Lecture 8.3
Triangulation by minimizing reprojection error

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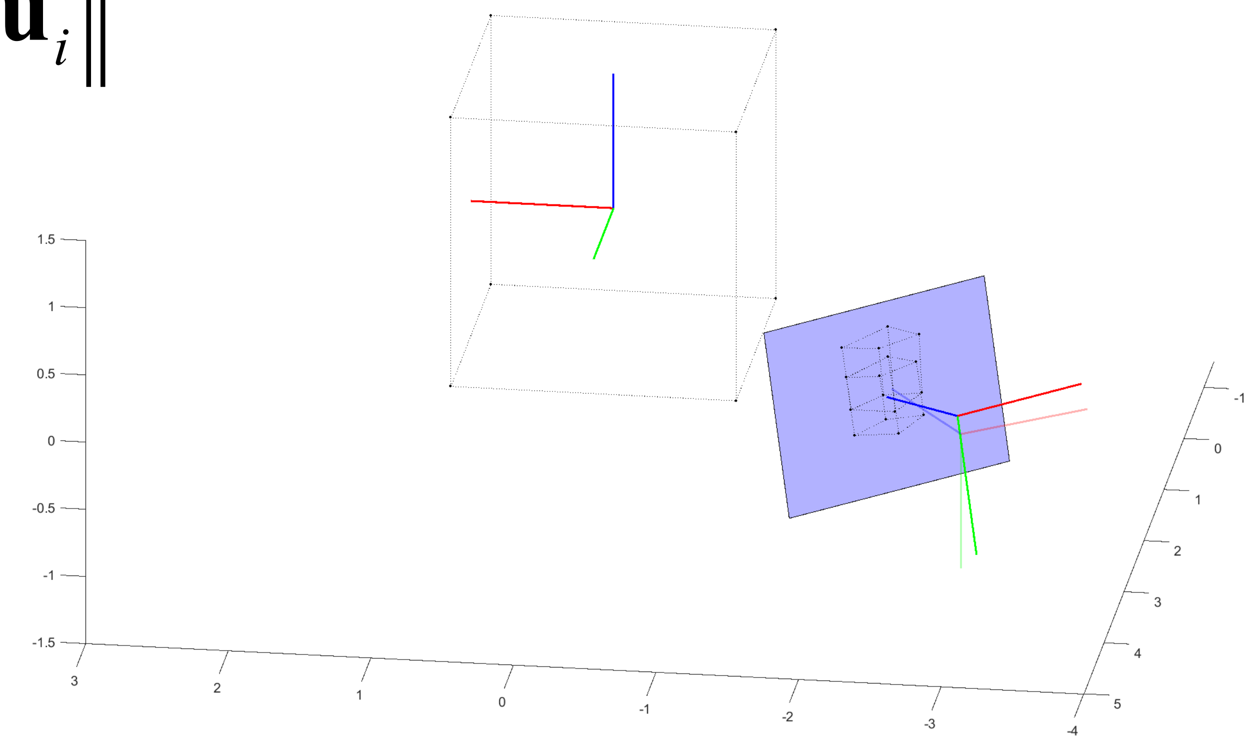
Pose estimation by minimizing reprojection error

Minimize **geometric error** over the **camera pose**

This is also sometimes called **Motion-Only Bundle Adjustment**

$$\mathbf{T}_{cw}^{*} = \underset{\mathbf{T}_{cw}}{\operatorname{argmin}} \sum_{i} \left\| \pi(\mathbf{T}_{cw}\tilde{\mathbf{x}}_{i}^{w}) - \mathbf{u}_{i} \right\|^{2}$$



Nonlinear state estimation

We have seen how we can find the **MAP estimate** of our unknown states given measurements

$$X^{MAP} = \underset{X}{\operatorname{argmax}}\ p(X \mid Z)$$

by representing it as
a **nonlinear least squares problem**

$$X^{*} = \underset{X}{\operatorname{argmin}} \sum_{i=1}^{m} \left\| h_i(X_i) - \mathbf{z}_i \right\|_{\mathbf{\Sigma}_i}^{2}$$

Choose a suitable inital estimate X^0

$$\mathbf{A}, \mathbf{b} \leftarrow \text{Linearize at } X^t$$

$$\mathbf{\Delta}^{*} \leftarrow \text{Solve } \underset{\mathbf{\Delta}}{\operatorname{argmin}} \left\| \mathbf{A}\mathbf{\Delta} - \mathbf{b} \right\|^{2}$$

