{\tau}} \\
&= \mathbf{I}.
\end{aligned}
$$

First-order Taylor approximation

We can linearise the function $f(\mathcal{X})$ with the first-order Taylor approximation

$$f(\mathcal{X} \oplus \boldsymbol{\tau}) \xrightarrow[\boldsymbol{\tau} \to 0]{} f(\mathcal{X}) \oplus \mathbf{J}_{\mathcal{X}}^{f(\mathcal{X})} \boldsymbol{\tau}$$

Elementary Lie group Jacobians

Jacobians for *SO*(3) and *SE*(3) are given in the compendium!

- Try computing a few of them by hand!
- They are also implemented in [pylie]()

5.4.1 Jacobian of the inverse operation

$$\mathbf{J}_{\mathbf{R}}^{\mathbf{R}^{-1}} = -\mathbf{Ad}_{\mathbf{R}} = -\mathbf{R}. \tag{5.50}$$

5.4.2 Jacobians of the composition operation

$$\mathbf{J}_{\mathbf{R}_a}^{\mathbf{R}_a\mathbf{R}_b} = \mathbf{Ad}_{\mathbf{R}_b}{}^{-1} = \mathbf{R}_b^\top \tag{5.51}$$
$$\mathbf{J}_{\mathbf{R}_b}^{\mathbf{R}_a\mathbf{R}_b} = \mathbf{I}. \tag{5.52}$$

5.4.3 Jacobians of the group action

From Example 5.14 we have

$$\mathbf{J}_{\mathbf{R}}^{\mathbf{R}\cdot\mathbf{x}} = -\mathbf{R}[\mathbf{x}]_\times \tag{5.53}$$
$$\mathbf{J}_{\mathbf{x}}^{\mathbf{R}\cdot\mathbf{x}} = \mathbf{R}. \tag{5.54}$$

5.4.4 Jacobians of the plus and minus operators

$$\mathbf{J}_{\mathbf{R}}^{\mathbf{R}\oplus\boldsymbol{\theta}} = \mathbf{Ad}_{\mathrm{Exp}(\boldsymbol{\theta})}{}^{-1} = \mathbf{R}(\boldsymbol{\theta})^\top \tag{5.55}$$
$$\mathbf{J}_{\boldsymbol{\theta}}^{\mathbf{R}\oplus\boldsymbol{\theta}} = \mathbf{J}_r(\boldsymbol{\theta}). \tag{5.56}$$

For $\boldsymbol{\theta} = \mathbf{R}_b \ominus \mathbf{R}_a$, we have

$$\mathbf{J}_{\mathbf{R}_a}^{\mathbf{R}_b\ominus\mathbf{R}_a} = -\mathbf{J}_l{}^{-1}(\boldsymbol{\theta}) \tag{5.57}$$
$$\mathbf{J}_{\mathbf{R}_b}^{\mathbf{R}_b\ominus\mathbf{R}_a} = \mathbf{J}_r{}^{-1}(\boldsymbol{\theta}). \tag{5.58}$$

5.5 Jacobian blocks for *SE*(3)

The left Jacobian and its inverse have the following closed form expressions:

$$\mathbf{J}_l(\boldsymbol{\xi}) = \begin{bmatrix} \mathbf{J}_l(\boldsymbol{\theta}) & \mathbf{Q}(\boldsymbol{\xi}) \\ \mathbf{0} & \mathbf{J}_l(\boldsymbol{\theta}) \end{bmatrix} \tag{5.59}$$
$$\mathbf{J}_l{}^{-1}(\boldsymbol{\xi}) = \begin{bmatrix} \mathbf{J}_l{}^{-1}(\boldsymbol{\theta}) & -\mathbf{J}_l{}^{-1}(\boldsymbol{\theta})\mathbf{Q}(\boldsymbol{\xi})\mathbf{J}_l{}^{-1}(\boldsymbol{\theta}) \\ \mathbf{0} & \mathbf{J}_l{}^{-1}(\boldsymbol{\theta}) \end{bmatrix}. \tag{5.60}$$

Vectors are also Lie groups!

The group of vectors under addition $(\mathbb{R}^n, +)$ is a trivial Lie group
where the group elements, the Lie algebra and the tangent spaces are all the same:

$$\mathbf{t} = \mathbf{t}^\wedge = \mathrm{Exp}(\mathbf{t})$$

This means that

$$\mathbf{t}_1 \oplus \mathbf{t}_2 = \mathbf{t}_1 + \mathbf{t}_2$$
$$\mathbf{t}_2 \ominus \mathbf{t}_1 = \mathbf{t}_2 - \mathbf{t}_1$$

and that everything we have developed for Lie groups also applies for vectors, including:

$$\mathbf{J} = \frac{{}^{\mathcal{X}}\partial f(\mathcal{X})}{\partial \mathcal{X}} \triangleq \lim_{\boldsymbol{\tau} \to 0} \frac{f(\mathcal{X} \oplus \boldsymbol{\tau}) \ominus f(\mathcal{X})}{\boldsymbol{\tau}} \in \mathbb{R}^{n \times m}$$