TWO-DIMENSIONAL METAMATERIAL

We have already seen how the concept of a metamaterial yields an analytic description of a planar periodic layer structure. Now let us apply the metamaterial concept to a doubly periodic array of cylinders. Let the relative permittivity be ε_1 inside the cylinders and ε_2 between the cylinders, and let the cylinder radius be a.

Problem 1

Let us first consider the case with the E field pointing in the z direction along the cylinders, *i.e.*, transverse magnetic (TM) polarization. In the metamatierial (low–frequency) limit, the E field is then approximately constant inside a unit cell of the photonic crystal. The **effective relative permittivity** ε_{zz} of the metamaterial is defined as the mean of the D field over the unit cell divided by the mean of the E field times ε_0 over the unit cell. Show that for a z-polarized field,

$$\varepsilon_{zz} = \varepsilon_2 + (\varepsilon_1 - \varepsilon_2) f, \tag{1}$$

where the fill factor f is the area of the cylinder relative to the area of the unit cell,

$$f = \pi a^2 / A_u = \pi a^2 / (bh). \tag{2}$$

The area A_u of the unit cell is the base line b times the height h.

Problem 2

Let us then consider a TE-polarized field, with the E field lying in the x-y plane, perpendicular to the cylinders, again in the metamaterial limit. Limiting our analysis to a small fill factor, we may consider the E field to be approximately constant inside and between the cylinders. There is then a region near the outside of each cylinder where the field is not constant, and where we may use the low-frequency approximation that the E field is the gradient of a potential $V(r,\varphi)$ that is continuous everywhere, and has the form

$$V_1(r,\varphi) = -E_0 \frac{2\varepsilon_2}{\varepsilon_2+\varepsilon_1} r\cos\varphi = -E_0 \frac{2\varepsilon_2}{\varepsilon_2+\varepsilon_1} x \quad \text{for } r < a \quad \text{(inside the cylinder)} \tag{3}$$

$$V_2(r,\varphi) = -E_0 \left(r + \frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1} \frac{a^2}{r} \right) \cos\varphi = \tag{4}$$

$$= -E_0 \left(x + \frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1} \frac{a^2 x}{x^2+y^2} \right) \quad \text{for } r > a \quad \text{(outside the cylinder)} \tag{5}$$

Show that the potential (3) inside the cylinder yields a constant E field that points in the x direction and is equal to

$$E_{x,1} = \frac{2\varepsilon_2}{\varepsilon_2+\varepsilon_1} E_0. \tag{6}$$

Problem 3

From (5), derive expressions for the x and y components of the E field outside the cylinder with radius a. Show that the mean of the E field points in the x direction, when the mean is taken over the cross-sectional area outside the sylinder of radius a and inside the rectangular unit cell. Show that this mean is equal to E_0, regardless of the size of the unit cell.

Problem 4

Show that when the cylinders are far from each other, we get the following approximations for the means of E_x and D_x over the unit cell,

$$\bar{E}_x = \left(1 + \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1} f\right) E_0, \tag{7}$$

$$\bar{D}_x = \varepsilon_2 \varepsilon_0 \left(1 - \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1} f\right) E_0, \tag{8}$$

resulting in the effective relative permittivity

$$\varepsilon_{xx} \approx \varepsilon_2 \left(1 - \frac{2(\varepsilon_2 - \varepsilon_1)}{\varepsilon_2 + \varepsilon_1} f\right). \tag{9}$$

Problem 5

Now let us consider the general case with cylinders that are not far from each other, but restrict ourselves to a rectangular unit cell with width b and height h. We note that if the E field is x-polarized in the center of the cylinder in a rectangular unit cell, the E field is purely x polarized in all the mirror planes of the structure, x-z planes and y-z planes going through the centers of the cylinders and in the middle between cylinders. We note that everywhere inside the unit cell,

$$r < d = \frac{1}{2}\sqrt{b^2 + h^2}. \tag{10}$$

Instead of a single cosine contribution like in (3), we then need a sum of cosine terms, a socalled multipole expansion, to represent the E field, both inside and outside the sylinder. Inside the cylinder (for $r < a$) we may use the following expressions for the x and y components of the E field

$$E_{x,1}(r,\varphi) = \sum_{m=0}^{M-1} E_m \frac{2\varepsilon_2}{\varepsilon_2 + \varepsilon_1} \frac{r^{2m}}{d^{2m}} \cos(2m\varphi), \tag{11}$$

$$E_{y,1}(r,\varphi) = -\sum_{m=0}^{M-1} E_m \frac{2\varepsilon_2}{\varepsilon_2 + \varepsilon_1} \frac{r^{2m}}{d^{2m}} \sin(2m\varphi). \tag{12}$$

The corresponding expressions for the E field outside of the cylinders (for $r > a$) are

$$E_{x,2}(r,\varphi) = \sum_{m=0}^{M-1} E_m \left(\frac{r^{2m}}{d^{2m}} \cos(2m\varphi) - \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1} \frac{a^{4m+2}}{d^{2m} r^{2m+2}} \cos(2m\varphi + 2\varphi)\right), \tag{13}$$

$$E_{y,2}(r,\varphi) = -\sum_{m=0}^{M-1} E_m \left(\frac{r^{2m}}{d^{2m}} \sin(2m\varphi) - \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1} \frac{a^{4m+2}}{d^{2m} r^{2m+2}} \sin(2m\varphi + 2\varphi)\right). \tag{14}$$

We note that for an x-polarized field in a rectangular unit cell, only terms with even multiples $2m$ of the angle φ are needed in the multipole expansions (11)-(14).

Show that with E_x and E_y given by the multipole expansions (11)-(14), the average of E_y over a rectangular unit cell is zero.

Problem 6 (Matlab)

We can find the expansion coefficients E_m in the series (11)-(14) via point matching. So let us require $E_y(r,\varphi)$ in (14) to be minimized in $2M-1$ different positions around the unit cell, given by $2M-1$ different values for the angle φ

$$\varphi_p = \frac{\pi}{4M} p, \quad p = 1, 2, ...(2M-1). \tag{15}$$

The corresponding distances from the origin are

$$r_p = \frac{b}{2\cos\varphi_p} \quad \text{if } \tan\varphi_p < h/b \quad \text{and} \quad r_p = \frac{h}{2\sin\varphi_p} \quad \text{if } \tan\varphi_p > h/b. \tag{16}$$

Use the $2M-1$ equations obtained by setting $E_y(r_p, \varphi_p)/E_0 = 0$ in (14) for $p = 1, 2, ...(2M-1)$ to set up an overdetermined set of linear equations in Matlab and determine E_m/E_0 for $m = 1, 2, ...(M-1)$. Then compute the field in the middle between the cylinders, $E_x(r = b/2, \varphi = 0)/E_0$. Do the calculation for $\varepsilon_1 = 2, \varepsilon_2 = 1$, and $M = 10$ terms in the series expansion, for two cases of a rectangular unit cell, a tall cell with $b = 3a$ and $h = 4a$, and a wide cell with $b = 4a$ and $h = 3a$.

Hint: An overdetermined system of linear equations can be solved in Matlab with the help of the matrix divide operation.

Finally, do a numerical average over the unit cell to obtain $\bar{E}_x$ and $\bar{D}_x$ for both the tall and the wide unit cells, and compare the numerically computed averages with the formulas (7) and (8).

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