TEK9540 – Quantum computation and quantum information
Problem set 3 – Solutions

Exercise 2.40

$$[X,Y] = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} - \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}2i & 0\\ 0 & -2i\end{pmatrix} = 2iZ$$

$$[Y,Z] = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \begin{pmatrix}0 & 2i\\ 2i & 0\end{pmatrix} = 2iX$$

$$[Z,X] = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}0 & 2\\ -2 & 0\end{pmatrix} = 2iY$$

We can write

$$[\sigma_j, \sigma_k] = 2i\sum_{l=1}^{3} \varepsilon_{jkl}\sigma_l$$

Exercise 2.41

Calculating

$$\{X,Y\} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

$$\{X,Z\} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

$$\{Y,Z\} = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

and

$$X^2 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$Y^2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$Z^2 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Therefore we can write

$$\{\sigma_j, \sigma_k\} = 2\delta_{jk}I$$

Exercise 2.42

We have that

$$AB = \frac{AB + AB}{2} + \frac{BA - BA}{2} = \frac{AB - BA}{2} + \frac{AB + BA}{2} = \frac{[A, B] + \{A, B\}}{2}$$

Exercise 2.43

From the previous tasks we see that

$$\sigma_j \sigma_k = \frac{[\sigma_j, \sigma_k] + \{\sigma_j, \sigma_k\}}{2} = \frac{2\delta_{jk} I + 2i \sum_{l=1}^{3} \varepsilon_{jkl} \sigma_l}{2} = \delta_{jk} I + i \sum_{l=1}^{3} \varepsilon_{jkl} \sigma_l$$

2.51

H is unitary if $H^\dagger = H^{-1}$ i.e. if $H^\dagger H = I$. It is easy to see that $H^\dagger = H$ so we have,

$$H^\dagger H = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

2.52

Since we have shown that $H^\dagger H = I$ and $H^\dagger = H$ we have $H^2 = I$

2.53

The eigenvalues λ we find by solving the characteristic equation $|H - \lambda I| = 0$.

$$(-\lambda + \frac{1}{\sqrt{2}})(-\lambda - \frac{1}{\sqrt{2}}) - \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} = \lambda^2 - 1 = 0 \Longrightarrow \lambda_1 = 1, \quad \lambda_2 = -1$$

We find the eigenvectors v by solving the equations $Hv - \lambda v = 0$. The unnormalized eigenvectors are

$$v_1 = \begin{pmatrix} 1 \\ \sqrt{2} - 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ -\sqrt{2} - 1 \end{pmatrix}$$

2.54

Assume that A and B are commuting, Hermetian operators. Show that $e^A e^B = e^{A+B}$.

We know that two commuting Hermitian operators may be diagonalized in the same basis.

$$A = \sum_i a_i |i\rangle\langle i| \qquad\qquad B = \sum_i b_i |i\rangle\langle i|.$$

Since A and B are Hermitian we have $e^A = \sum_i e^{a_i} |i\rangle\langle i|$ and $e^B = \sum_i e^{b_i} |i\rangle\langle i|$. Then we get

$$\begin{aligned} e^A e^B &= \sum_i e^{a_i} |i\rangle\langle i| \sum_j e^{b_j} |j\rangle\langle j| = \sum_{i,j} e^{a_i} e^{b_j} |i\rangle \underbrace{\langle i|j\rangle}_{\delta_{ij}} \langle j| \\ &= \sum_i e^{a_i + b_i} |i\rangle\langle i| = e^{A+B}. \end{aligned}$$

2.57

Assume that we initially have a system in a state $|\psi\rangle$. After a measurement with measurement operators $\{L_l\}$ the system is in the state

$$|\phi_l\rangle = \frac{L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}}$$

with probababilty

$$p_\phi(l) = \langle\psi|L_l^\dagger L_l|\psi\rangle.$$

If we then do a measurement with $\{M_m\}$ the system will be in the state

$$\begin{aligned} |\xi_{ml}\rangle = \frac{M_m|\phi_l\rangle}{\langle\phi_l|M_m^\dagger M_m|\phi_l\rangle} &= \frac{M_m L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}\sqrt{\frac{\langle\psi|L_l^\dagger}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}} M_m^\dagger M_m \frac{L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}}}} \\ &= \frac{M_m L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger M_m^\dagger M_m L_l|\psi\rangle}} = \frac{N_{lm}|\psi\rangle}{\sqrt{\langle\psi|N_{lm}^\dagger N_{lm}|\psi\rangle}} \end{aligned}$$

with probability

$$p_\xi(l,m) = \langle\phi_l|M_m^\dagger M_m|\phi_l\rangle p_\phi(l) = \langle\psi|N_{lm}^\dagger N_{lm}|\psi\rangle.$$

This shows that the two measurements $\{M_m\}$ and $\{L_l\}$ is equivalent to the measurement $\{N_{lm}\}$.

2.59

Expectation value:

$$\langle X\rangle = \langle 0|\, X\, |0\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

Standard deviation:

$$\sigma = \langle X^2\rangle - \langle X\rangle^2 = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 0 = 1$$

2.63

By singular value decomposition $M_m = UDV$ and $M_m^\dagger = V^\dagger D U^\dagger$ with D diagonal and non-negative and U and V unitary. We then have

$$\begin{aligned} \sqrt{E_m} &= \sqrt{M_m^\dagger M_m} = \sqrt{V^\dagger D U^\dagger U D V^\dagger} = \sqrt{V^\dagger D^2 V} \\ &= V^\dagger D V = V^\dagger U^\dagger U D V = U_m^\dagger M_m \end{aligned}$$

with $U_m = UV$. Since U and V are unitary U_m is also unitary.

This shows that if we know a POVM-element, we know the corresponding measurement operator only up to a unitary transformation. Accordingly measurement operators only distinguished by a unitary transformation corresponds to the same POVM-element.

Note that the different POVM-elements of a measurement generally is related to the measurement operators by different unitary transformations.

2.66

We choose to use matrix notation, it is also possible to use outer product notation.

$$\frac{\langle 00| + \langle 11|}{\sqrt{2}} X \otimes Z \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0$$