TEK9540 – Quantum computation and quantum information
Problem set 2: Tensor products and traces – Solutions

Exercise 2.26

Outer product notation:

$$
\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \\
|\psi\rangle^{\otimes 2} &= \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |\psi\rangle + |1\rangle \otimes |\psi\rangle\right) = \frac{1}{2}\left(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle\right) \\
|\psi\rangle^{\otimes 3} &= \frac{1}{2}\left(|0\rangle|0\rangle \otimes |\psi\rangle + |0\rangle|1\rangle \otimes |\psi\rangle + |1\rangle|0\rangle \otimes |\psi\rangle + |1\rangle|1\rangle \otimes |\psi\rangle\right) \\
&= \frac{1}{2}\sqrt{2}\left(|0\rangle|0\rangle|0\rangle + |0\rangle|0\rangle|1\rangle + |0\rangle|1\rangle|0\rangle + |0\rangle|1\rangle|1\rangle + |1\rangle|0\rangle|0\rangle + |1\rangle|0\rangle|1\rangle + |1\rangle|1\rangle|0\rangle + |1\rangle|1\rangle|1\rangle\right)
\end{aligned}
$$

Kronecker product notation:

$$
|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}
$$

$$
|\psi\rangle^{\otimes 2} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 \otimes |\psi\rangle\\1 \otimes |\psi\rangle\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}
$$

$$
|\psi\rangle^{\otimes 3} = \frac{1}{2}\begin{pmatrix}1 \otimes |\psi\rangle\\1 \otimes |\psi\rangle\\1 \otimes |\psi\rangle\\1 \otimes |\psi\rangle\end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}
$$

Exercise 2.27

$$X \otimes Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad = \begin{pmatrix} 0 \otimes Z & 1 \otimes Z \\ 1 \otimes Z & 0 \otimes Z \end{pmatrix} \qquad = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$I \otimes X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad = \begin{pmatrix} 1 \otimes X & 0 \otimes X \\ 0 \otimes X & 1 \otimes X \end{pmatrix} \qquad = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$X \otimes I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad = \begin{pmatrix} 0 \otimes I & 1 \otimes I \\ 1 \otimes I & 0 \otimes I \end{pmatrix} \qquad = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

From the last two result we see that the tensor product is not commutative.

Exercise 2.28

$$\begin{aligned}
(A \otimes B)^* &= \begin{pmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B \end{pmatrix}^* \\
&= \begin{pmatrix} (A_{11}B)^* & (A_{12}B)^* & \cdots & (A_{1n}B)^* \\ (A_{21}B)^* & (A_{22}B)^* & \cdots & (A_{2n}B)^* \\ \vdots & \vdots & \ddots & \vdots \\ (A_{m1}B)^* & (A_{m2}B)^* & \cdots & (A_{mn}B)^* \end{pmatrix} \\
&= \begin{pmatrix} A_{11}^*B^* & A_{12}^*B^* & \cdots & A_{1n}^*B^* \\ A_{21}^*B^* & A_{22}^*B^* & \cdots & A_{2n}^*B^* \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}^*B^* & A_{m2}^*B^* & \cdots & A_{mn}^*B^* \end{pmatrix} \\
&= A^* \otimes B^*
\end{aligned}$$

$$
\begin{aligned}
(A \otimes B)^{\mathrm{T}} &= \begin{pmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B \end{pmatrix}^{\mathrm{T}} \\
&= \begin{pmatrix} (A_{11}B)^{\mathrm{T}} & (A_{21}B)^{\mathrm{T}} & \cdots & (A_{m1}B)^{\mathrm{T}} \\ (A_{12}B)^{\mathrm{T}} & (A_{22}B)^{\mathrm{T}} & \cdots & (A_{m2}B)^{\mathrm{T}} \\ \vdots & \vdots & \ddots & \vdots \\ (A_{1n}B)^{\mathrm{T}} & (A_{2n}B)^{\mathrm{T}} & \cdots & (A_{nm}B)^{\mathrm{T}} \end{pmatrix} \\
&= \begin{pmatrix} A_{11}B^{\mathrm{T}} & A_{21}B^{\mathrm{T}} & \cdots & A_{m1}B^{\mathrm{T}} \\ A_{12}B^{\mathrm{T}} & A_{22}B^{\mathrm{T}} & \cdots & A_{m2}B^{\mathrm{T}} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n}B^{\mathrm{T}} & A_{2n}B^{\mathrm{T}} & \cdots & A_{nm}B^{\mathrm{T}} \end{pmatrix} \\
&= A^{\mathrm{T}} \otimes B^{\mathrm{T}}
\end{aligned}
$$

