Periodic Film Structure as a Metamaterial

Again, we consider a periodic film structure with two layers in the period. Layer 1 has index of refraction n_1, relative permittivity $\varepsilon_1 = n_1^2$, and thickness d_1, whereas layer 2 has n_2, ε_2 and d_2. The period of the layer structure is

$$d = d_1 + d_2. \tag{1}$$

The layers are oriented in the x-y plane, perpendicular to the z direction, and have unity relative magnetic permeability. In each layer harmonic electromagnetic waves with frequency f travel up and down in the z direction. Let us define

$$\begin{aligned} k_0 &= 2\pi f/c = \omega/c, & (2)\\ k_1 &= n_1 k_0 \;\text{ and }\; k_2 = n_2 k_0, & (3) \end{aligned}$$

where ω is the angular frequency, c is the speed of light in vacuum, and k_i, $i = 0, 1, 2$ are the angular repetency of plane waves propagating in vacuum, layer 1 and layer 2, respectively. Let us introduce a layer number p, which may be equal to 1 or 2: Inside each layer p the electic field is a superposition of two plane waves, one of them propagating in positive z direction and one of them in the negative z direction. If we choose the electric E field to to point in the x direction, the magnetic H field points in the y direction. Within layer p we may obtain the following expressions for the E and H fields:

$$\begin{aligned} E_{xp}(z) &= E_{+p}\exp(ik_p z) + E_{-p}\exp(-ik_p z) = & (4)\\ &= \eta_p H_{+p}\exp(ik_p z) - \eta_p H_{-p}\exp(-ik_p z), & (5)\\ H_{yp}(z) &= H_{+p}\exp(ik_p z) - H_{-p}\exp(-ik_p z). & (6) \end{aligned}$$

E_{+p} and H_{+p} are the field amplitues for plane waves going up, and E_{-p} and H_{-p} are the field amplitudes for plane waves going down, and we have introduce the plane-wave impedance η_p for pure up waves or down waves in layer p:

$$\eta_p = E_{+p}/H_{+p} = E_{-p}/H_{-p} = \eta_0/n_p = \sqrt{\mu_0/(\varepsilon_0\varepsilon_p)}. \tag{7}$$

We want Eqs. (5) and (6) to represent a Bloch wave going up with angular repetency k, effective index n, and Bloch-wave impedance η:

$$k = n k_0. \tag{8}$$

Let us choose our origin for z so that layer 1 is between $z = -d_1$ and $z = 0$, and layer 2 between $z = 0$ and $z = d_2$, and let us define the position-dependent impedance for the Bloch wave

$$\begin{aligned} \eta_z(z) &= \frac{E_{x1}(z)}{H_{y1}(z)} = \eta_1 \frac{1 + \Gamma_1 \exp(-2ik_1 z)}{1 - \Gamma_1 \exp(-2ik_1 z)} \quad \text{for } z < 0, & (9)\\ \text{and}\quad \eta_z(z) &= \eta_2 \frac{1 + \Gamma_2 \exp(-2ik_2 z)}{1 - \Gamma_2 \exp(-2ik_2 z)} \quad \text{for } z > 0 & (10) \end{aligned}$$

In (9) and (10) we have introduced reflection coeffiecients in each of the two layers:

$$\begin{aligned} \Gamma_1 &= E_{-1}/E_{+1} = H_{-1}/H_{+1}, & (11)\\ \Gamma_2 &= E_{-2}/E_{+2} = H_{-2}/H_{+2}. & (12) \end{aligned}$$

An optical metamaterial is a description valid at *low frequency*,

$$k_i d << 1 \quad \text{for } i = 0, 1 \text{ og } 2. \tag{13}$$

For the metamaterial in the limit of zero frequency, we get an impedance that is independent of position, so this impedance is then also the impedance of the Bloch wave :

$$\eta = \eta_z(0). \tag{14}$$

If we have different materials in the two layers, there is a discontinuity in the index of refraction and permittivity when we go in the z direction from one material into the other material. Maxwell's equations imply that $E_{xp}(z), E'_{xp}(z) = dE_{xp}/dz$ and $H_{yp}(z)$ are alle continuous across the layer interfaces, whereas $H'_{yp}(z) = dH_{yp}/dz$ is discontinuous across the interfaces. Since E_x and H_y must be continuous when we cross the interface at $z = 0$, the impedance $\eta_z(z)$ must be the same on both sides of the interface at $z = 0$:

$$\eta_2 \frac{1+\Gamma_2}{1-\Gamma_2} = \eta_z(0) = \eta_1 \frac{1+\Gamma_1}{1-\Gamma_1} \tag{15}$$

Problem 1)

Use (15) to derive expressions for Γ_1 and Γ_2 in terms of $\eta = \eta_z(0)$.

Problem 2)

Use (9) to get the approximation

$$\eta_z(z) = \frac{E_{x1}(z)}{H_{y1}(z)} \approx \eta_1 \frac{1+\Gamma_1}{1-\Gamma_1}\left(1 + 2ik_1 z\Gamma_1\left(\frac{1}{1+\Gamma_1} + \frac{1}{1-\Gamma_1}\right)\right) \quad \text{for } z < 0. \tag{16}$$

Hint: Make a power series expansion of $\eta_z(z)$ to first order in z in (9).

