11.1

1) $f(x) = e^{x^2}$ i $x=0$

Vi skal finne Taylorpolynomet av grad 4

$T_4 f(x)$

$f'(x) = 2xe^{x^2}$

$f''(x) = 2e^{x^2} + 2x\,2xe^{x^2} = (2+4x^2)e^{x^2}$

$f'''(x) = 8xe^{x^2} + (2+4x^2)2xe^{x^2} = (12x+8x^3)e^{x^2}$

$f^{(4)}(x) = (12+24x^2)e^{x^2} + (12x+8x^3)2xe^{x^2} = (12+48x^2+16x^4)e^{x^2}$

$f'(0) = 0$

$f''(0) = 2$

$f'''(0) = 0$

$f^{(4)}(0) = 12$

$T_4 f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)}{4!}x^4$

$= 1 + 0 + \frac{2}{2!}x^2 + 0 + \frac{12}{24}x^4$

$= 1 + x^2 + \frac{1}{2}x^4$

$(e^x = 1 + x + \frac{1}{2}x^2 + \dots)$

11.1.2 $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ om $x = 1$, grad 3

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$
$$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-\frac{3}{2}} = -\frac{1}{4}x^{-\frac{3}{2}}$$
$$f'''(x) = -\frac{1}{4}\left(-\frac{3}{2}\right)x^{-\frac{5}{2}} = \frac{3}{8}x^{-\frac{5}{2}}$$

$$f'(1) = \frac{1}{2}$$
$$f''(1) = -\frac{1}{4}$$
$$f'''(1) = \frac{3}{8}$$

$$T_3\sqrt{x} = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$
$$= 1 + \frac{1}{2}(x-1) - \frac{1}{4\cdot 2}(x-1)^2 + \frac{3}{8\cdot 6}(x-1)^3$$
$$= \underline{1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3}$$

11.1.4 $f(x) = \tan x$ om $x=0$, grad 3

$$f'(x) = \frac{1}{\cos^2 x}$$

$$f''(x) = \frac{-2\cos x(-\sin x)}{\cos^4 x} = \frac{2\sin x}{\cos^3 x}$$

$$f'''(x) = \frac{2\cos x(\cos^3 x) - 3\cos^2 x(-\sin x)2\sin x}{\cos^6 x}$$

$$= \frac{2\cos^4 x + 6\cos^2 x\sin^2 x}{\cos^6 x}$$

$f(0) = 0$

$f'(0) = 1$

$f''(0) = 0$

$f'''(0) = 2$

$$T_3 \tan(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$= x + \frac{2}{6}x^3 = \underline{x + \frac{1}{3}x^3}$$

11.2.1 $f(x)=e^x$ om 0, grad 4 Taylorpolynom og restledd

$f^{(k)}(x)=e^x$ for alle k $\qquad f^{(k)}(0)=1$

$$T_4 e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$R_4 e^x = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$ for en c mellom 0 og $\underset{b}{\underset{\uparrow}{x}}$.

siden e^x er voksende

$$\frac{e^c}{(n+1)!} b^{n+1} \le \frac{e^b}{(n+1)!} b^{n+1}$$ siden $c \le b$

Skal regne ut $|R_4 e^x(b)|$ for en $b \geq 0$

Derfor er $|R_4 e^x(b)| \le \frac{e^b}{(4+1)!} b^{4+1} = \frac{e^b}{120} b^5$

11.2.9 $f(x) = e^x$

regn ut $\int_0^1 \frac{1-e^{-t}}{t}\,dt$ med nøyaktighet 10^{-3}

c mellom 0 og x

$$e^x = T_n(x) + R_n(x) = 1 + x + \dots + \frac{x^n}{n!} + \frac{e^c}{(n+1)!}x^{n+1}$$

setter inn $x = -t$ $(0 \le t \le 1)$ → mellom 0 og $-t$, $c \in [-t, 0]$

$$\frac{1-e^{-t}}{t} = \frac{1 - \left(1 + (-t) + \dots + \frac{(-t)^n}{n!} + \frac{e^c}{(n+1)!}(-t)^{n+1}\right)}{t}$$

$$= \frac{t + \dots + \frac{(-1)^{n+1}t^n}{n!} + \frac{(-1)^{n+2}e^c}{(n+1)!}t^{n+1}}{t}$$

$$= 1 + \dots + (-1)^{n+1}\frac{t^{n-1}}{n!} + \frac{(-1)^{n+2}e^c}{(n+1)!}t^n$$

$$\int_0^1 \frac{1-e^{-t}}{t}\,dt = \int_0^1 \Big(\underbrace{1 + \dots + (-1)^{n+1}\frac{t^{n-1}}{n!}}_{\text{tilnærming}} + \underbrace{(-1)^{n+2}\frac{e^{c(t)}}{(n+1)!}t^n}_{\text{feil}}\Big)\,dt$$

$-t \le c(t) \le 0$

$c(t) \le 0$

$e^{c(t)} \le 1$

feilen: $$\left|\int_0^1 (-1)^{n+2}\frac{e^{c(t)}}{(n+1)!}t^n\,dt\right| \le \int_0^1 \left|(-1)^{n+2}\frac{e^{c(t)}}{(n+1)!}t^n\right|dt$$

$$\le \int_0^1 \frac{1}{(n+1)!}t^n\,dt = \left[\frac{1}{(n+1)!}\frac{1}{n+1}t^{n+1}\right]_0^1 = \frac{1}{(n+1)!(n+1)}$$

Når er $\frac{1}{(n+1)!(n+1)} < 10^{-3}$?

$n=4$: $5! \cdot 5 = 120 \cdot 5 = 600 < 100$

$n=5$: $6! \cdot 6 = 720 \cdot 6 > 1000$

$(n+1)!(n+1) > 1000$

Vi må velge $n \ge 5$. Da blir tilnærmingen:

$$\int_0^1 \frac{1-e^{-t}}{t}\,dt \approx \int_0^1 \left(1 - \frac{t}{2} + \frac{t^2}{6} - \frac{t^3}{24} + \frac{t^4}{120}\right)dt$$

$$= \left[t - \frac{t^2}{2\cdot 2} + \frac{t^3}{3\cdot 6} - \frac{t^4}{4\cdot 24} + \frac{t^5}{5\cdot 120}\right]_0^1$$

$$= 1 - \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3!} - \frac{1}{4\cdot 4!} + \frac{1}{5\cdot 5!} = \dots = \frac{5737}{7200}$$

($5\cdot 5! \to 600$)

$$\approx \underline{\underline{0.7968}}$$