$$
(A \otimes B)^{\dagger} = (A \otimes B)^{*\mathrm{T}} = \left(A^{\mathrm{T}} \otimes B^{\mathrm{T}}\right)^{*} = A^{\dagger} \times B^{\dagger}
$$

Exercise 2.29

Let A and B be unitary and $C{=}A \otimes B$

$$
CC^{\dagger} = (A \otimes B)\,(A \otimes B)^{\dagger} = (A \otimes B)\left(A^{\dagger} \otimes B^{\dagger}\right) = AA^{\dagger} \otimes BB^{\dagger} = I \otimes I = I
$$

So C is unitary.

Exercise 2.30

Let A and B be Hermitain and $C = A \otimes B$

$$
C^{\dagger} = (A \otimes B) \dagger = A^{\dagger} \otimes B^{\dagger} = A \otimes B = C
$$

So C is Hermitian

Exercise 2.31

Let A and B be positive operators with spectral decomposition

$$
A = \sum_i \lambda_i |i_A\rangle \langle i_A| \qquad B = \sum_i \kappa_i |i_B\rangle \langle i_B| \qquad \lambda_i, \kappa_i \geq 0.
$$

Define $C{=}A{\otimes}B$ and let $|\psi\rangle$ be a arbitary vector in the vector space spanned by $\{i_A\}{\otimes}\{i_B\}$. Then $|\psi\rangle = \sum_{ij} c_{ij}|i_A\rangle \otimes |j_B\rangle$ for some c_{ij}. We then have

$$\begin{aligned}
\langle\psi| C|\psi\rangle &= \sum_{ij} c^*_{ij}(\langle i_A| \otimes \langle j_B|)(A \otimes B) \sum_{kl} c_{kl}(|k_A\rangle \otimes |l_B\rangle) \\
&= \sum_{ijkl} c^*_{ij}c_{kl}\langle i_A| A|k_A\rangle \langle j_B| B|l_B\rangle = \sum_{ijkl} c^*_{ij}c_{kl}\langle i_A| \lambda_i|k_A\rangle \langle j_B| \kappa_j|l_B\rangle \\
&= \sum_{ij} |c_{ij}|^2\lambda_i\kappa_j \geq 0
\end{aligned}$$

So C is a positive operator.

Exercise 2.32

Let $P = \sum_i|i_P\rangle \langle i_P|$ and $Q = \sum_i|i_Q\rangle \langle i_Q|$ be projectors. Then

$$R = P \otimes Q = \sum_{ij}(|i_Q\rangle \langle i_Q| \otimes (|j_Q\rangle \langle j_Q|) = \sum_{ij}|i_P j_Q\rangle \langle i_P j_Q|$$

is clearly a projector on the space spanned by $\{|i_P\rangle\} \otimes \{|i_Q\rangle\}$.

Exercise 2.33

Note first that x and y in the formula for $H^{\otimes n}$ are vectors in $\{0,1\}^{\otimes n}$, or put another way, they are bit-strings of n bits. We sum over all different strings.

