2D+1 Metamaterial

Let us consider an optical metamaterial that is uniform in the z direction, and has a unit cell in the x-y plane. Let the length of the longest of the two diagonals of the unit cell be d_u. In a metamaterial, d_u is much smaller than $1/k = c/\omega$, where c is the speed of light in vacuum, ω is the angular frequancy of the light, and k is the angular repetency of the light in vacuum. In the textbook by Joannopolous et al we find the Maxwell equations for a linear nonmagnetic dielectric material as number (6) in Chapter 2:

$$\begin{aligned} c\nabla \times \boldsymbol{E}(\boldsymbol{r}) - ik\eta_0 \boldsymbol{H}(\boldsymbol{r}) &= 0, &(1)\\ c\eta_0 \nabla \times \boldsymbol{H}(\boldsymbol{r}) + ik\varepsilon(\boldsymbol{r})\boldsymbol{E}(\boldsymbol{r}) &= 0. &(2)\end{aligned}$$

To make the two equations above look similar, we have introduced the vacuum wave impedance $\eta_0 = \sqrt{\mu_0/\varepsilon_0} = 377\Omega$, so that $\eta_0\boldsymbol{H}$ and $\boldsymbol{E}$ are measured in the same units. Let us consider waves propagating in the posive z direction with angular repetency k_z. Then we may write

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{\rho},z) &= \mathrm{Re}\left(\boldsymbol{E}(\boldsymbol{\rho})e^{ik_z z}\right), &(3)\\ \boldsymbol{H}(\boldsymbol{\rho},z) &= \mathrm{Re}\left(\boldsymbol{H}(\boldsymbol{\rho})e^{ik_z z}\right), &(4)\end{aligned}$$

Waves propagating in the z-direction are called modes of the electromagnetic field, and come in two classes, EH modes and HE modes. The main electromagnetic field components of EH modes are E_z, H_x and H_y, and the main electromagnetic field components of HE modes are H_z, E_x and E_y. For a given frequency, EH modes propagate with an angular repetency $k_E = n_E k$, where n_E is called the mode index, and HE modes propagate with a different angular repetency k_H and mode index n_H.

The EH modes are represented by (2):

$$\begin{aligned} \eta_0\partial_y H_z(\boldsymbol{\rho}) - ik_E\eta_0 H_y(\boldsymbol{\rho}) &= -ik\varepsilon(\boldsymbol{\rho})E_x(\boldsymbol{\rho}), &(5)\\ ik_E\eta_0 H_x(\boldsymbol{\rho}) - \eta_0\partial_x H_z(\boldsymbol{\rho}) &= -ik\varepsilon(\boldsymbol{\rho})E_y(\boldsymbol{\rho}), &(6)\\ \eta_0\left(\partial_x H_y(\boldsymbol{\rho}) - \eta_0\partial_y H_x(\boldsymbol{\rho})\right) &= -ik\varepsilon(\boldsymbol{\rho})E_z(\boldsymbol{\rho}), &(7)\end{aligned}$$

(1) implies that $\boldsymbol{H}(\boldsymbol{r},t)$ is proportional to a curl, and hence has vanishing divergence:

$$\partial_x H_x(\boldsymbol{\rho}) + \partial_y H_y(\boldsymbol{\rho}) + ik_E H_z(\boldsymbol{\rho}) = 0. \quad (8)$$

With the frequency equal to zero, $k = k_E = k_E = 0$, and the EH mode fields have the simple static transverse magnetic (TM) form for nonmagnetic materials

$$\begin{aligned} \nabla_{||}^2\Phi_E(\boldsymbol{\rho}) &= 0 \quad \eta_0 H_{0x} = E_{0y} = E_{0x} = E_{0z} = 0, &(9)\\ \eta_0\boldsymbol{H}_{0||}(\boldsymbol{\rho}) &= -\nabla_{||}\Phi_E(\boldsymbol{\rho}), \quad \eta_0 H_{0z}(\boldsymbol{\rho}) = -ik_E\Phi_E(\boldsymbol{\rho}). &(10)\end{aligned}$$

For the TM field in a nonmagnetic metamaterial, the magnetic field is the gradient of a potential $\Phi_E(\boldsymbol{\rho})$ that depends linearly on x and y, and gives the orientation of a uniform static field $\boldsymbol{H}_{0||}$ in the x-y-plane.

In contrast, the HE modes are represented by (1)

$$\begin{aligned} \partial_y E_z(\boldsymbol{\rho}) - ik_H E_y(\boldsymbol{\rho}) &= ik\eta_0 H_x(\boldsymbol{\rho}), &(11)\\ ik_H E_x(\boldsymbol{\rho}) - \partial_x E_z(\boldsymbol{\rho}) &= ik\eta_0 H_y(\boldsymbol{\rho}), &(12)\\ \partial_x E_y(\boldsymbol{\rho}) - \partial_y E_x(\boldsymbol{\rho}) &= ik\eta_0 H_z(\boldsymbol{\rho}), &(13)\end{aligned}$$

(2) implies that $\varepsilon(\boldsymbol{\rho})\boldsymbol{E}(\boldsymbol{r},t)$ is proportional to a curl, and hence has vanishing divergence:

$$\partial_x\left(\varepsilon(\boldsymbol{\rho})E_x(\boldsymbol{\rho})\right) + \partial_y\left(\varepsilon(\boldsymbol{\rho})E_y(\boldsymbol{\rho})\right) + ik_H\varepsilon(\boldsymbol{\rho})E_z(\boldsymbol{\rho}) = 0. \quad (14)$$

With the frequency equal to zero, the HE mode field is a static transverse electric (TE) field. This electric mode field is the gradient of a potential $\Phi_H(\boldsymbol{\rho})$ satisfying the following equations:

$$\nabla_{||} \cdot \left(\varepsilon(\boldsymbol{\rho}) \nabla_{||} \Phi_H(\boldsymbol{\rho})\right) = 0 \quad E_{0z} = \eta_0 H_{0y} = \eta_0 H_{0x} = \eta_0 H_{0z} = 0, \tag{15}$$

$$\boldsymbol{E}_{0||}(\boldsymbol{\rho}) = -\nabla_{||} \Phi_H(\boldsymbol{\rho}), \quad E_{0z} = -ik_H \Phi_H. \tag{16}$$

The formulas given above for the TM and TE mode fields represent a very compact description of the metamaterial. (16) is a very useful relation for small nonvanishing frequency, and allows us to give an estimate of the ratio between the maximum variation of $\Phi_H(\boldsymbol{\rho})$ and the maximum magnitude of $\boldsymbol{E}_{0||}(\boldsymbol{\rho})$, the variation obtained by varying the position $\boldsymbol{\rho}$ inside the unit cell of size d_u. An order-of-magnitude estimate of this ratio is $k_H d_u$. This ratio approaces zero linearly with frequency. Similar reasoning allows us say that the ratio between the maximum variation of $E_{0z}(\boldsymbol{\rho})$ and the maximum magnitude of $\boldsymbol{E}_{0||}(\boldsymbol{\rho})$ approaches zero quadratically with frequency.

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