10.4.2 ; Kalkulus

a) $y' = 3yx \qquad y(0) = 4$

$u = y(x)$

$du = \frac{dy}{dx}dx$

$\int \frac{y'}{y} dx = \int \frac{1}{y} \frac{dy}{dx} dx = \int \frac{1}{y} dy$

$\frac{y'}{y} = 3x \rightarrow \boxed{\int \frac{y'}{y} dx = \int 3x\, dx}$

integrerer

$\ln|y| = \frac{3}{2}x^2 + C$

$y(0)=4 \Rightarrow \ln 4 = \frac{3}{2}\cdot 0 + C \Rightarrow C = \ln 4$

$x=0 \Rightarrow y=4$

$e^{\ln|y|} = e^{\frac{3}{2}x^2 + \ln 4} \rightarrow e^{a+b} = e^a e^b$

$|y| = e^{\frac{3}{2}x^2} e^{\ln 4} = \underline{\underline{4e^{\frac{3}{2}x^2}}}$

e) $y' + 2e^y = 0 \qquad y(0)=0$

$y' = -2e^y \rightarrow \frac{1}{e^y} = e^{-y}$

$e^{-y}y' = -2$

$\int e^{-y}\frac{dy}{dx}\cancel{dx} = \int e^{-y} dy$

$\int e^{-y}y'\,dx = \int -2\,dx$

$-e^{-y} = -2x + C \qquad y(0)=0 \Rightarrow -1 = -0 + C \Rightarrow C = -1$

$-e^{-y} = -2x - 1$

$e^{-y} = 2x + 1$

$\ln e^{-y} = \ln(2x+1)$

$-y = \ln(2x+1)$

$\underline{\underline{y = -\ln(2x+1)}}$

10.4.10

vekstraten til $x(t)$

vekstrate = fødselsrate − dødsrate

a)

$$\frac{dx}{dt} = \underbrace{bx^2}_{\text{fødselsrate}} - \underbrace{ax}_{\text{dødsrate}}$$

b) $x(0) = x_0$

$$\frac{1}{bx^2-ax}\,\underbrace{\frac{dx}{dt}}_{=x'(t)} = 1$$

$$\Downarrow$$

$$\int \frac{1}{bx^2-ax}\frac{dx}{\cancel{dt}}\cancel{dt} = \int 1\,dt$$

$$\int \frac{1}{bx^2-ax}\,dx = \int \frac{1}{x(bx-a)}\,dx$$

$$\frac{1}{x(bx-a)} = \frac{A}{x} + \frac{B}{bx-a}$$

$$= \frac{A(bx-a)+Bx}{x(bx-a)}$$

$$= \frac{(Ab+B)x - aA}{x(bx-a)}$$

ser at $Ab+B=0 \Rightarrow -\frac{b}{a}+B=0 \Rightarrow B=\frac{b}{a}$

$-aA = 1 \Rightarrow A = -\frac{1}{a}$

$$\Rightarrow \int \frac{1}{bx^2-ax}\,dx = \int\left(-\frac{1}{ax} + \frac{b}{a(bx-a)}\right)dx$$

$$= \frac{1}{a}\int\left(-\frac{1}{x} + \frac{b}{bx-a}\right)dx$$

$$= \frac{1}{a}\int\left(-\frac{1}{x} + \frac{1}{x-\frac{a}{b}}\right)dx$$

$$= \frac{1}{a}\left(-\ln|x| + \ln\left|x-\frac{a}{b}\right|\right) + C$$

$$\frac{1}{a}\left(-\ln|x| + \ln\left|x-\frac{a}{b}\right|\right) = t + C$$

$$\left(\ln\left(\frac{a}{b}\right) = \ln a - \ln b\right)$$

$$\frac{1}{a}\left(\ln\left|\frac{x-\frac{a}{b}}{x}\right|\right) = t + C$$

$$\ln\left|\frac{x-\frac{a}{b}}{x}\right| = at + C'$$

$$\left|\frac{x-\frac{a}{b}}{x}\right| = e^{at+C'} = e^{at}\underbrace{e^{C'}}_{k}$$