$$X^{t+1} \leftarrow X^{t} + \mathbf{\Delta}^{*}$$

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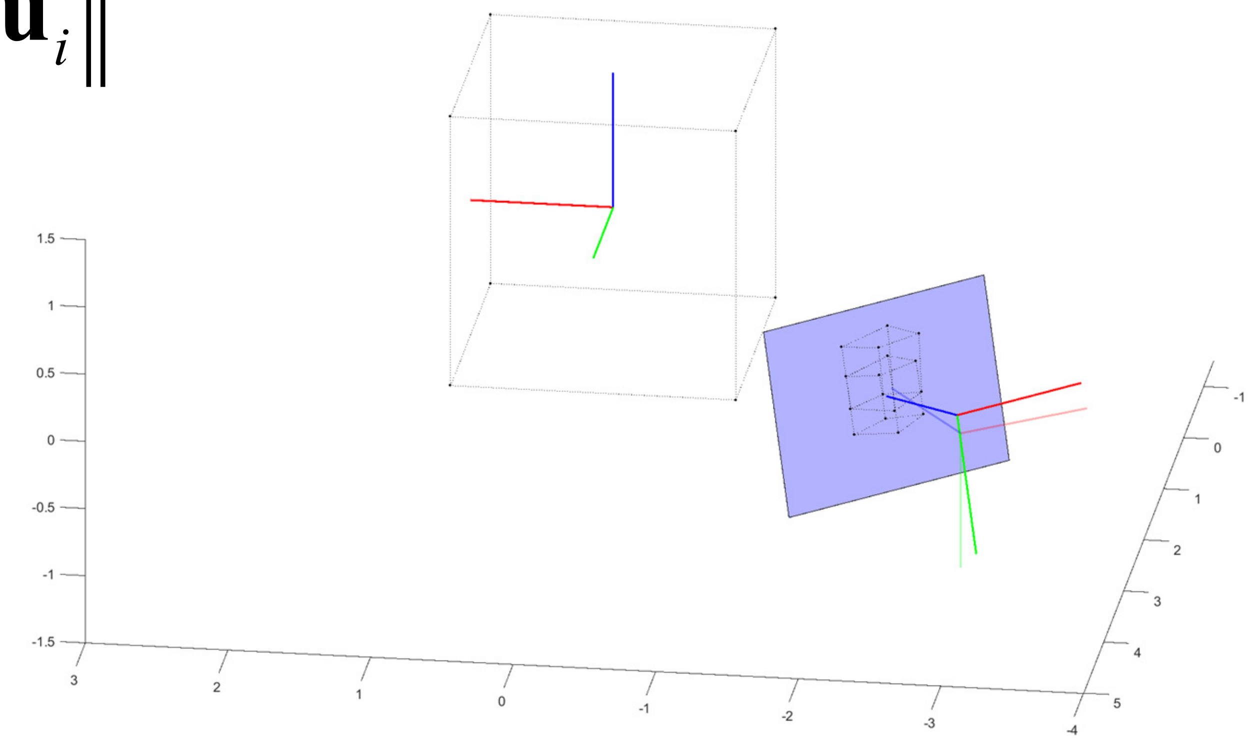

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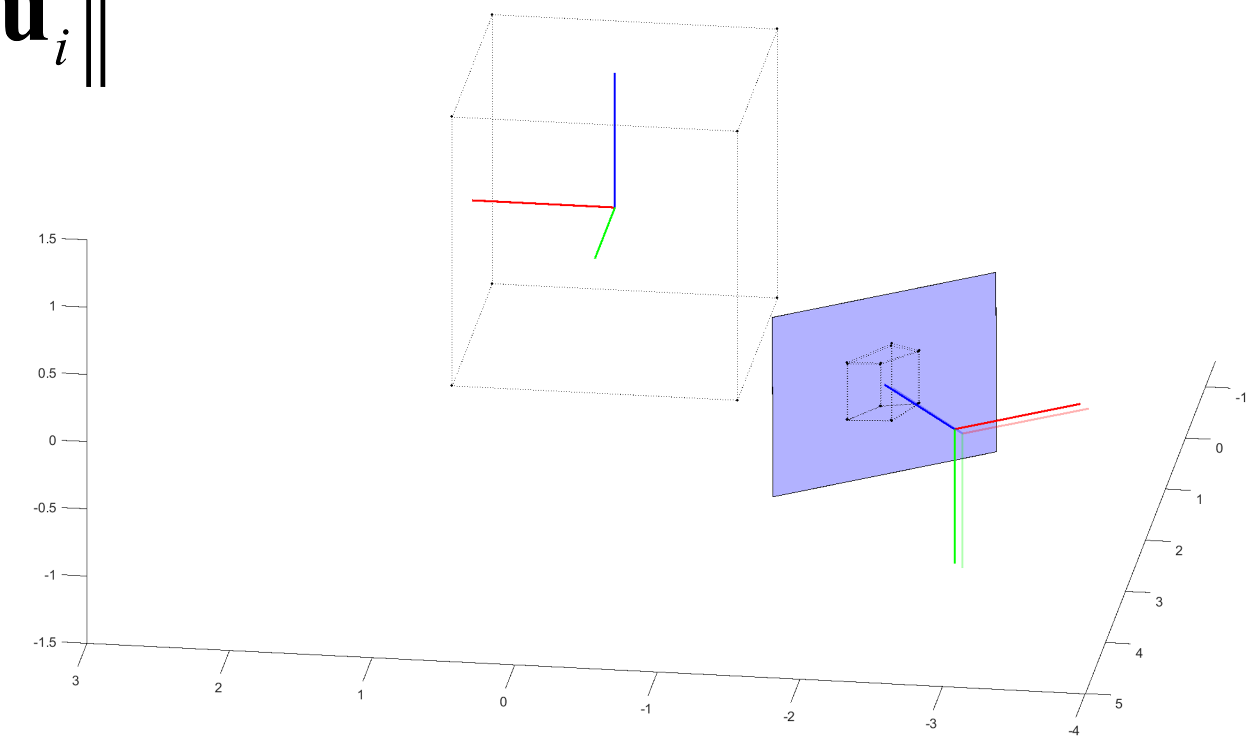