We are going to prove the formula by induction. For $n = 1$ we get

$$H = \frac{1}{\sqrt{2}}\left(|0\rangle \langle 0| + |0\rangle \langle 1| + |1\rangle \langle 0| - |1\rangle \langle 1|\right)$$

which is correct. Assuming the formula is correct for n we get for $n + 1$

$$\begin{aligned}
H^{\otimes n} \otimes H &= \frac{1}{\sqrt{2^n}} \sum_{\substack{x,y\in\\ \{0,1\}^{\otimes n}}} (-1)^{x\cdot y}|x\rangle \langle y| \otimes \frac{1}{\sqrt{2}}\left(|0\rangle \langle 0| + |0\rangle \langle 1| + |1\rangle \langle 0| - |1\rangle \langle 1|\right) \\
&= \frac{1}{\sqrt{2^{n+1}}} \sum_{\substack{x,y\in\\ \{0,1\}^{\otimes n}}} (-1)^{x\cdot y}\left(|x0\rangle \langle y0| + |x0\rangle \langle y1| + |x1\rangle \langle y0| - |x1\rangle \langle y1|\right) \\
&= \frac{1}{\sqrt{2^{n+1}}} \sum_{\substack{x,y\in\\ \{0,1\}^{\otimes n+1}}} (-1)^{x\cdot y}|x\rangle \langle y| \\
&= H^{\otimes n+1}
\end{aligned}$$

The formula is thus valid for any $n \geq 1$ For $n = 2$ we get

$$H^{\otimes 2} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Exercise 2.37

We use that the elements of the matrix product AB are given by $AB_{ik} = \sum_j A_{ij} B_{jk}$

$$\operatorname{Tr}(AB) = \sum_i (AB)_{ii} = \sum_{ij} A_{ij} B_{ji} = \sum_{ij} B_{ji} A_{ij} = \sum_j (BA)_j = \operatorname{Tr}(BA)$$

Exercise 2.38

$$\operatorname{Tr}(A+B) = \sum_i (A+B)_{ii} = \sum_i A_{ii} + B_{ii} = \operatorname{Tr}(A) + \operatorname{Tr}(B)$$
$$\operatorname{Tr}(zA) = \sum_i (zA)_{ii} = z \sum_i A_{ii} = z \operatorname{Tr}(A)$$

Exercise 2.39

1. We need to show the three criteria on p65 in the book.
   (a) $\operatorname{Tr}\left(A^\dagger \sum_i \lambda_i B_i\right) = \sum_i \lambda_i \operatorname{Tr}\left(A^\dagger B_i\right)$
   by Exc. 2.38
   (b) $(A, B) = \operatorname{Tr}\left(A^\dagger B\right) = \operatorname{Tr}\left(BA^\dagger\right) = \operatorname{Tr}\left(B^\dagger A\right)^\dagger = (B, A)^\dagger = (B, A)^*$
   by Exc. 2.37
   (c) $(A, A) = \operatorname{Tr}\left(A^\dagger A\right) = \sum_{ij} A^\dagger_{ij} A_{ji} = \sum_{ij} A^*_{ji} A_{ji} = \sum_{ij} |A_{ji}|^2 \geq 0$
   We only get equality in the last expression if all elements $A_{ij} = 0$ i.e. $A = 0$

2. The matrix representation of a operator taking a vector from V to V is a element of $\mathbb{C}^{d\times d}$ where d is the dimension of V. Thus the dimension of L_V is d^2

3. The simplest basis for L_V is $\{A_{ij}\}$ where $A_{ij} =$ is a matrix where the elements row i and column j equals one and all other elements are zero. However these matrices are not Hermitian when $i \neq j$. To make a Hermitian basis we notice that the σ-matrices and the identity matrix is a Hermitian orthogonal basis for $\mathbb{C}^{2\times 2}$. For $i < j$ define

$$B_{ij} = \tfrac{A_{ij} + A_{ji}}{\sqrt{2}}$$
$$C_{ij} = \tfrac{\mathrm{i}(A_{ij} - A_{ji})}{\sqrt{2}}$$

   Together $\{B_{ij}\}$, $\{C_{ij}\}$ and $\{A_{ii}\}$ is a Herimitian orthonormal basis for L_V