2D Photonic Crystal, Analytically Solvable

There are very few analytically solvable models of photonic crystal in two dimensions. One such model was presented by Chen in 1981, and is mentioned at the end of Ch. 5, on page 93 in the textbook.

Let the relative permittivity $\varepsilon(x, y)$ be independent of z, with period a in the x direction and b in the y direction. We introduce the angular repetency in vacuum, $k_0 = \omega/c$, for light waves with angular frequency ω and let us consider z-polarized Bloch waves propagating in the x-y plane, perpendicular to the z axis. For a z-polarized wave, the E vector points in the z direction and the H vector lies in the x-y plane, so we have transverse magnetic (TM) polarization. Maxwells equations then imply that the z component $\varphi(x, y)$ of the E field satisfies the equation

$$\left[\partial_x^2 + \partial_y^2 + \varepsilon(x, y) k_0^2\right] \varphi = 0. \tag{1}$$

We may solve Eq. (1) by the method of separation variables, if $\varepsilon(x, y)$ can be written as a sum of two functions, one a function of x only, and the other a function of y only:

$$\varepsilon(x, y) = \varepsilon_x(x) + \varepsilon_y(y). \tag{2}$$

Problem 1

Show that when $\varepsilon(x, y)$ satisfies (2), there are Bloch-wave solutions of Eq. (1) that are separable in product form:

$$\varphi(x, y) = \varphi_x(x)\varphi_y(y) = \exp(ik_x x + ik_y y)\, u_x(x) u_y(y). \tag{3}$$

In (3) k_x and k_y are the components of the Bloch wave vector in the x-y plane, $u_x(x)$ is a periodic function with period a, and $u_y(y)$ is a periodic functions with period b. Show that $\varphi_x(x)$ and $\varphi_y(y)$ satisfy

$$d^2\varphi_x/dx^2 + (\varepsilon_x(x) - \varepsilon) k_0^2 \varphi_x = 0 \tag{4}$$

$$\text{and} \quad d^2\varphi_y/dy^2 + (\varepsilon_y(y) + \varepsilon) k_0^2 \varphi_y = 0, \tag{5}$$

and show that ε may be chosen freely. Note that we may subtract any number ε from $\varepsilon_x(x)$ and add the same number to $\varepsilon_y(y)$ without affecting $\varepsilon(x, y)$ in Eq. (2). Also note that k_x, k_y, $\varphi_x(x)$, and $\varphi_y(y)$ all depend on ε.

Problem 2

Let us consider a unit cell consisting of four rectangular areas, each a homogeneous area, and let the permittivities in the four rectangles be ε_{11}, ε_{12}, ε_{21} and ε_{22}. The permittivity has the form (2) if

$$\varepsilon_{22} = \varepsilon_{12} + \varepsilon_{21} - \varepsilon_{11}. \tag{6}$$

Then we may choose

$$\varepsilon_x(x) = \varepsilon_{11} \text{ for } 0 < x < a_1 \quad \text{and} \quad \varepsilon_x(x) = \varepsilon_{12} \text{ for } a_1 < x < a_1 + a_2 = a \tag{7}$$

$$\text{and} \quad \varepsilon_y(y) = 0 \text{ for } 0 < y < b_1 \quad \text{and} \quad \varepsilon_y(y) = \varepsilon_{21} - \varepsilon_{11} \text{ for } b_1 < y < b_1 + b_2 = b. \tag{8}$$

We have already derived dispersion equations for Bloch waves propagating perpendicular to a periodic planar two-layer structure, where the E field satisfies an equation of the same form as (4) and (5), and where the layer structure has the same permittivity distribution as given in

Eqs. (7) or (8). From our treatment of the periodic two-layer structure we obtain the following expressions for $\varphi_x(x)$ and its derivative $\varphi_x'(x)$ for $0 < x < a_1$:

$$\begin{aligned}\varphi_x(x) &= \varphi_x(0)\cos(k_{x1}x) + (\varphi_x'(0)/k_{x1})\sin(k_{x1}x), \quad (9)\\ \varphi_x'(x) &= -k_{x1}\varphi_x(0)\sin(k_{x1}x) + \varphi_x'(0)\cos(k_{x1}x), \quad (10)\end{aligned}$$

and for $a_1 < x < a_1 + a_2 = a$ we get

$$\begin{aligned}\varphi_x(x) &= \varphi_x(a_1)\cos(k_{x2}x) + (\varphi_x'(a_1)/k_{x2})\sin(k_{x2}x), \quad (11)\\ \varphi_x'(x) &= -k_{x2}\varphi_x(a_1)\sin(k_{x2}x) + \varphi_x'(a_1)\cos(k_{x2}x), \quad (12)\end{aligned}$$

$$\text{with} \quad k_{x1} = \sqrt{\varepsilon_{11} - \varepsilon}k_0 \quad \text{and} \quad k_{x2} = \sqrt{\varepsilon_{12} - \varepsilon}k_0. \quad (13)$$

Show that we get four equations for $\varphi_y(y)$ and its derivative $\varphi_y'(y)$, equations very similar to (9)-(12), with

$$k_{y1} = \sqrt{\varepsilon}k_0 \quad \text{and} \quad k_{y2} = \sqrt{\varepsilon_{21} - \varepsilon_{11} + \varepsilon}k_0. \quad (14)$$

Problem 3

Use our earlier results on Bloch waves propagating in a planar layer structure to write down an expression for $\cos k_x a$ when φ_x satiesfies Eq. (4) with the permittivity (7), and an expression for $\cos k_y b$ when φ_y satiesfies Eq. (5) with the permittivity (8).

Problem 4 (Matlab)

For band structure calculations, we are usually not interested in taking k_0 and ε as given and computing k_x and k_y. We prefer to specify k_x and k_y, and compute k_0 and the corresponding auxiliary variable ε. The analytic expressions for k_x and k_y as functions of k_0 and ε may, however, be considered to be two equations connecting four unknowns k_0, k_x, k_y og ε, and the two equations allow us to compute k_0 and ε numerically if k_x and k_y are given.

Let $\varepsilon_{11} = 1$, $\varepsilon_{12} = \varepsilon_{21} = 2$, $\varepsilon_{22} = 3$, and let $a_1 = a_2 = b_1 = b_2 = a/2 = b/2$. Then the unit cell of the photonic crystal lattice is a square with side a. Use the two dispersion equations for k_x and k_y to find k_0 and ε for the two lowest bands of the band structure, for six positions in the first Brillouin zone, specifically in the points Γ, X and M, and in three points between these three, along the sides of the triangular irreducible Brillouin zone. Use the results to outline the corresponding band structure diagram, as done for TM modes in Fig. 2 in Chapter 5 in the textbook.

Hint: Use 2D false-color plots of the dispersion equations as functions of k_0 and ε, to find the zeros of the equations. Plot ranges: $0 < k_0 < 10/a$ and $-1 < \varepsilon < 2$

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