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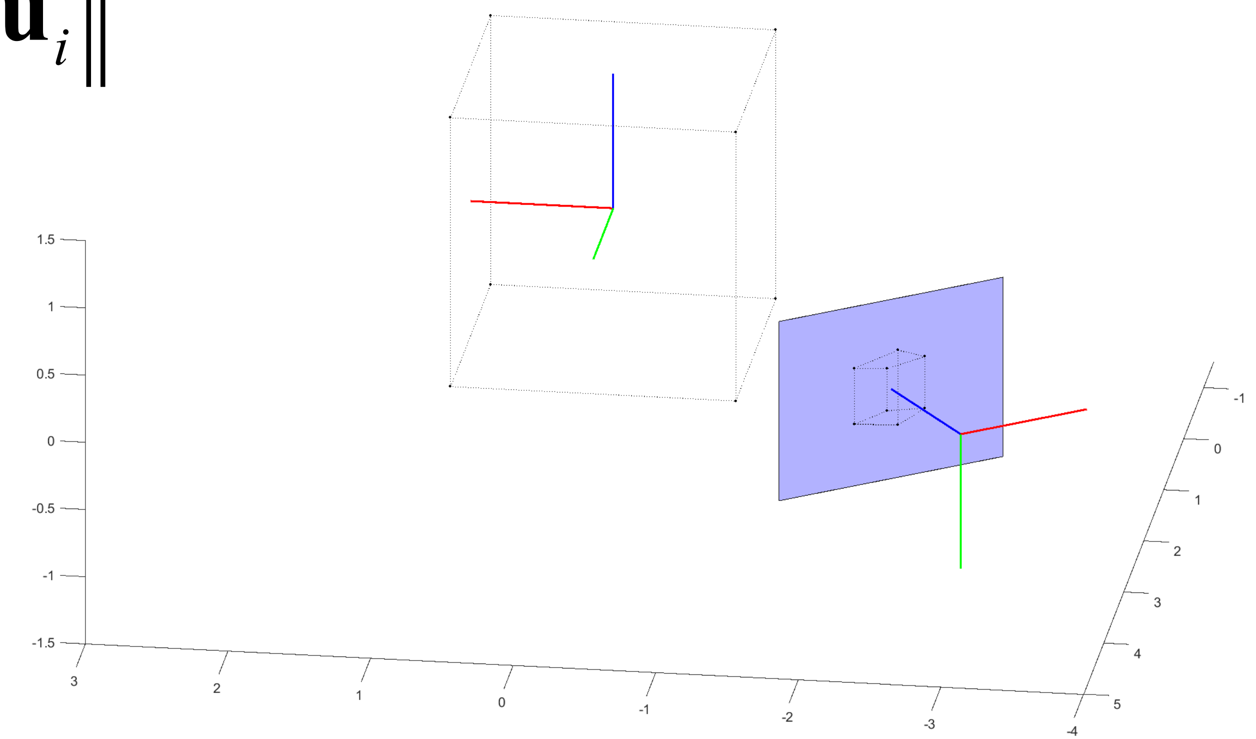

