TEK9540 – Quantum computation and quantum information
Problem set 4: Density matrices – Solutions

2.69

It should be easy to verify that the Bell states $|\beta_i\rangle$ are orthonormal, i.e. $\langle\beta_i|\beta_j\rangle = \delta_{ij}$. Since the state space spanned by two qubits is four-dimensional, the four Bell states form a basis.

2.70

E is positive and therefore Hermitian and can be written

$$E = \begin{pmatrix} a+d & b+\mathrm{i}c \\ b-\mathrm{i}c & a-d \end{pmatrix}.$$

The total operator is then given by

$$E \otimes I = \begin{pmatrix} a+d & a+d & b+\mathrm{i}c & b+\mathrm{i}c \\ a+d & a+d & b+\mathrm{i}c & b+\mathrm{i}c \\ b-\mathrm{i}c & b-\mathrm{i}c & a-d & a-d \\ b-\mathrm{i}c & b-\mathrm{i}c & a-d & a-d \end{pmatrix}.$$

Doing the inner product we find

$$\langle\psi|\, E \otimes I|\psi\rangle = a$$

for all ψ.

Thus for any measurement Eve can do the probability of getting a certain measurement result m is the same for all states. This means that she cannot get any information about which bit string Alice is trying to send.

2.71

Since ρ is Hermitian, it can be diagonalized in an orthogonal basis basis $|i\rangle$, such that $\rho = \sum_i \lambda_i |i\rangle\langle i|$ with $\lambda_i \geq 0$, andăTr$(\rho) = \sum_i \lambda_i = 1$. Then $\rho^2 = \sum_i \lambda_i^2 |i\rangle\langle i|$, which gives

$$\mathrm{Tr}(\rho^2) = \sum_n \sum_i \lambda_i^2 \langle n|i\rangle\langle i|n\rangle = \sum_i \lambda_i^2.$$

If ρ is pure then $\lambda_i = 1$ for one and only one i and zero for all others so $\mathrm{Tr}(\rho^2) = 1$. If ρ is mixed then more than one eigenvalue is different from zero and then $\sum_i \lambda_i^2 < 1$, since the eigenvalues are positive and $\sum_i \lambda_i = 1$.

2.72

- A general Hermitian matrix ρ can be written as

$$\rho = \alpha I + \beta \vec{r} \cdot \vec{\sigma} = \begin{pmatrix} \alpha + \beta r_z & \beta(r_x - \mathrm{i} r_y) \\ \beta(r_x + \mathrm{i} r_y) & \alpha - \beta r_z \end{pmatrix}$$

  with α, β, r_x, r_y and r_z real. Note that we have introduced a scaling factor β which we might assign any value by adjusting $\vec{r}$

  Since $\mathrm{Tr}(\rho) = 1$ we have $\alpha = \frac{1}{2}$. We also know that $\mathrm{Tr}\left(\rho^2\right) \leq 1$ which gives:

$$\begin{aligned} \mathrm{Tr}\left(\rho^2\right) &= (\alpha + \beta r_z)^2 + 2\beta^2 (r_x + \mathrm{i} r_y)(r_x - \mathrm{i} r_y) + (\alpha - \beta r_z)^2 \\ &= 2(\alpha^2 + \beta^2 (r_x^2 + r_y^2 + r_z^2)) \leq 1 \end{aligned}$$

  If we choose $\beta = \frac{1}{2}$ this gives $\rho = \frac{I + \vec{r} \cdot \vec{\sigma}}{2}$ and

$$\begin{aligned} r_x^2 + r_y^2 + r_z^2 = \|\vec{r}\|^2 &\leq 1 \\ \|\vec{r}\| &\leq 1. \end{aligned}$$

- For $\rho = I/2$ we have $\vec{r} = 0$. This is a maximally mixed state.

- From the relations above we have:

$$\rho \text{ is pure} \quad \Longleftrightarrow \quad \mathrm{Tr}(\rho^2) = 1 \quad \Longleftrightarrow \quad \|\vec{r}\|^2 = 1 \quad \Longleftrightarrow \quad \|\vec{r}\| = 1.$$

- A pure state can be written

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle.$$

  From Section 1.2 a Bloch-vector is defined by the angles φ and θ and having unit length. From Figure 1.3 in N&C we see that $r_z = \cos\theta$, $r_x = \sin\theta\,\cos\varphi$ and $r_y = \sin\theta\,\sin\varphi$.

