TEK9540 – Quantum computation and quantum information
Problem set 1: Linear algebra – Solutions

Exercise 2.2

It should be easy to see that A is given by

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Let us try to find what A looks like in the $\{|+\rangle, |-\rangle\}$-basis. We have

$$|0\rangle = \frac{|0\rangle + |1\rangle}{2} + \frac{|0\rangle - |1\rangle}{2} = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

$$|1\rangle = \frac{|0\rangle + |1\rangle}{2} - \frac{|0\rangle - |1\rangle}{2} = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

So A takes $\frac{1}{\sqrt{2}}\binom{1}{1}_{\pm}$ into $\frac{1}{\sqrt{2}}\binom{1}{-1}_{\pm}$ and vice versa. It should be easy to see that

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\pm}$$

does the job.

Exercise 2.5

(1) $$\left(|v\rangle, \sum_j \lambda_j |w_j\rangle\right) = \sum_i v_i^* \sum_j \lambda_j w_{ij} = \sum_j \lambda_j \sum_i v_i^* w_{ij} = \sum_j (|v\rangle, |w_j\rangle)$$

(2) $$(|v\rangle, |w\rangle) = \sum_i v_i^* w_i = \sum_i w_i^{**} v_i^* = (|w\rangle, |v\rangle)^*$$

(3) $$(|v\rangle, |v\rangle) = \sum_i v_i^* v_i = \sum_i |v_i|^2 \geq 0$$

We get equality if and only if all $v_i = 0$ which means that $|v\rangle = 0$

Exercise 2.7

Two vectors are orthogonal if their inner product is zero.

$$\langle v|w\rangle = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 - 1 = 0$$

Normalized forms:

$$|w_{\mathrm{N}}\rangle = \frac{|w\rangle}{\sqrt{\langle w|w\rangle}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|v_{\mathrm{N}}\rangle = \frac{|v\rangle}{\sqrt{\langle v|v\rangle}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Exercise 2.9

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |1\rangle\langle 0| + |0\rangle\langle 1|$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i|1\rangle\langle 0| - i|0\rangle\langle 1|$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

Exercise 2.11

The eigenvalues are denoted λ_1 and λ_2, and their corresponding normalized eigenvectors are $|v_1\rangle$ and $|v_2\rangle$.

$$\sigma_x: \quad \lambda_1 = 1, \quad \lambda_2 = -1, \quad |v_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |v_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\sigma_y: \quad \lambda_1 = 1, \quad \lambda_2 = -1, \quad |v_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad |v_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\sigma_z: \quad \lambda_1 = 1, \quad \lambda_2 = -1, \quad |v_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |v_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The diagonal representation is given by $\sigma = \sum_i \lambda_i |v_i\rangle\,\langle v_i|$

$$\begin{aligned}
\sigma_x &= \frac{1}{2}\left[\Big(|0\rangle + |1\rangle\Big)\Big(\langle 0| + \langle 1|\Big) - \Big(|0\rangle - |1\rangle\Big)\Big(\langle 0| - \langle 1|\Big)\right] \\
\sigma_y &= \frac{1}{2}\left[\Big(|0\rangle + i|1\rangle\Big)\Big(\langle 0| + i\langle 1|\Big) - \Big(|0\rangle - i|1\rangle\Big)\Big(\langle 0| - i\langle 1|\Big)\right] \\
\sigma_z &= |0\rangle\langle 0| - |1\rangle\langle 1|.
\end{aligned}$$

Note that all the sigma matrices can be written written as

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

when we use the corresponding eigenvectors as basis

Exercise 2.17

Any normal matrix A has a spectral decomposition

$$A = \sum_i \lambda_i |i\rangle\langle i|$$

where λ_i are the eigenvalues and $|i\rangle$ the corresponding eigenvectors. We then have

$$A^\dagger = \sum_i \lambda_i^* |i\rangle\langle i|$$

If then A is hermitian, $A = A^\dagger$, we have $\lambda_i = \lambda_i^*$. Then all λs has to be real. The implication the other way is just as easy, when all eigenvalues of a normal matrix A are real we see from the spectral composition that $A = A^\dagger$ so A is hermitian

Exercise 2.18

Let $|u\rangle$ be a eigenvector of U with eigenvalue λ_u. Then

$$U|u\rangle = \lambda_u|u\rangle \text{ and } \langle u|\, U^\dagger = \langle u|\, \lambda_u^*$$

This gives

$$\langle u|\, U^\dagger U|u\rangle = \langle u|\, |\lambda_u|^2 |u\rangle = |\lambda_u|^2$$

But since $U^\dagger = U^{-1}$

$$\langle u|\, U^\dagger U|u\rangle = 1$$

So $|\lambda_u|^2 = 1$ and $\lambda_u = e^{i\theta}$.

Exercise 2.22

We know that any Hermitian matrix A has spectral decomposition

$$A = \sum_i \lambda_i |i\rangle \langle i|$$

where λ_i are the real eigenvalues and $|i\rangle$ are the corresponding eigenvectors. Then

$$\begin{aligned} 0 = \langle i|\, A|j\rangle - \langle i|\, A|j\rangle &= \langle i|\, A|j\rangle - \langle i|\, A^\dagger|j\rangle = \langle i|\, \lambda_j|j\rangle - \langle i|\, \lambda_i|j\rangle \\ &= \langle i|j\rangle\, (\lambda_j - \lambda_i) \end{aligned}$$

Therefore, if $\lambda_i \neq \lambda_j$ then $\langle i|j\rangle = 0$.

Exercise 2.23

A projector P is Hermitian, so it has spectral decomposition

$$P = \sum_i \lambda_i |i\rangle \langle i|$$

where λ_i are the real eigenvalues and $\{|i\rangle\}_i$ is an orthonormal basis. Since $P = P^2$ we have

$$\langle i| P|i\rangle = \langle i| \left(\sum_j \lambda_j |j\rangle \langle j|\right) |i\rangle = \lambda_i$$

and

$$\langle i| P|i\rangle = \langle i| P^2|i\rangle = \langle i| \left(\sum_j \lambda_j |j\rangle \langle j|\right) \left(\sum_k \lambda_k |k\rangle \langle k|\right) = \lambda_i^2$$

so that $\lambda_i = \lambda_i^2$, meaning that $\lambda_i = 0$ or $1 \forall i$.

Exercise 2.24 and 2.25

Note that if a matrix T is Hermitian we have:

$$\langle v| T|v\rangle = \langle v| T^\dagger|v\rangle = (\langle v| T|v\rangle)^\dagger = (\langle v| T|v\rangle)^*$$

for any $|v\rangle$. This means that $\langle v| T|v\rangle$ must be real for any Hermitian T. We then define

$$B = \frac{1}{2}\left(A + A^\dagger\right) \qquad \text{and} \qquad C = -\frac{\mathrm{i}}{2}\left(A - A^\dagger\right).$$

It is easy to check that $A = B + \mathrm{i}C$, and that B and C are Hermitian. If A is positive, $\langle v|A|v\rangle \geq 0$ for any $|v\rangle$, we have

$$\langle v|A|v\rangle = \langle v|B|v\rangle + \mathrm{i}\langle v|C|v\rangle = \beta + \mathrm{i}\gamma$$

Since B og C are Hermitian, β and γ must be real. To keep $\langle v| A|v\rangle$ positive we must have $\gamma = 0$. Since $|v\rangle$ is arbitary, this means that C must be zero, which gives $A = A^\dagger$, i.e. A is Hermitian.