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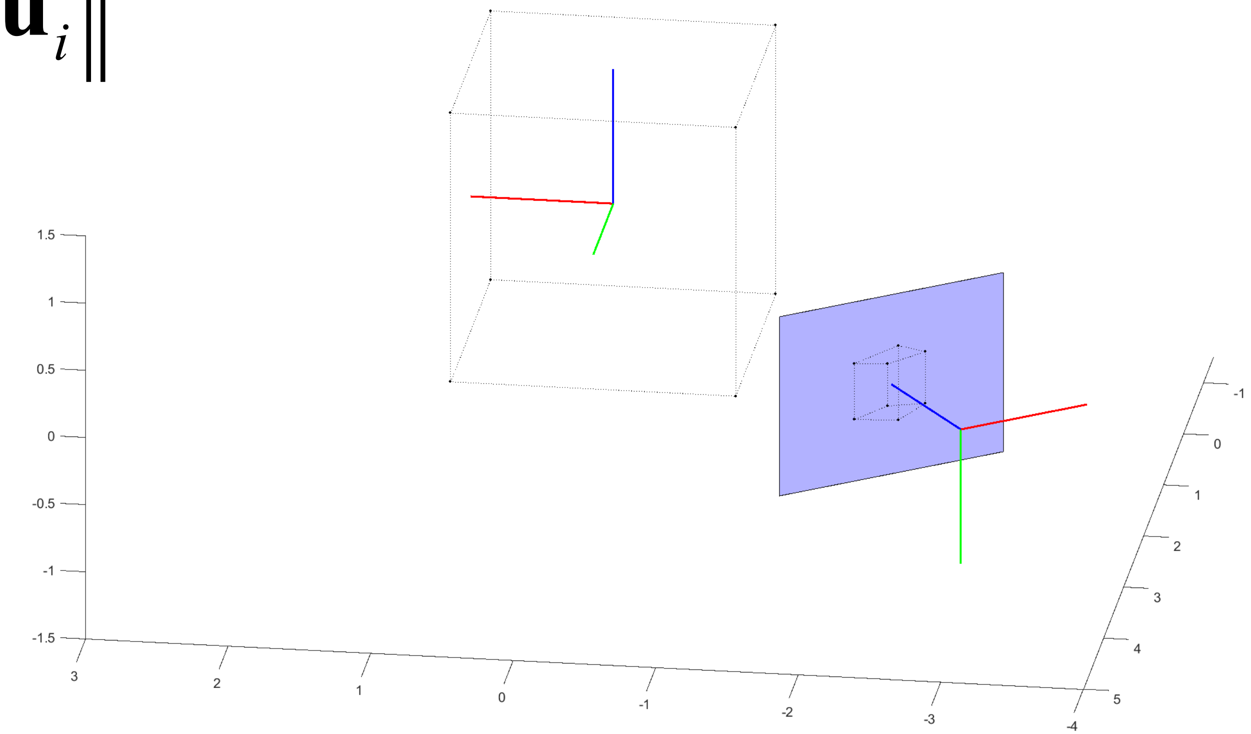
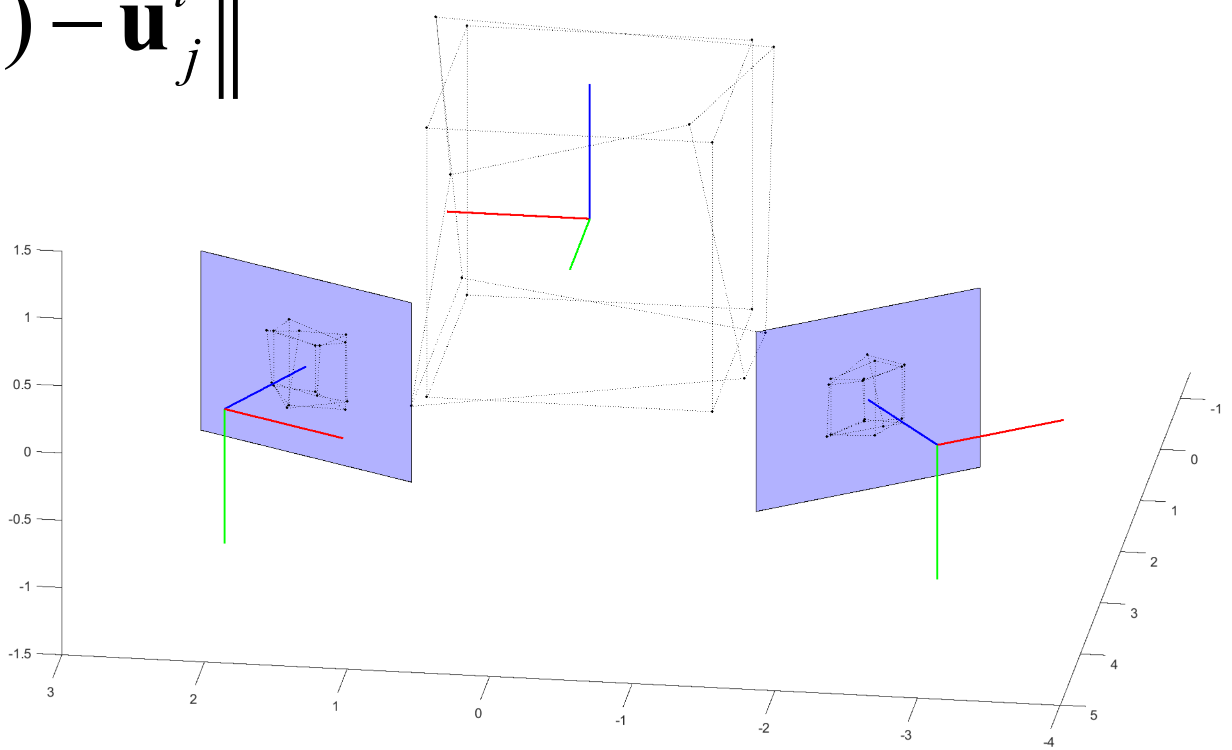


Triangulation by minimizing reprojection error

Minimize **geometric error** over the **world points**

This is also sometimes called **Structure-Only Bundle Adjustment**

$$\mathbf{x}_j^{w*} = \underset{\mathbf{x}_j^{w*}}{\operatorname{argmin}} \sum_i \sum_j \left\| \pi_i(\mathbf{T}_{cw_i} \tilde{\mathbf{x}}_j^w) - \mathbf{u}_j^i \right\|^2$$



Objective function

Minimize error over the **state variables** $X = \{\mathbf{x}_j^w\}$
with the **measurements** $Z = \{\mathbf{u}_j^i\} = \{\mathbf{x}_{n_j}^i\}$

The optimization problem is

$$X^* = \underset{X}{\operatorname{argmin}} \sum_i \sum_j \left\| \pi(g(\mathbf{T}_{wc_i}, \mathbf{x}_j^w)) - \mathbf{x}_{n_j}^i \right\|_{\mathbf{\Sigma}_{ij}}^2$$

For simpler notation,
we assume that the measurements are pre-calibrated to normalized image coordinates

$$\mathbf{x}_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{K}^{-1} \begin{bmatrix} \mathbf{u} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{u - c_u}{f_u} \\ \frac{v - c_v}{f_v} \end{bmatrix}$$

