FORELESNING 10/11 - 2015

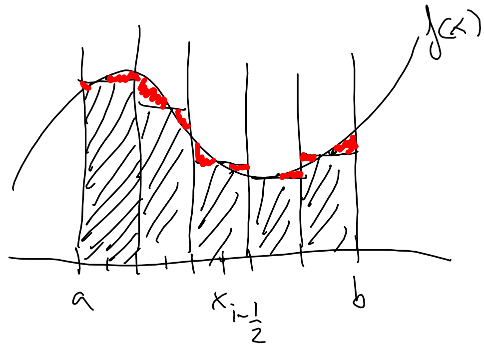
IDAG: NUMERISK INTEGRASJON, FEIL-ANALYSE (12)

NESTE UKE: LØSNING AV LIGNINGER (kap 10)

MIDTPUNKTSMETODEN

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} h\, f(x_{i-\frac{1}{2}})$$

$h = \frac{b-a}{n}$



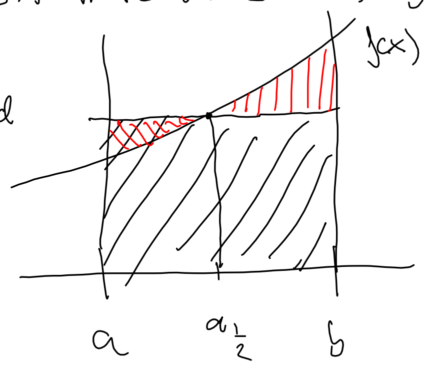
Hva blir feilen? Skal se på feilen

$$\int_a^b f(x)\,dx - \sum_{i=1}^{n} h\, f(x_{i-\frac{1}{2}})$$

som funksjon av h, b−a og f selv

FEILANALYSE PÅ ETT INTERVALL (LOKAL)

$$\int_a^b f(x)\,dx \approx (b-a)\cdot f(a_{\frac{1}{2}}) = I_{mid}$$



Feil: $\int_a^b f(x)\,dx - (b-a)\,f(a_{\frac{1}{2}})$

Analyserer med Taylor polynomer av grad 1 om $a_{\frac{1}{2}}$

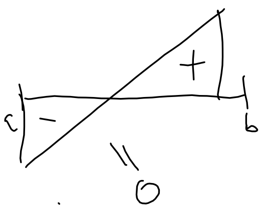
$$f(x) = f(a_{\frac{1}{2}}) + (x-a_{\frac{1}{2}})\,f'(a_{\frac{1}{2}}) + \frac{1}{2}(x-a_{\frac{1}{2}})^2 f''(\xi) \quad \text{for } \xi=\xi_x\in[a,b]$$

Integrerer dette

$$\int_a^b f(x)\,dx \approx \int_a^b f(a_{\frac{1}{2}})\,dx + \underbrace{\int_a^b (x-a_{\frac{1}{2}})\,f'(a_{\frac{1}{2}})\,dx}_{\parallel} + \int_a^b \frac{1}{2}(x-a_{\frac{1}{2}})^2 f''(\xi)\,dx$$

$$= (b-a)\cdot f(a_{\frac{1}{2}}) + 0 + \int_a^b \frac{1}{2}(x-a_{\frac{1}{2}})^2 f''(\xi)\,dx$$

Forklaring



$$= (b-a)\,f(a_{\frac{1}{2}}) + \int_a^b \frac{1}{2}(x-a_{\frac{1}{2}})^2 f''(\xi)\,dx$$

Feilen blir (stabler om)

$$\int_a^b f(x)\,dx - (b-a)\,f(a_{\frac{1}{2}}) = \frac{1}{2}\int_a^b (x-a_{\frac{1}{2}})^2 f''(\xi)\,dx$$

Har funnet et uttrykk for feilen. Litt upraktisk

$\Rightarrow$ forsøker å finne en øvre grense

$$\left|\int_a^b f(x)\,dx - (b-a) f(a_{\frac{1}{2}})\right| = \left|\frac{1}{2}\int_a^b (x-a_{\frac{1}{2}})^2 \cdot f''(\xi)\,dx\right|$$

