Systemer av differensialligninger

Ex. 13.34 (Seksjon 13.8 i Kompl.

Vi har systemet

$$x' = xy - \cos z, \qquad x(0) = x_0$$
$$y' = 2 - t^2 + z^2 y, \qquad y(0) = y_0$$
$$z' = \sin t - x + y, \qquad z(0) = z_0$$

$x(t)$, $y(t)$ og $z(t)$ er ukjente funksjoner

Vi innfører vektoren $\bar{x} = (x, y, z)$, $\bar{x}_0 = (x_0, y_0, z_0)$
og $\bar{f}(t, \bar{x}) = (f_1(t, \bar{x}), f_2(t, \bar{x}), f_3(t, \bar{x}))$ definert ved

$$x' = f_1(t, \bar{x}) = f_1(t, x, y, z) = xy + \cos z$$
$$y' = f_2(t, \bar{x}) = f_2(t, x, y, z) = 2 - t^2 + z^2 y$$
$$z' = f_3(t, \bar{x}) = f_3(t, x, y, z) = \sin t - x + y$$

Da kan vi skrive det opprinnelige systemet som

$$\boxed{\bar{x}' = \bar{f}(t, \bar{x}), \quad \bar{x}(0) = \bar{x}_0}$$

Generelt: $\bar{x}' = \bar{f}(t,\bar{x})$, $\bar{x}(a) = \bar{x}_0$

der $\bar{x} = (x_1, x_2, \ldots, x_M)$,

$$\bar{f}(t,\bar{x}) = (f_1(t,\bar{x}), f_2(t,\bar{x}), \ldots, f_M(t,\bar{x}))$$

Eulers metode for systemer:

$$\bar{x}_{k+1} = \bar{x}_k + h\,\bar{f}(t_k, \bar{x}_k), \quad k = 0,1,2,3,\ldots$$

$$t_0 = a, \quad t_{k+1} = t_k + h$$

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Eks 13.37

$$\bar{x}' = \bar{f}(t, \bar{x}),$$

$$\bar{f}(t,\bar{x}) = \left(f_1(t, x_1, x_2, x_3),\ f_2(t, x_1, x_2, x_3),\ f_3(t, x_1, x_2, x_3)\right)$$

$$= \left(x_1 x_2 + \cos x_3,\ 2 - t^2 + x_3^2 x_2,\ \sin t - x_1 + x_2\right)$$

Systemet er altså

$$x_1' = x_1 x_2 + \cos x_3$$

$$x_2' = 2 - t^2 + x_3^2 x_2$$

$$x_3' = \sin t - x_1 + x_2$$

Eulers metode: $\bar{X}_{k+1} = \bar{X}_k + h \bar{f}(t_k, \bar{X}_k)$

Skrevet ut: $\bar{X}_{k+1}$

$$\bar{X}_{k+1} = \begin{pmatrix} x_1^{k+1} \\ x_2^{k+1} \\ x_3^{k+1} \end{pmatrix}$$

$$x_1^{k+1} = x_1^k + h f_1(t_k, x_1^k, x_2^k, x_3^k)$$

$$= x_1^k + h(x_1^k x_2^k + \cos x_3^k)$$

$$x_2^{k+1} = x_2^k + h f_2(t_k, x_1^k, x_2^k, x_3^k)$$

$$= x_2^k + h(2 - t_k^2 + (x_3^k)^2 x_2^k$$

$$x_3^{k+1} = x_3^k + h(\sin t_k - x_1^k + x_2^k)$$

Euler, Euler midtpkt, Runge-Kutta etc kan alle brukes til å løse systemer av differensialligninger.

Høyere ordens ligninger som
system av 1. ordens ligninger.

Eks. $x'' = t^2 + \sin(x + x')$, $x(0) = 1$, $x'(0) = 0$

Vi innfører $x_2 = x'$, da er $x_2' = x''$

Så vi har $x_2' = t^2 + \sin(x + x_2)$, $x(0) = 1$

Sett $x_1 = x$. Da har vi totalt $x_2(0) = 0$

$$x_1' = x_2, \qquad x_1(0) = 1$$

$$x_2' = t^2 + \sin(x_1 + x_2), \quad x_2(0) = 0$$

Feilanalyse for Euler

$f(t,x) = t^2 + x$
$f_x(t,x) = 1$

Euler: $x_{k+1} = x_k + h\, f(t_k, x_k)$, $\quad x' = f(t, x)$

Taylor for $x(t)$ om t_k. $\quad x(a) = x_0$.

$$x(t_{k+1}) = x(t_k) + h\, x'(t_k) + \frac{h^2}{2} x''(\xi_k), \quad \xi_k \in (t_k, t_{k+1})$$

Subtraher: $\quad h = t_{k+1} - t_k$

$$\underbrace{x(t_{k+1}) - x_{k+1}}_{\varepsilon_{k+1}} = \underbrace{x(t_k) - x_k}_{\varepsilon_k} + h\, x'(t_k) - h\, f(t_k, x_k) + \frac{h^2}{2} x''(\xi_k)$$

$$\varepsilon_{k+1} = \varepsilon_k + h\, f(t_k, x(t_k)) - h\, f(t_k, x_k) + \frac{h^2}{2} x''(\xi_k)$$

$$= \varepsilon_k + h\big(f(t_k, x(t_k)) - f(t_k, x_k)\big) + \frac{h^2}{2} x''(\xi_k)$$

Husk at $\quad g(x) - g(y) = g'(\theta)(x - y), \quad \theta \in (x, y)$

$$= \varepsilon_k + h\, f_x(t_k, \theta_k) \underbrace{(x(t_k) - x_k)}_{\varepsilon_k} + \frac{h^2}{2} x''(\xi_k)$$

$$\varepsilon_{k+1} = \big(1 + h\, f_x(t_k, \theta_k)\big)\varepsilon_k + \frac{h^2}{2} x''(\xi_k)$$

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