Objective function

Minimize error over the **state variables** $X = \left\{ \mathbf{x}_j^w \right\}$
with the **measurements** $Z = \left\{ \mathbf{u}_j^i \right\} = \left\{ \mathbf{x}_{n_j}^i \right\}$

i: Camera index
j: World point index

The optimization problem is

$$X^* = \underset{X}{\operatorname{argmin}} \sum_i \sum_j \left\| \pi(g(\mathbf{T}_{wc_i}, \mathbf{x}_j^w)) - \mathbf{x}_{n_j}^i \right\|_{\mathbf{\Sigma}_{ij}}^2$$

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Measurement prediction

This gives us the **measurement prediction function**

$$\hat{\mathbf{x}}_n = h(\mathbf{x}^w; \mathbf{T}_{wc}) = \pi(g(\mathbf{T}_{wc}, \mathbf{x}^w))$$

where

$$g(\mathbf{T}_{wc}, \mathbf{x}^w) = \mathbf{R}_{wc}^T(\mathbf{x}^w - \mathbf{t}_{wc}^w) = \begin{bmatrix} x^c \\ y^c \\ z^c \end{bmatrix} = \mathbf{x}^c$$

(Coordinate transformation)

$$\pi(\mathbf{x}^c) = \frac{1}{z^c}\begin{bmatrix} x^c \\ y^c \end{bmatrix} = \begin{bmatrix} \hat{x}_n \\ \hat{y}_n \end{bmatrix} = \hat{\mathbf{x}}_n$$

(Camera model)

Linearization

We can **linearize** the measurement prediction function with a local first order Taylor expansion

$$h(\mathbf{x}^w + \boldsymbol{\delta}_\Delta; \mathbf{T}_{wc}) \approx h(\mathbf{x}^w; \mathbf{T}_{wc}) + \mathbf{G}\boldsymbol{\delta}_\Delta$$

where $\boldsymbol{\delta}_\Delta$ is a small perturbation in on the point in the world frame.
The **measurement Jacobian** is now given by

$$\mathbf{G} = \left.\frac{\partial h(\mathbf{x}^w + \boldsymbol{\delta}; \mathbf{T}_{wc})}{\partial \boldsymbol{\delta}}\right|_{\boldsymbol{\delta}=\mathbf{0}} = \left.\frac{\partial \pi(\mathbf{x}^c)}{\partial \mathbf{x}^c}\right|_{\mathbf{x}^c = g(\mathbf{T}_{wc}, \mathbf{x}^w)} \left.\frac{\partial g(\mathbf{T}_{wc}, \mathbf{x}^w + \boldsymbol{\delta})}{\partial \boldsymbol{\delta}}\right|_{\boldsymbol{\delta}=\mathbf{0}}$$

Jacobians

$$\left.\frac{\partial g(\mathbf{T}_{wc}, \mathbf{x}^{w} + \boldsymbol{\delta})}{\partial \boldsymbol{\delta}}\right|_{\boldsymbol{\delta}=\mathbf{0}}$$

$$g(\mathbf{T}_{wc}, \mathbf{x}^{w}) = \mathbf{R}_{wc}^{T}(\mathbf{x}^{w} - \mathbf{t}_{wc}^{w}) = \mathbf{x}^{c}$$

Jacobians

$$\left.\frac{\partial g(\mathbf{T}_{wc}, \mathbf{x}^w + \boldsymbol{\delta})}{\partial \boldsymbol{\delta}}\right|_{\boldsymbol{\delta}=\mathbf{0}} = \left.\frac{\partial (\mathbf{T}_{wc} \exp(\boldsymbol{\xi}^\wedge))^{-1} \oplus (\mathbf{x}^w + \boldsymbol{\delta})}{\partial \boldsymbol{\delta}}\right|_{\boldsymbol{\delta}=\mathbf{0}}$$

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$$= \left.\frac{\partial (\mathbf{T}_{wc})^{-1} \oplus (\mathbf{x}^w + \boldsymbol{\delta})}{\partial \boldsymbol{\delta}}\right|_{\boldsymbol{\delta}=0}$$

$$= \mathbf{R}_{wc}^T$$

Jacobians

$$\mathbf{G} = \left.\frac{\partial h(\mathbf{x}^w + \boldsymbol{\delta}; \mathbf{T}_{wc})}{\partial \boldsymbol{\delta}}\right|_{\boldsymbol{\delta}=\mathbf{0}} = \left.\frac{\partial \pi(\mathbf{x}^c)}{\partial \mathbf{x}^c}\right|_{\mathbf{x}^c = g(\mathbf{T}_{wc}, \mathbf{x}^w)} \left.\frac{\partial g(\mathbf{T}_{wc}, \mathbf{x}^w + \boldsymbol{\delta})}{\partial \boldsymbol{\delta}}\right|_{\boldsymbol{\delta}=\mathbf{0}}$$

$$= d \begin{bmatrix} 1 & 0 & -x_n \\ 0 & 1 & -y_n \end{bmatrix} \mathbf{R}_{wc}^T$$