Forklaring

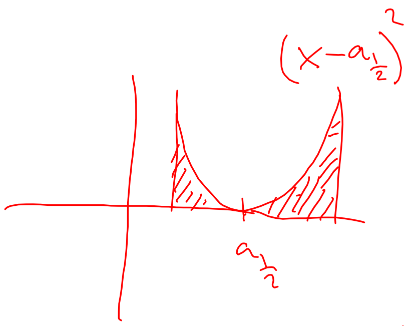
Trekant-ulikhet

$$\leq \frac{1}{2}\int_a^b \left|(x-a_{\frac{1}{2}})^2 f''(\xi)\right| dx$$

$$= \frac{1}{2}\int_a^b (x-a_{\frac{1}{2}})^2 \, |f''(\xi)| \, dx$$

$$\leq \frac{1}{2}\int_a^b (x-a_{\frac{1}{2}})^2 \cdot \max_{z\in[a,b]} |f''(z)| \, dx$$

konstant



$$= \max_{z\in[a,b]} |f''(z)| \cdot \frac{1}{2}\int_a^b (x-a_{\frac{1}{2}})^2 dx$$

$$\int_a^b (x-a_{\frac{1}{2}})^2 dx = \frac{1}{3}(x-a_{\frac{1}{2}})^3 \Big|_a^b = \frac{1}{3}\left(\left(\frac{h}{2}\right)^3 - \left(-\frac{h}{2}\right)^3\right) = \frac{h^3}{12}$$

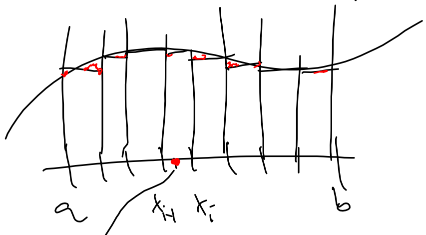
$$= \max_{z\in[a,b]} |f''(z)| \cdot \frac{(b-a)^3}{24}$$

ØVRE GRENSE FOR FEILEN I MIDTPUNKTS-METODEN MED $h = b-a$

GLOBAL FEILANALYSE (FLERE INTERVALLER)

Deler $[a,b]$ i n biter, hver med lengde $h = \frac{b-a}{n}$

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\,dx, \quad x_i = a + ih$$



$$\approx \sum_{i=1}^{n} \underbrace{(x_i - x_{i-1})}_{=h} f(x_{i-\frac{1}{2}}) \qquad \text{hvor } x_{i-\frac{1}{2}} = \frac{x_{i-1}+x_i}{2}$$

Hvilken feil gjør vi? Bruker forrige resultat

Forklaringer:

$$\text{Feil} = \left| \sum_{i=1}^{n} \left(\int_{x_{i-1}}^{x_i} f(x)\,dx - h f(x_{i-1/2}) \right) \right|$$

Trekant-ulikhet

$$\leq \sum_{i=1}^{n} \left| \int_{x_{i-1}}^{x_i} f(x)\,dx - h f(x_{i-\frac{1}{2}}) \right|$$

Forrige estimat

$$\leq \sum_{i=1}^{n} \max_{z\in[x_{i-1},x_i]} |f''(z)| \, \frac{(x_i - x_{i-1})^3}{24}$$

$$= \frac{h^3}{24} \cdot \sum_{i=1}^{n} \max_{z\in[x_{i-1},x_i]} |f''(z)|$$

$\leq \max_{z\in[a,b]} |f''(z)|$

$$\leq \frac{h^3}{24} \cdot \sum_{i=1}^{n} \max_{z\in[a,b]} |f''(z)|$$

$h \cdot n = \frac{b-a}{\cancel{n}} \cancel{n}$

$$= \frac{h^3}{24} \cdot n \cdot \max_{z\in[a,b]} |f''(z)|$$

$$\boxed{= \frac{h^2}{24}(b-a) \cdot \max_{z\in[a,b]} |f''(z)|}$$

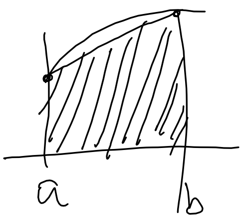
Dette er en øvre grense for feilen som funksjon av h (og a, b, f)

Hvis vi halverer h, blir feilen mindre med faktor 4

FEIL I ANDRE METODER

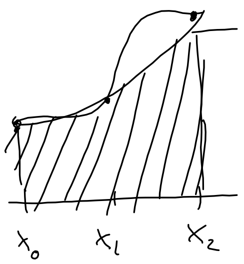
MIDTPUNKTSMETODEN: $\left| I - I_{mid}(h) \right| \leq \frac{h^2}{24} \cdot (b-a) \cdot \max_{z \in [a,b]} \left| f''(z) \right|$

TRAPES-METODEN: Bruker lineære approksimasjoner



$$\left| I - I_{trapes}(h) \right| \leq \frac{h^2}{12} (b-a) \cdot \max_{z \in [a,b]} \left| f''(z) \right|$$

SIMPSONS METODE: Bruker kvadratiske approksimasjoner



$$\left| I - I_{simp}(h) \right| \leq \frac{h^4}{180} (b-a) \max_{z \in [a,b]} \left| f''''(z) \right|$$