Problem 3)

Use the fact that the impedance is conserved when we cross an interface to show that $\eta_z(-d_1) = \eta_z(d_2)$ for Bloch waves propagating perpendicular the layers of a periodic layer structure. Then use the results in Problem 2) to show that the effective impedance for the metamaterial is

$$\eta = \eta_z(0) = \eta_0/n, \tag{17}$$

where n can be expressed as

$$n = k/k_0 = \sqrt{\left(n_1^2 d_1 + n_2^2 d_2\right)/d}. \tag{18}$$

Note that Eq. (18) is the expression for the effective index that can be found in Problem 5) in the first problem set on the periodic multilayer film.

Problem 4)

Let us then consider Bloch waves propagating along the layers in the y direction Inside layer number p the electic field is a superposition of two plane waves. One of them propagates in a direction in the y-z plane given by an angle φ_p relative to the z axis, and one in a direction in the y-z plane given by the angle $-\varphi_p$. relative to the z axis. Within layer p we obtain the following expressions for the E and H fields:

$$E_{xp}(y,z) = E_{+p}\exp(ik_y y + ik_{zp}z) + E_{-p}\exp(ik_y y - ik_{zp}z) = \tag{19}$$

$$= \eta_p H_{+p}\exp(ik_y y + ik_{zp}z) + \eta_p H_{-p}\exp(ik_y y - ik_{zp}z), \tag{20}$$

$$H_{yp}(y,z) = H_{+p}\exp(ik_y y + ik_{zp}z)\cos\varphi_p - H_{-p}\exp(ik_y y - ik_{zp}z)\cos\varphi_p, \tag{21}$$

$$H_{zp}(y,z) = -H_{+p}\exp(ik_y y + ik_{zp}z)\sin\varphi_p - H_{-p}\exp(ik_y y - ik_{zp}z)\sin\varphi_p \tag{22}$$

In the above equations we have chosen to represent what is called **transverse electric (TE) polarisation**, where the E field is perpendicular to the z direction. We have introduced the z component k_{zp} of the wave vector of the two plane waves in layer number p,

$$k_{zp} = k_p\cos\varphi_p = \sqrt{k_p^2 - k_y^2} \tag{23}$$

We have also introduced the y component k_y of the wave vector of the plane waves,

$$k_y = k_1\sin\varphi_1 = \sqrt{k_1^2 - k_{z1}^2} \tag{24}$$

$$k_y = k_2\sin\varphi_2 = \sqrt{k_2^2 - k_{z2}^2}. \tag{25}$$

k_y is also the y-component of the Bloch wave vector, proportional to the effective index n of the Bloch wave:

$$k_y = nk_0.$$

k_y must be the same in the two layers, so we obtain what is known as Snell's law in optics:

$$n_1\sin\varphi_1 = n_2\sin\varphi_2. \tag{26}$$

For Bloch waves propagating in the y direction, the fields must be periodic in the z direction with period d, and symmetric about the positions $z_1 = -d_1/2$ in layer 1 and $z_2 = d_2/2$ in layer 2,

$$E_{xp}(z) = 2\eta_p H_{+p}\cos\left(k_{zp}(z - z_p)\right), \tag{27}$$

$$H_{yp}(z) = 2H_{+p}\sin\left(k_{zp}(z - z_p)\right)\cos\varphi_p \tag{28}$$

$$H_{zp}(z) = -2H_{+p}\cos\left(k_{zp}(z - z_p)\right)\sin\varphi_p. \tag{29}$$

Show that by requring E_x and H_y to be continuous across the interface at $z = 0$ we obtain the following dispersion equation for k_y:

$$k_{z1}\tan\left(k_{z1}d_1/2\right) + k_{z2}\tan\left(k_{z2}d_2/2\right) = 0 \tag{30}$$

Problem 5)

The effctive index of the metamaterial **for TE polarization** is the low-frequency approximation to k_y. Show that, the expression for the effective index n obtained from Eq.(30) for waves propagating in the y direction in the metamaterial is is the same as that given in Eq. (18) for waves propagating in the z direction in the metamaterial .

Problem 6)

Let us finally consider **the case of TM waves propagatin along the layers**. Using the same procedure as in the derivation of Eqs. (27) and (28), we obtain the following expressions for the nonzero components of the E and H fields in layer p for Bloch waves propagating in the y direction

$$\begin{aligned} H_{xp}(z) &= 2H_{+p}\cos\left(k_{zp}\left(z-z_p\right)\right) & (31) \\ E_{yp}\left(z\right) &= 2\eta_p H_{+p}\sin\left(k_{zp}\left(z-z_p\right)\right)\cos\varphi_p, & (32) \\ E_{zp}(z) &= 2\eta_p H_{+p}\cos\left(k_{zp}\left(z-z_p\right)\right)\sin\varphi_p. & (33) \end{aligned}$$

Show that by requring E_x and H_y to be continuous across the interface at $z=0$ we obtain the following dispersion equation for k_y:

$$n_1^{-2}k_{z1}\tan\left(k_{z1}d_1/2\right)+n_2^{-2}k_{z2}\tan\left(k_{z2}d_2/2\right)=0 \qquad (34)$$

Problem 7)

The effctive index of the metamaterial **for TM polarization** is given by the low-frequency approximation to k_y as given by Eq.(34). Show that the resulting expression for k_y is given by

$$\frac{d}{k_y^2}\approx\frac{d_1}{k_1^2}+\frac{d_2}{k_2^2}, \qquad (35)$$

instead of Eq. (18).

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