Linear least-squares

We can then obtain a **linear least-squares problem**

$$\boldsymbol{\delta}_\Delta^* = \underset{\boldsymbol{\delta}_\Delta}{\operatorname{argmin}} \sum_i \sum_j \left\| h(\mathbf{x}_j^w; \mathbf{T}_{wc_i}) + \mathbf{G}_{ij}\boldsymbol{\delta}_j - \mathbf{x}_{n_j}^i \right\|_{\boldsymbol{\Sigma}_{ij}}^2$$

$$= \underset{\boldsymbol{\delta}_\Delta}{\operatorname{argmin}} \sum_i \sum_j \left\| \mathbf{G}_{ij}\boldsymbol{\delta}_j - \left\{ \mathbf{x}_{n_j}^i - h(\mathbf{x}_j^w; \mathbf{T}_{wc_i}) \right\} \right\|_{\boldsymbol{\Sigma}_{ij}}^2$$

$$= \underset{\boldsymbol{\delta}_\Delta}{\operatorname{argmin}} \sum_i \sum_j \left\| \mathbf{A}_{ij}\boldsymbol{\delta}_j - \mathbf{b}_{ij} \right\|^2$$

$$= \underset{\boldsymbol{\delta}_\Delta}{\operatorname{argmin}} \left\| \mathbf{A}\boldsymbol{\delta}_\Delta - \mathbf{b} \right\|^2$$

Linear least-squares

The **measurement Jacobian** $\mathbf{A}$ is now a **block sparse matrix**.

For an example with *two cameras* and *three points* we have

$$\mathbf{A} = \begin{bmatrix} \mathbf{G}_{11} & & \\ & \mathbf{G}_{12} & \\ & & \mathbf{G}_{13} \\ \mathbf{G}_{21} & & \\ & \mathbf{G}_{22} & \\ & & \mathbf{G}_{23} \end{bmatrix} \qquad \boldsymbol{\delta}_{\Delta} = \begin{bmatrix} \boldsymbol{\delta}_1 \\ \boldsymbol{\delta}_2 \\ \boldsymbol{\delta}_3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} \mathbf{b}_{11} \\ \mathbf{b}_{12} \\ \mathbf{b}_{13} \\ \mathbf{b}_{21} \\ \mathbf{b}_{22} \\ \mathbf{b}_{23} \end{bmatrix}$$

Solution to the linearized problem

The solution can be found by solving **the normal equations**

$$\left(\mathbf{A}^T\mathbf{A}\right)\boldsymbol{\delta}_{\boldsymbol{\Delta}}^* = \mathbf{A}^T\mathbf{b}$$

Since $\mathbf{A}$ is sparse,
a sparse solver should be used.

$$\text{Choose a suitable inital estimate } X^0$$

$$\mathbf{A}, \mathbf{b} \leftarrow \text{Linearize at } X^t$$

$$\boldsymbol{\Delta}^* \leftarrow \mathit{S}\text{olve } \underset{\boldsymbol{\Delta}}{\operatorname{argmin}} \left\| \mathbf{A}\boldsymbol{\Delta} - \mathbf{b} \right\|^2$$

$$X^{t+1} \leftarrow X^t + \boldsymbol{\Delta}^*$$

Gauss-Newton optimization

Given a good initial estimate $X^0 = \left\{ \mathbf{x}_j^{w,0} \right\}$.

