1.2.2 i Kalkulus

Skal vise: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ (P_n)

Skal vise: P_1 $(n=1)$: VS : 1

HS: $\frac{1(1+1)(2+1)}{6} = 1$

så P_1 stemmer

anta at vi har vist at $P_1, P_2, \dots, P_n$ er sanne.

Vi må da vise at P_{n+1} også er sann (induksjonssteget)

Vi skal vise: P_{n+1} : $\sum_{i=1}^{n+1} i^2 = \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= (n+1)\left(\frac{n(2n+1)}{6} + n+1\right)$$

$$= (n+1)\left(\frac{2n^2+n+6(n+1)}{6}\right)$$

$$= (n+1)\frac{2n^2+7n+6}{6}$$

$$= \frac{(n+1)(n+2)(n+\frac{3}{2})2}{6}$$

$$= \frac{(n+1)(n+2)(2n+3)}{6}$$

$$= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$$

røtter: $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \frac{-7 \pm \sqrt{49-48}}{4} = \frac{-7 \pm 1}{4}$

$\rightarrow$ røttene er -2 og $-\frac{3}{2}$

Dette fullfører induksjonssteget.

0 0 1 0 0 0

0 1 1 0 0

1 2 1

1 3 3 1

$(n+1)^4$ 1 4 6 4 1 $n^4 + 4n^3 + 6n^2 + 4n + 1$

$(n+1)^5$ 1 5 10 10 5 1 $n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1$

$$(n+1)^5 - (n+1) = \underline{n^5} + 5n^4 + 10n^3 + 10n^2 + 5n + \cancel{1} \underline{-n} - \cancel{1}$$

$$= \underline{n^5 - n} + \underline{\underline{5n^4 + 10n^3 + 10n^2 + 5n}}$$

11.1.1

$f(x) = e^{x^2}$

Skal finne Taylorpolynomet av grad 4 om $x=0$

$f'(x) = 2x e^{x^2}$ $\quad f''(x) = 2e^{x^2} + 2x \cdot 2x e^{x^2}$

$= (2 + 4x^2) e^{x^2}$

$f'''(x) = (12x + 8x^3) e^{x^2}$

$f^{(4)}(x) = (12 + 48x^2 + 16x^4) e^{x^2}$

$f'(0) = 0 \quad f''(0) = 2 \quad f'''(0) = 0 \quad f^{(4)}(0) = 12$

$f(0) = 1$

$T_4(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^{(4)}(0)}{4!}(x-0)^4$

$= 1 + \frac{2}{2!}x^2 + \frac{12}{24}x^4 = \underline{\underline{1 + x^2 + \frac{1}{2}x^4}}$

Se på 11.1.3

$T_4(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{f^{(3)}\left(\frac{\pi}{4}\right)}{3!}\left(x - \frac{\pi}{4}\right)^3 + \frac{f^{(4)}\left(\frac{\pi}{4}\right)}{4!}\left(x - \frac{\pi}{4}\right)^4$

11.2.5 $\quad (x^n)' = nx^{n-1}$

$f(x) = \sqrt{x} = x^{\frac{1}{2}}$

Bruk Taylor av grad 2 om 100 til å finne en tilnærming til $\sqrt{101}$. Hvor upresis er denne tilnærmingen?

$\sqrt{101} = T_2(101) + R_2(101)$

$f(x) = T_2(x) + R_2(x) \qquad R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

(c er et tall mellom a og x)

$f(x) = x^{\frac{1}{2}}$ | $f(100) = 10$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$ | $f'(100) = \frac{1}{20}$

$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$ | $f''(100) = -\frac{1}{4}\frac{1}{(100)^{\frac{3}{2}}} = -\frac{1}{4}\frac{1}{100\cdot 10} = -\frac{1}{4000}$

$f'''(x) = \frac{3}{8}x^{-\frac{5}{2}}$

Taylor av grad 2: $T_2(x) = f(100) + f'(100)(x-100) + \frac{f''(100)}{2!}(x-100)^2$

$= 10 + \frac{1}{20}(x-100) - \frac{1}{8000}(x-100)^2$

tilnærmingen: $T_2(101) = 10 + \frac{1}{20}(101-100) - \frac{1}{8000}(101-100)^2$

$= 10 + \frac{1}{20} - \frac{1}{8000} = \frac{80399}{8000} \approx 10.049875$

$R_2(x) = \frac{f^{(3)}(c)}{3!}(x-100)^3$

$R_2(101) = \frac{f^{(3)}(c)}{6}$ (c er et tall mellom 100 og 101)

$= \frac{1}{6}\frac{3}{8}c^{-\frac{5}{2}} \rightarrow$ denne er avtagende, så den har sin største verdi for $c = 100$

$\Rightarrow R_2(101) \leq \frac{1}{6}\cdot\frac{3}{8}100^{-\frac{5}{2}} = \frac{1}{16}\frac{1}{100^{5/2}} = \frac{1}{16\cdot(10^2)^{5/2}} = \frac{1}{16\cdot 10^5}$

$= \frac{1}{16}10^{-5} = 0.625\cdot 10^{-6}$