Mock exam, MAT-INF 1100, Autumn 2003

This mock exam has the same format as the real exam, and contains problems of the same type and difficulty. The first part of the exam consists of 10 multiple choice questions which count 3 points each. There is only one correct answer to each of these questions. If your answer is wrong or you do not answer a question, you get no points. In other words, there is no penalty for guessing. The second part of the exam consists of traditional questions. In this part, each of the 7 questions counts 10 points. The maximum total score is therefore 100 points. In the second part of the exam you must explain how you arrived at your answer, if not you will be given 0 points even if the answer is correct!

The only permitted aid at the final exam is a calculator. The sheet of formulas will be attached to the exam paper.

Part 1: Multiple choice problems

Problem 1. The Taylor polynomial of degree 3 for the function $f(x) = e^{-x}$ based at $a = 0$ is given by

☐ $1 + x + x^2/2 + x^3/6$ ☐ $1 + x + x^2/2 + x^3/3$ ☐ $1 - x - x^2/2 - x^3/6$
☐ $1 - x + x^2/2 - x^3/6$ ☐ $1 - x + x^2/2 - x^3/3$

Problem 2. The Taylor polynomial of degree 3 for the function $f(x) = \arctan x$ based at $a = 0$ is given by

☐ $1 + x^2/2$ ☐ $x - x^3/3$ ☐ $1 + x/2 + x^2/3 + x^3/4$ ☐ $x - x^3/6$
☐ $x + x^3/6$

Problem 3. The coefficient multiplying x^3 in the Taylor polynomial of the function $f(x) = \int_0^x e^{\sin u}\,du$ based at $a = 0$ is

☐ 1 ☐ 0 ☐ $1/3$ ☐ $-1/3$ ☐ $1/6$

Problem 4. The Bernstein polynomial $b_{i,n}(x) = \binom{n}{i}(1-x)^i x^{n-i}$, where $0 \leq i \leq n$ and $n \geq 0$ has the property

☐ $b_{i,n}(x) \leq 1$ for all $x \in [0,1]$ ☐ $b_{i,n}(0) = 1/2$
☐ $b_{i,n}(x) \leq 0.1$ for all $x \in [0,1]$ ☐ $b_{i,n}(1) = 1/2$
☐ $b_{i,n}(x) \leq 2$ for all $x \in \mathbb{R}$

Problem 5. The differential equation $y' + y = x$ has the solution

☐ $y(x) = 1 + Cx$ ☐ $y(x) = -1 + x + Ce^{-x}$ ☐ $y(x) = x + Cx^2/2$
☐ $y(x) = x + Ce^{-x}$ ☐ $y(x) = x^2/2 + Ce^{x}$

where C is an arbitrary, real number.

Problem 6. The differential equation $y' + y/x = x^3$ with initial value $y(1) = 1$ has the solution

□ $y(x) = x^4/5 + 4/(5x)$ □ $y(x) = x^4$ □ $y(x) = x^2/2 + 1/(2x)$
□ $y(x) = x$ □ $y(x) = 1/x$

Problem 7. The differential equation $y' - 2xy^{-1/2} = 0$ has the general solution

□ $y(x) = \sqrt{x + C}$ □ $y(x) = x + C$ □ $y(x) = \left(\frac{3}{2}\right)^{2/3} (x^2 + C)^{2/3}$
□ $y(x) = (x^3 + C)^{1/3}$ □ $y(x) = e^{x+C}$

where C is an arbitrary, real number.

Problem 8. The differential equation $y'' + 3y' + 2y = 0$ has the general solution

□ $y(x) = \cos x$ □ $y(x) = C_1 \cos x + C_2 \sin x$ □ $y(x) = C_1 e^{-2x} + C_2 e^{-x}$
□ $y(x) = C_1 e^{-2x} + C_2 e^{x}$ □ $y(x) = C_1 e^{-x} + C_2 e^{-4x}$

where C_1 and C_2 are arbitrary, real numbers.

Problem 9. The differential equation $y'' + 2y' + 10y = 0$ with initial values $y(0) = 0$, $y'(0) = 3$ has the solution

□ $y(x) = 3x$ □ $y(x) = 3e^x - 3$ □ $y(x) = e^{-x} \sin(3x)$
□ $y(x) = e^x \cos(3x)$ □ $y(x) = e^x$

Problem 10. The differential equation $y' = y \cos x$ with initial value $y(0) = 1$ has the solution

□ $y(x) = e^{\sin x}$ □ $y(x) = x$ □ $y(x) = \cos x$ □ $y(x) = e^x$
□ $y(x) = e^{-x}$

Part 2

Remember that an explanation must accompany all your answers!

Problem 1. In this problem we are going to study the Taylor polynomials of the logarithm, based at $a = 1$.

a) Determine the Taylor polynomials of degree 3 of the function $f(x) = \ln x$ based at $a = 1$, and use this to compute an approximation to $\ln 1.1$.

b) Let $T_n f(x)$ denote the Taylor polynomial of f of degree n, based at $a = 1$, and let $R_n f(x) = f(x) - T_n f(x)$ denote the remainder term. Show that the remainder term can be written

$$R_n f(x) = (-1)^n \frac{(x-1)^{n+1}}{n+1} c^{-n-1},$$

where c is a number in the interval $(1, x)$.
We wish to employ $T_n f$ to compute an approximate value of $\ln 1.1$. Find a value of n such that the error is guaranteed to be smaller than 10^{-10} in absolute value.

Hint: You may find Lagranges version of the remainder term useful. This says that if we compute the Taylor polynomial of f based at a, then the remainder can be written as

$$R_n f(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c).$$

Here c is a number in the interval (a, x)

Problem 2. Find the solution of the differential equation

$$y'' + 2y' + 2y = x$$

that satisfies the initial values $y(0) = 1$ and $y'(0) = 0$.

Problem 3. In this problem we are going to study a mathematical model of saving.

a) You have decided to save some money and wish to have NOK 300 000 available after 10 years. You start saving by depositing an amount b with your bank at the end of a year. You receive interest at a fixed annual rate of 5 % which is added at the end of every year. In addition you save an amount s which is added to the account at the end of each year, starting one year after the initial deposit.
Explain why the amount x_n in your account after n years is given by

$$x_n = 1.05\, x_{n-1} + s, \quad x_0 = b.$$

If $s = 18\ 000$ (NOK), how large must b be if you are going to have NOK 300 000 after 10 years?

b) You consider other saving schemes and would like the initial amount b to be the same as the annual amount s. How much do you have to save every year with this scheme to end up with NOK 300 000 after 10 years?

c) You reckon with an annual price increase of 3 %, and believe that you should be able to let the annual savings amount increase at the same rate as the prices. Explain why the amount you now have after n years is given by

$$y_n = 1.05\, y_{n-1} + 1.03^n s.$$

If $s = 18\ 000$ (NOK) as in (a), what is the initial capital b that is needed to end up with NOK 300 000 after 10 years?

Problem 4. A sequence is given by the difference equation $x_n = x_{n-1}/2 + (n+1)/n$ for $n \geq 1$, where $x_0 = 2$. Show by induction that $2 \leq x_n \leq 3$ for all integers $n \geq 2$.

Good luck!