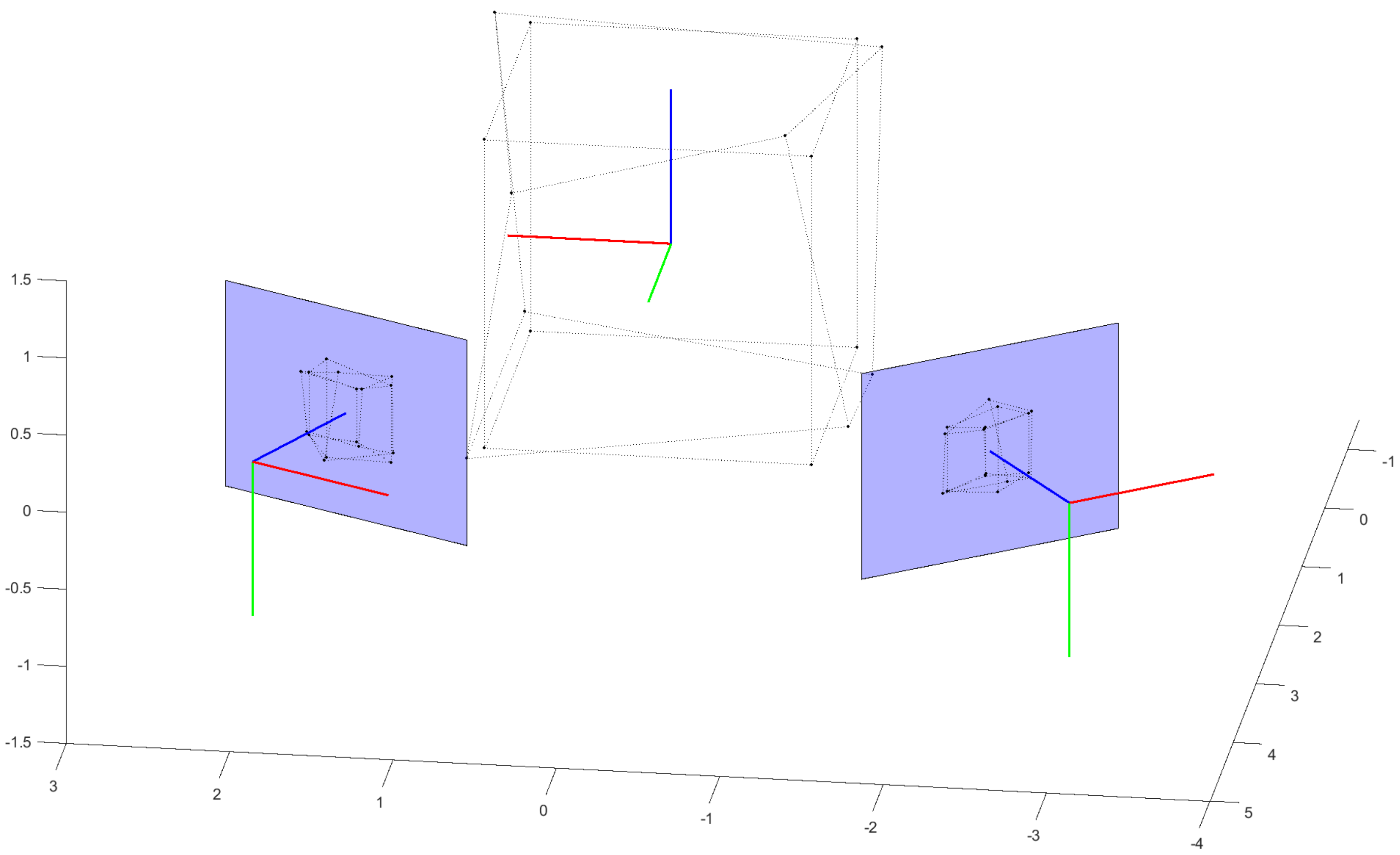
For $t = 0, 1, \ldots, t^{max}$

$\mathbf{A}, \mathbf{b} \leftarrow$ Linearize at X^t

$\boldsymbol{\delta}_\Delta^* \leftarrow$ Solve the linearized problem with $\left(\mathbf{A}^T \mathbf{A} \right) \boldsymbol{\delta}_\Delta^* = \mathbf{A}^T \mathbf{b}$

$\mathbf{x}_j^{w,t+1} \leftarrow \mathbf{x}_j^{w,t} + \boldsymbol{\delta}_j^*$

Example



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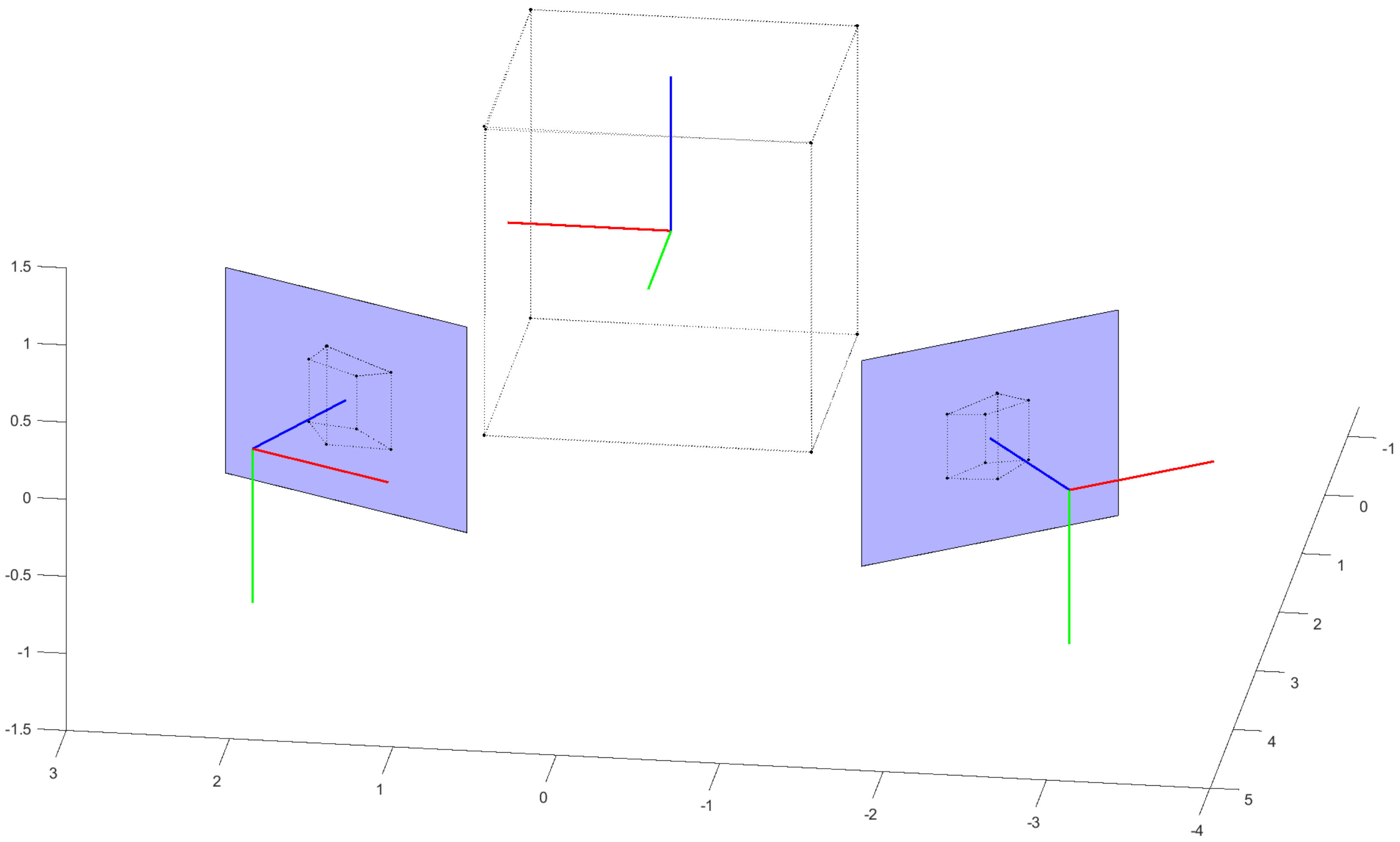