  Using the trigonometric identities

$$\begin{aligned} \sin\frac{\theta}{2}\cos\frac{\theta}{2} &= \frac{1}{2}\sin\theta \\ \cos^2\frac{\theta}{2} &= \frac{1}{2} - \left(\frac{1}{2} - \cos^2\frac{\theta}{2}\right) = \frac{1}{2} + \frac{1}{2}\cos\theta \\ \sin^2\frac{\theta}{2} &= \frac{1}{2} - \left(\frac{1}{2} - \sin^2\frac{\theta}{2}\right) = \frac{1}{2} - \frac{1}{2}\cos\theta \end{aligned}$$

the density matrix $\rho = |\psi\rangle\langle\psi|$ is given by

$$\begin{aligned}\rho &= \begin{pmatrix} \cos^2\frac{\theta}{2} & \frac{1}{2}e^{-i\varphi}\sin\theta \\ \frac{1}{2}e^{i\varphi}\sin\theta & \sin^2\frac{\theta}{2} \end{pmatrix} \\ &= \frac{1}{2}\left[I + \begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix}\right] \\ &= \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma})\end{aligned}$$

So for pure states the two definitions of the Bloch vector are equivalent.

2.74

The reduced density operator of system A is

$$\rho^A = \mathrm{Tr}_B(\rho) = \mathrm{Tr}_B(|a\rangle\langle a| \otimes |b\rangle\langle b|) = |a\rangle\langle a| \, \mathrm{Tr}(|b\rangle\langle b|) = |a\rangle\langle a|,$$

which is a pure state.

2.75

The density operator of the Bell state $|\beta\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is $\rho = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 11|)$ The density operator for the first qubit is

$$\begin{aligned}\rho_1 &= \mathrm{Tr}_2(|\beta\rangle\langle\beta|) \\ &= \frac{1}{2}\left(|0\rangle\langle 0| \,\mathrm{Tr}(|0\rangle\langle 0|) + |1\rangle\langle 0| \,\mathrm{Tr}(|1\rangle\langle 0|) + |0\rangle\langle 1| \,\mathrm{Tr}(|0\rangle\langle 1|) + |1\rangle\langle 1| \,\mathrm{Tr}(|1\rangle\langle 1|)\right) \\ &= \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \frac{I}{2}.\end{aligned}$$

The other seven calculations are done the same way, and the answer is $\frac{I}{2}$ for all of them. Thus for any Bell state the state of just one of the qubits is a maximally mixed state.

2.79

The proof of Theorem 2.7 also gives us a recipe for finding the Schmidt decomposition. Writing $|\psi\rangle = \sum_{jk} a_{jk}|j\rangle|k\rangle$ we need to find the singular value decomposition of the matrix a, $a_{jk} = \sum_i u_{ji}d_{ii}v_{ik}$, with u and v unitary and d diagonal. If a is diagonalizable this amounts to finding the eigenvalues and normalized eigenvectors. The Schmidt decomposition is given by $|\psi\rangle = \sum_i \lambda_i|i_A\rangle|i_B\rangle$ with $\lambda_i = d_{ii}$, $|i_A\rangle = \sum_j u_{ji}|j\rangle$, and $|i_B\rangle = \sum_k v_{ik}|k\rangle$.

- $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is in the form of a Schmidt decomposition. ($\lambda_i = \{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\}$, $|i_A\rangle = \{|0\rangle, |1\rangle\}$, $|i_B\rangle = \{|0\rangle, |1\rangle\}$)
- $\psi = \frac{1}{2}(|00\rangle + |01\rangle + |01\rangle + |11\rangle)$ gives

$$a = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Thus a has eigenvalues and eigenvectors,

$$\lambda_i = \{1, 0\} \quad u_i = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

Since a is normal $v = u^{-1} = u^\dagger$, which gives

$$|i_A\rangle = \left\{ \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \right\}$$
$$|i_B\rangle = \left\{ \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \right\}$$

The second vector of the sets $|i_A\rangle$ and $|i_B\rangle$ is not needed as λ_2 is zero, but is included for completeness. Note that it is possible to see the Schmidt decomposition directly from the expression for $|\psi\rangle$ by noting that $|\psi\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}} \frac{|0\rangle+|1\rangle}{\sqrt{2}}$.