$$\frac{x-\frac{a}{b}}{x} = ke^{at} \Rightarrow 1 - \frac{a}{bx} = ke^{at} \Rightarrow 1 - ke^{at} = \frac{a}{bx} \Rightarrow x = \frac{a}{b(1-ke^{at})}$$

$$x(0) = x_0 \Rightarrow \frac{x_0 - \frac{a}{b}}{x_0} = k \Rightarrow k = 1 - \frac{a}{bx_0}$$

$$x(t) = \frac{a}{b - bke^{at}}$$

$$= \frac{a}{b - b\left(1-\frac{a}{bx_0}\right)e^{at}}$$

$$= \frac{a}{b - \left(b - \frac{a}{x_0}\right)e^{at}}$$

13.3.3 a) $t_0 = 0 \quad x_0 = 1 \qquad x' = f(t, x)$

$h = 0.1 \Rightarrow t_1 = 0.1, \; t_2 = 0.2 \qquad = t + x$

$x_1 = x_0 + hf(t_0, x_0)$

$= 1 + 0.1(t_0 + x_0) = 1 + 0.1(0+1) = 1 + 0.1 = \underline{1.1}$

$x_2 = x_1 + hf(t_1, x_1)$

$= 1.1 + 0.1(t_1 + x_1) = 1.1 + 0.1(0.1 + 1.1)$

$= 1.1 + 0.1 \cdot 1.2 = 1.1 + 0.12 = \underline{1.22}$

$x_3 = x_2 + hf(t_2, x_2)$

$= 1.22 + 0.1(t_2 + x_2) = 1.22 + 0.1(0.2 + 1.22)$

$= 1.22 + 0.1 \cdot 1.42 = 1.22 + 0.142 = \underline{1.362}$

$x' = t + x$

13.3.6

$$\frac{x(t+h)-x(t)}{h} \approx x'(t) = f(t,x(t))$$

ganget opp med h.

$$x(t+h)-x(t) \approx h f(t,x(t))$$

$$x(t+h) \approx x(t) + h f(t,x(t))$$

et steg med Euler fra t til t+h

$$x_1 = x_0 + h f(t_0,x_0)$$

13.4.4

a) $x' = -\lambda x$ (uavhengig av t) $x(0)=1$ $t_0 = 0$

Euler: $x_1 = x_0 + h f(t_0, y_0)$

$= x_0 + h(-\lambda x_0) = x_0(1-\lambda h) = 1-\lambda h$

$\vdots$

$x_{k+1} = x_k + h f(t_k, x_k) = x_k + h(-\lambda x_k)$

$= x_k(1-\lambda h)$

Ser at: $\underline{\underline{x_k = (1-h\lambda)^k}}$

når er $|1-h\lambda| \le 1$, dvs $-1 \le 1-h\lambda \le 1$

$-2 \le -h\lambda \le 0$

$2 \ge h\lambda \ge 0$

$\frac{2}{\lambda} \ge h \ge 0$

b) $x_{k+\frac{1}{2}} = x_k + \frac{h}{2} f(t_k, x_k)$ $t_k = kh$

$x_{k+1} = x_k + h f(t_{k+\frac{1}{2}}, x_{k+\frac{1}{2}})$ $t_{k+\frac{1}{2}} = (k+\frac{1}{2})h$

$x_{k+\frac{1}{2}} = x_k - \frac{h}{2}\lambda x_k = (1-\frac{h\lambda}{2})x_k$

$x_{k+1} = x_k - h\lambda x_{k+\frac{1}{2}} = x_k - h\lambda(1-\frac{h\lambda}{2})x_k$

$= (1 - h\lambda + \frac{h^2\lambda^2}{2})x_k$

$= \cdots (1 - h\lambda + \underbrace{\frac{h^2\lambda^2}{2}})^k$