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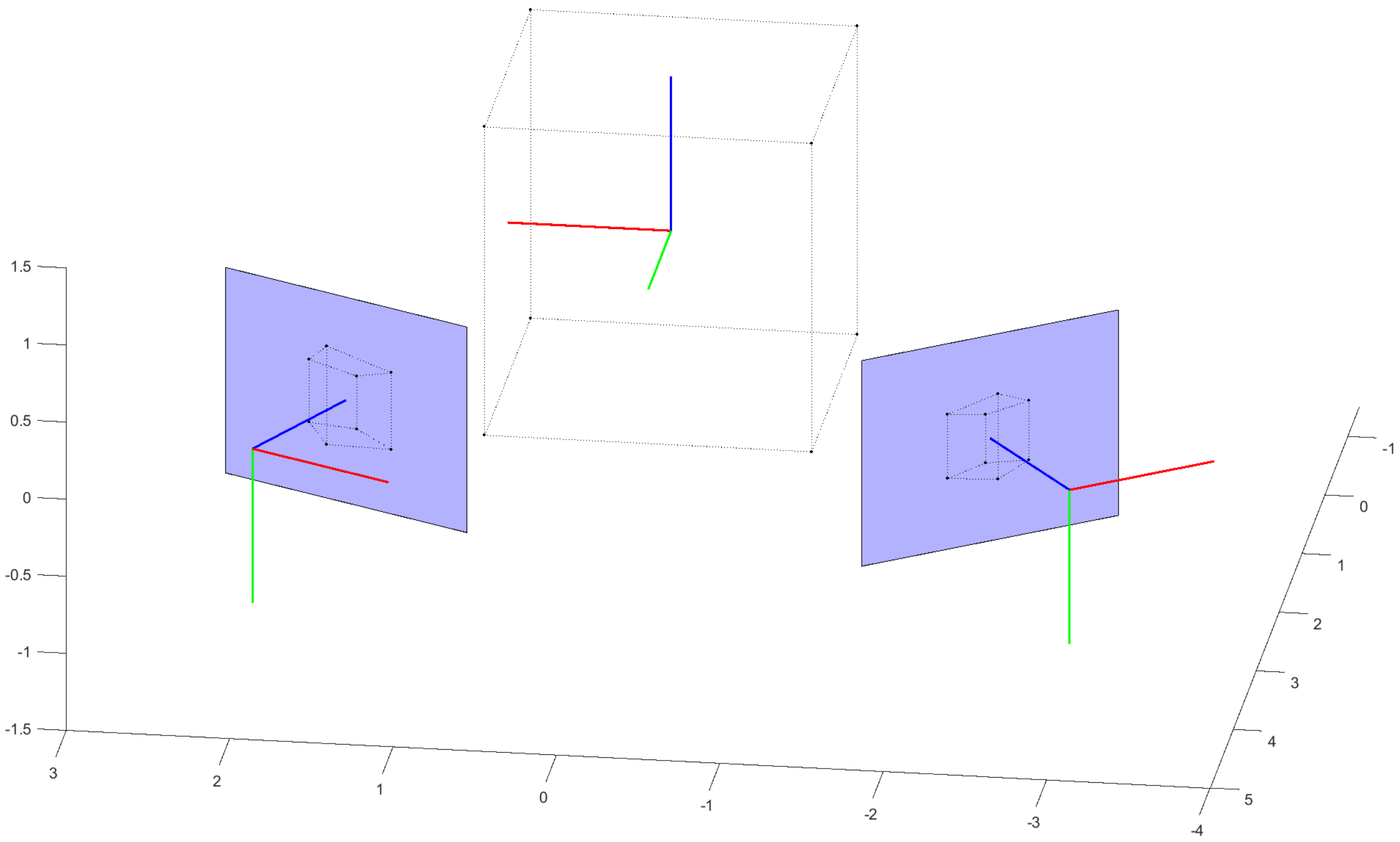

Summary

Triangulation by minimizing reprojection error

- Obtain 2D-2D point correspondences between at least two images

- Find an initial estimate, for example based on the linear method from lecture 8.3

$$\begin{aligned}\tilde{\mathbf{u}}^a &= \mathbf{P}_a \tilde{\mathbf{x}}^w \\ \tilde{\mathbf{u}}^b &= \mathbf{P}_b \tilde{\mathbf{x}}^w\end{aligned} \longrightarrow \mathbf{A}\tilde{\mathbf{x}} = 0 \xrightarrow{\text{SVD}} \mathbf{x}$$

- Minimize reprojection error iteratively using nonlinear least squares

$$X^* = \underset{X}{\operatorname{argmin}} \sum_i \sum_j \left\| \pi(g(\mathbf{T}_{wc_i}, \mathbf{x}_j^w)) - \mathbf{x}_{n_j}^i \right\|_{\boldsymbol{\Sigma}_{ij}}^2$$