- $\psi = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |01\rangle)$ gives

  $$a = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

  a has eigenvalues and eigenvectors,

  $$\lambda_i = \{\frac{1 \pm \sqrt{5}}{2\sqrt{3}}\} \quad u_i = \left\{ \sqrt{\frac{2}{5+\sqrt{5}}} \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}, \sqrt{\frac{2}{5+\sqrt{5}}} \begin{pmatrix} 1 \\ -\frac{1+\sqrt{5}}{2} \end{pmatrix} \right\}$$

  Again a is normal so $v = u^\dagger$. We then get

  $$|i_A\rangle = \left\{ \sqrt{\frac{2}{5+\sqrt{5}}} \left(\frac{1+\sqrt{5}}{2}|0\rangle + |1\rangle \right), \sqrt{\frac{2}{5+\sqrt{5}}} \left(|0\rangle - \frac{1+\sqrt{5}}{2}|1\rangle \right) \right\}$$
  $$|i_B\rangle = \left\{ \sqrt{\frac{2}{5+\sqrt{5}}} \left(\frac{1+\sqrt{5}}{2}|0\rangle + |1\rangle \right), \sqrt{\frac{2}{5+\sqrt{5}}} \left(|0\rangle - \frac{1+\sqrt{5}}{2}|1\rangle \right) \right\}$$

2.81

Let $|AR_1\rangle$ and $|AR_2\rangle$ be to purifications of a system A in the state ρ. Since they are pure states the have Schmidt decompositions,

$$|AR_1\rangle = \sum_i \sqrt{p_i}|i_A\rangle\,|i_R\rangle\,, \quad |AR_2\rangle = \sum_i \sqrt{q_i}|\tilde{i}_A\rangle\,|\tilde{i}_R\rangle\,.$$

We have then two different expression for the density matrix of system A

$$\rho = \sum_i p_i|i_A\rangle\,\langle i_A| \sum_i q_i|\tilde{i}_A\rangle\,\langle \tilde{i}_A|$$

By using Theorem 2.6 in N&C we can relate the two different expressions by a unitary matrix U

$$\sum_i \sqrt{p_i}|i_A\rangle = \sum_{ij} u_{ij}\sqrt{q_j}|\tilde{j}_A\rangle\,.$$

We also know that since $\{|i_R\rangle\}$ and $\{|\tilde{i}_R\rangle\}$ are bases for the same state space there exist a unitary matrix V such that $|i_R\rangle = \sum_{ij} v_{ij}|\tilde{i}_R\rangle$ This gives us,

$$
\begin{aligned}
\text{ă}|AR_1\rangle &= \sum_{ij} u_{ij}\sqrt{q_j}|\tilde{j}_A\rangle \sum_{ik} v_{ik}|k_R\rangle \\
&= \sum_{j} \sqrt{q_j}|\tilde{j}_A\rangle \sum_{ijk} u_{ij}v_{ik}|k_R\rangle \\
&= (I \otimes U_R)|AR_2\rangle
\end{aligned}
$$

where $U_R = U^T V$.

2.82

1. We need to show that when we trace away system R from the state $\rho_{AR} = \sum_{ij}\sqrt{p_i p_j}|\psi_i\rangle|i\rangle\langle\psi_j|\langle j|$ the resulting state of system A is ρ

$$
\begin{aligned}
\text{Tr}_R \sum_{ij} \sqrt{p_i p_j}|\psi_i\rangle|i\rangle\langle\psi_j|\langle j| &= \sum_{ijk}\sqrt{p_i p_j}\langle k||\psi_i\rangle|i\rangle\langle\psi_j|\langle j||k\rangle \\
&= \sum_k p_k|\psi_k\rangle\langle\psi_k| = \rho
\end{aligned}
$$

   Note that we need introduce new summation variables j and k. Generally the density matrix of a state $\sum_i \sqrt{p_i}|\psi_i\rangle$ is *not* $\sum_i p_i|\psi_i\rangle\langle\psi_i|$

2. The measurement operator corresponding to this result is $M_i = I \otimes |i\rangle\langle i|$. The probability is

$$
\begin{aligned}
p(i) &= \text{Tr}\, M_i \rho_{AR} M_i^\dagger \\
&= \text{Tr}\,(I \otimes |i\rangle\langle i|)\Big(\sum_{jk}\sqrt{p_j p_k}|\psi_j\rangle|j\rangle\langle\psi_k|\langle k|\Big)(I \otimes |i\rangle\langle i|) \\
&= \text{Tr}\, p_i|\psi_i\rangle\langle\psi_i| \otimes |i\rangle\langle i| = p_i
\end{aligned}
$$

   The resulting state of system A is

$$
\rho_i = \frac{\text{Tr}_R\, M_i\rho_{AR}M_i^\dagger}{p_i} = \frac{\text{Tr}_R\, p_i|\psi_i\rangle\langle\psi_i| \otimes |i\rangle\langle i|}{p_i} = |\psi_i\rangle\langle\psi_i|
$$

   Thus by measuring the auxiliary system R, we can implicitly measure the system A.

3. From Exercise 2.81 we know that any purification can be written on the form $|AR\rangle = \sum_i p_i|\psi_i\rangle U|i\rangle$. By measuring in the basis given by $\{\sum_j u_{ij}|j\rangle\}$ system A is in the state $|\psi_i\rangle$ with probability p_i after the measurement.