11.1.2 i kompendiet:

Newton-kvotienten: $f'(a) \approx \dfrac{f(a+h)-f(a)}{h}$

(a) Hvis $f(x) = \cos(x)$, hva er feilen.

førsteordens Taylor med restledd:

$$f(a+h) = f(a) + f'(a)h + f''(c)h^2/2 \quad , \text{der } c \in [a, a+h]$$

$$\Downarrow$$

$$\frac{f(a+h)-f(a)}{h} - f'(a) = f''(c)h/2$$

$\Rightarrow$ feilen er mindre enn $\max_{c\in[a,a+h]} |f''(c)|\, h/2$

Her: $f''(x) = -\cos x \Rightarrow |f''(x)| \leq 1$

$\Rightarrow$ feilen mindre enn $h/2$ (alternativ 4)

(b) maskinen runder av $f(a+h)$ til $\overline{f(a+h)}$, $f(a)$ til $\overline{f(a)}$

$\varepsilon_1, \varepsilon_2 \leq \vec{\varepsilon}$

Vi har lært at: $\overline{f(a)} = f(a)(1+\varepsilon_1)$ $\quad \overline{f(a+h)} = f(a+h)(1+\varepsilon_2)$

Total feil:
$$f'(a) - \frac{\overline{f(a+h)} - \overline{f(a)}}{h} = f'(a) - \frac{f(a+h)(1+\varepsilon_2) - f(a)(1+\varepsilon_1)}{h}$$

$$= \underline{f'(a) - \frac{f(a+h)-f(a)}{h}} - \frac{\underline{f(a+h)\varepsilon_2} \; \underline{- f(a)\varepsilon_1}}{h}$$

$$|\text{totalfeil}| \leq \left| f'(a) - \frac{f(a+h)-f(a)}{h} \right| + \frac{|f(a+h)|\varepsilon_2}{h} + \frac{|f(a)|\varepsilon_1}{h}$$

$$\leq \frac{h}{2} \max_{x\in[a,a+h]} |f''(x)| + \max_{x\in[a,a+h]} |f(x)| \frac{2\varepsilon^*}{h} \Rightarrow \text{alternativ 4.}$$

10.6.2

(a) $y'' - 2y' - 8y = 0$

$r^2 - 2r - 8 = 0 \Rightarrow r = \frac{2 \pm \sqrt{4+32}}{2} = \frac{2 \pm 6}{2}$

$\Rightarrow r = 4$ eller $r = -2$

$\Rightarrow$ generell løsning: $y(x) = Ce^{4x} + De^{-2x}$

b) $y'' - 2y' - 8y = 6 - 8x$ Vi prøver $y_p = Ax + B$:

$-2A - 8(Ax + B) = -8Ax - 2A - 8B = 6 - 8x$

ser at $A = 1$. $-2A - 8B = 6 \Rightarrow -2 - 8B = 6 \Rightarrow B = -1$

$\Rightarrow \underline{\underline{y_p(x) = x - 1}}$

c) generell løsning: $y = \underset{(a)}{y_h} + \underset{(b)}{y_p} = Ce^{4x} + De^{-2x} + x - 1$

$y'(x) = 4Ce^{4x} - 2De^{-2x} + 1$

$y(1) = 0: \quad Ce^4 + De^{-2} = 0 \Rightarrow Ce^4 + De^{-2} = 0$

$y'(1) = 1: \quad 4Ce^4 - 2De^{-2} + 1 = 1 \Rightarrow 4Ce^4 - 2De^{-2} = 0$

$\Rightarrow C = D = 0$

$\underline{\underline{y(x) = x - 1}}$

10.6.4

(a) $y'' - 4y' + 4y = 0$ $\quad r^2 - 4r + 4 = 0 \quad \Rightarrow r = 2$ dobbelrot.

$$\Rightarrow y(x) = \underline{Ce^{2x} + Dxe^{2x}}$$

(b) $y'' - 4y' + 4y = x$ $\quad y(0) = 0, \; y'(0) = 1$

prøver: $y(x) = Ax + B$:

$$-4A + 4Ax + 4B = 4Ax - 4A + 4B = x$$

$$\Rightarrow A = \frac{1}{4}, \quad B = A = \frac{1}{4}$$

$$\Rightarrow y_p = \frac{1}{4}(x+1)$$

$$y = Ce^{2x} + Dxe^{2x} + \frac{1}{4}(x+1)$$

$$y'(x) = 2Ce^{2x} + (D + 2Dx)e^{2x} + \frac{1}{4}$$

$y(0) = 0$: $\quad C + \frac{1}{4} = 0 \quad \Rightarrow C = -\frac{1}{4}$

$y'(0) = 1$: $\quad 2C + D + \frac{1}{4} = 1 \Rightarrow D = \frac{3}{4} - 2C = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$

$$\Rightarrow \underline{\underline{y(x) = -\frac{1}{4}e^{2x} + \frac{5}{4}xe^{2x} + \frac{1}{4}(x+1)}}$$

11.2.3 $f(x) = \ln x$ Taylorrekke av grad 3 om $a=1$

$f(x) = \ln x \quad f'(x) = \frac{1}{x} \quad f''(x) = -\frac{1}{x^2} \quad f'''(x) = \frac{2}{x^3} \quad f^{(iv)}(x) = -\frac{6}{x^4}$

$f(1) = 0 \quad f'(1) = 1 \quad f''(1) = -1 \quad f'''(1) = 2$

$$\Rightarrow T_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{6}(x-1)^3$$

$$= \underline{\underline{(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3}}$$

c mellom 1 og $b \geq 1$

Vi skal vise at $|R_3 f(b)| \leq \frac{|b-1|^4}{4}$, $b \geq 1$: ✓

Vi har at $R_3 f(x) = \frac{f^{(iv)}(c)}{24}(x-1)^4 \quad \left(\frac{f^{(iv)}(c)}{(n+1)!}(x-1)^{n+1} \quad n=3 \right)$

$$= |R_3 f(x)| = \frac{6}{c^4 24}(x-1)^4 = \frac{1}{4c^4}(x-1)^4 \leq \frac{(x-1)^4}{4}$$

Det var dette vi skulle vise $(x = b)$

Eksamen H12 del 2, oppgave 2:

Vi skal vise at, for $n \geq 1$, så er $\frac{1}{3}n^3 \leq \sum_{k=1}^{n} k^2$

P_n: $\frac{1}{3}n^3 \leq \sum_{k=1}^{n} k^2$ (induksjonshypotesen)

P_1: ser at $\frac{1}{3} \leq 1$, som er opplagt sant $\Rightarrow P_1$ er sann.

Anta at vi har vist at P_k er sann, for $k = 1, 2, \ldots, n$

Vi skal vise at P_{n+1} er sann:

$$\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2 \overset{P_n \text{ sann}}{\geq} \frac{1}{3}n^3 + (n+1)^2$$

$$= \frac{1}{3}n^3 + n^2 + 2n + 1$$

$$\geq$$

$$\frac{1}{3}n^3 + n^2 + n + \frac{1}{3}$$

$$= \frac{1}{3}(n+1)^3$$

$\Rightarrow P_{n+1}$ er også sann.