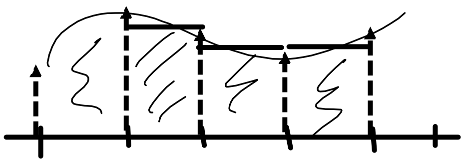
Feilanalyse for numerisk integrasjon.

Midtpunkt metoden.

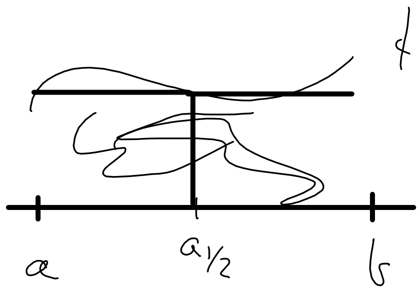


Deler i intervaller.
På hvert intervall
tilnærmer vi f med
en konstant gitt ved funksjonsverdien i
midtpunktet. Integralet av denne konstanten
(arealet av en stolpe) er da en tilnærming
til integralet av f.

Feilanalyse på <u>ett intervall</u>

Tilnærmer med

$$\int_a^b f(x)\,dx \approx (b-a)\, f(a_{1/2})$$



$$a_{1/2} = \frac{a+b}{2}$$

Feil:

$$\int_a^b f(x)\,dx - (b-a)\, f(a_{1/2})$$

Taylor om $a_{1/2}$ for $f(x)$

$$f(x) = f(a_{1/2}) + (x - a_{1/2})\, f'(a_{1/2}) + \frac{1}{2}(x - a_{1/2})^2 f''(\xi)$$

$$\xi \in (a_{1/2}, x)$$

Erstatter $f(x)$ med sitt Taylorpolynom og regner

$$\int_a^b f(x)\,dx - (b-a)\,f(a_{1/2})$$

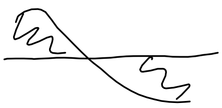
$$= \int_a^b \left(f(a_{1/2}) + (x-a_{1/2})\,f'(a_{1/2}) + \frac{1}{2}(x-a_{1/2})^2 f''(\xi)\right) dx - (b-a)\,f(a_{1/2})$$

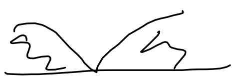
$$= \cancel{(b-a)\,f(a_{1/2})} + \frac{1}{2}(x-a_{1/2})^2 f'(a_{1/2})\Big|_a^b + \frac{1}{2}\int_a^b (x-a_{1/2})^2 f''(\xi)\,dx - \cancel{(b-a)\,f(a_{1/2})}$$

$$= \frac{1}{2}\cancel{(b-a_{1/2})^2 f'(a_{1/2})} - \cancel{\frac{1}{2}(a-a_{1/2})^2 f'(a_{1/2})} + \frac{1}{2}\int_a^b (x-a_{1/2})^2 f''(\xi)\,dx$$

$$= \frac{1}{2}\int_a^b (x-a_{1/2})^2 f''(\xi)\,dx$$

$$\left|\int_a^b f(x)\,dx - (b-a)\,f(a_{1/2})\right| = \left|\frac{1}{2}\int_a^b (x-a_{1/2})^2 f''(\xi)\,dx\right|$$





$$\leq \frac{1}{2}\int_a^b \left|(x-a_{1/2})^2 f''(\xi)\right| dx$$

$$= \frac{1}{2}\int_a^b (x-a_{1/2})^2 \left|f''(\xi)\right| dx$$

$$\leq \frac{1}{2}\int_a^b (x-a_{1/2})^2 \max_{z\in[a,b]} \left|f''(z)\right| dx$$

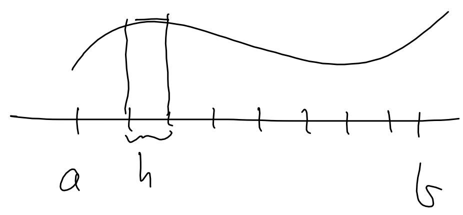
$$= \max_{z\in[a,b]} \left|f''(z)\right| \frac{1}{2}\int_a^b (x-a_{1/2})^2\,dx$$

$$= \max_{z\in[a,b]} \left|f''(z)\right| \frac{(b-a)^3}{24}$$

Global feilanalyse

Deler $[a,b]$ i små biter og bruker midtpunkt-metoden på hvert delintervall

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\,dx \qquad x_i = a + i h \qquad h = \frac{b-a}{n}$$



$$\approx \sum_{i=1}^{n} \underbrace{(x_i - x_{i-1})}_{h} f(x_{i-1/2})$$

Feil: $$\left| \sum_{i=1}^{n} \left(\int_{x_{i-1}}^{x_i} f(x)\,dx - f(x_{i-1/2})\,h \right) \right|$$

$$\leq \sum_{i=1}^{n} \left| \int_{x_{i-1}}^{x_i} f(x)\,dx - f(x_{i-1/2})\,h \right|$$

$$\leq \sum_{i=1}^{n} \frac{(x_i - x_{i-1})^3}{24} \max_{z \in [x_{i-1}, x_i]} |f''(z)|$$

$$= \sum_{i=1}^{n} \frac{h^3}{24} \max_{z \in [x_{i-1}, x_i]} |f''(z)|$$

$$\leq \frac{h^3}{24} \sum_{i=1}^{n} \max_{z \in [a,b]} |f''(z)| \qquad M$$

$$= \frac{h^3}{24}\, n \cdot M$$

$$= \frac{h^2}{24} \cdot M \cdot n \cdot h = \frac{h^2}{24} \cdot M \cdot \cancel{n}\, \frac{b-a}{\cancel{n}}$$

$$= (b-a) \frac{h^2}{24} \max_{z \in [a,b]} |f''(z)|$$