1.1.3 a) Skriv uten summetegn og regn ut:

$$\sum_{n=1}^{6} 2^n = 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6$$

$$= 2 + 4 + 8 + 16 + 32 + 64$$

$$= 126$$

1.1.5 a) $5+7+9+11+13$ Skriv med summetegn.

Oddetall er på formen $2k+1$ for noen k, så vi skriver

$$\sum_{k=2}^{6}(2k+1) = 5+7+9+11+13$$

c) $4+8+12+\dots+64 = \sum_{n=1}^{16} 4n$

e) $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\frac{1}{81}-\frac{1}{243} = \frac{1}{3^0}-\frac{1}{3^1}+\frac{1}{3^2}-\frac{1}{3^3}+\frac{1}{3^4}-\frac{1}{3^5}$

$= \sum_{n=0}^{5}(-1)^n\left(\frac{1}{3}\right)^n = \sum_{n=0}^{5}\left(-\frac{1}{3}\right)^n$

1.2.2 Vis ved induksjon at $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

$$P_n: \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

P_1: v.s. $\sum_{i=1}^{1} i^2 = 1^2 = 1$ h.s. $\frac{1(1+1)(2\cdot 1+1)}{6} = \frac{1\cdot 2\cdot 3}{6} = 1$. P_1 sann!

P_2: v.s. $\sum_{i=1}^{2} i^2 = 1^2+2^2 = 5$ h.s. $\frac{2(2+1)(2\cdot 2+1)}{6} = \frac{2\cdot 3\cdot 5}{6} = 5$. P_2 sann!

Anta P_n holder. Vil vise at da holder P_{n+1}.

$$P_{n+1}: \sum_{i=1}^{n+1} i^2 = \frac{(n+1)(n+2)(2n+3)}{6} \leftarrow$$

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2 \overset{P_n}{=} \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6}$$

$$= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}$$

$$= \frac{(n+1)\left(n(2n+1) + 6(n+1)\right)}{6}$$

$$= \frac{(n+1)(2n^2+n+6n+6)}{6}$$

$$= \frac{(n+1)(2n^2+7n+6)}{6}$$

$$\sum_{i=1}^{n+1} i^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$: P_{n+1} er sann!

Induksjon OK. □

1.2.5 Vis at $n^5 - n$ delelig med 5 for alle naturlige tall n.

P_n: $n^5 - n$ er delelig med 5.

Sjekke:

P_1: $1^5 - 1 = 0$, og 0 er delelig med 5. OK!

P_2: $2^5 - 2 = 32 - 2 = 30$, som er delelig med 5. OK!

Induksjonssteget:

Anta at P_k er sann: $k^5 - k$ er delelig med 5.

Se på uttrykket for $k+1$:

$$(k+1)^5 - (k+1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k \cancel{+1} - k \cancel{-1}$$

$$= \underbrace{5(k^4 + 2k^3 + 2k^2 + k)}_{\text{delelig med 5}} + \underbrace{(k^5 - k)}_{\text{delelig med 5 } (P_k)}$$

Dermed er $(k+1)^5 - (k+1)$ delelig med 5: P_{k+1} er sann. Induksjon OK. $\square$

Eksempel 1.2.4

Vis at $(1+x)^n \geq 1+nx$ for alle naturlige tall n hvis $x \geq -1$.

P_n: $(1+x)^n \geq 1+nx$.

Sjekk P_1: v.s. $1+x$, h.s. $1+x$. $1+x \geq 1+x$. OK!

Anta P_k: $(1+x)^k \geq 1+kx$.

Vil vise at $(1+x)^{k+1} \geq 1+(k+1)x$.

pga. P_k og $x \geq -1$.

Vi ser at

$$(1+x)^{k+1} = (1+x)^k(1+x) \geq (1+kx)(1+x)$$

$$= 1+(k+1)x+kx^2 \geq 1+(k+1)x. \qquad \square$$

$x^2 \geq 0,\ k \geq 0.$

1.2.10 a)

$$1 = 1$$
$$4 = 1 + 3$$
$$9 = 1 + 3 + 5$$
$$16 = 1 + 3 + 5 + 7$$
$$25 = 1 + 3 + 5 + 7 + 9$$

Hypotese: $\sum_{i=1}^{n} (2i-1) = n^2$.

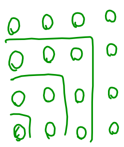
b) La $P_k: \sum_{i=1}^{k}(2i-1) = k^2$. Anta P_k.

Vil vise $P_{k+1}: \sum_{i=1}^{k+1}(2i-1) = (k+1)^2$.

$$\sum_{i=1}^{k+1}(2i-1) = \sum_{i=1}^{k}(2i-1) + (2(k+1)-1)$$

$$\overset{P_k}{=} k^2 + 2k + 1$$

$$= (k+1)^2.$$ P_{k+1} er sann! $\square$



Ekstraoppgave

Bevis $\sum_{i=1}^{n}(-1)^{i+1} i^2 = (-1)^{n+1} n \frac{n+1}{2}$ for $n \geq 1$.

P_k: $\sum_{i=1}^{k}(-1)^{i+1} i^2 = (-1)^{k+1} k \frac{k+1}{2}$.

P_1: v.s. $\sum_{i=1}^{1}(-1)^{i+1} i^2 = 1^2 = 1$. h.s. $(-1)^2 \cdot 1 \cdot \frac{2}{2} = 1$. OK!

Anta nå P_k, og vil vise P_{k+1}: $\sum_{i=1}^{k+1}(-1)^{i+1} i^2 = (-1)^{k+2} \frac{(k+1)(k+2)}{2}$

$$\sum_{i=1}^{k+1}(-1)^{i+1} i^2 = \sum_{i=1}^{k}(-1)^{i+1} i^2 + (-1)^{k+2}(k+1)^2$$

$$\overset{P_k}{=} (-1)^{k+1}\frac{k(k+1)}{2} + (-1)^{k+2}(k+1)^2$$

$$= (-1)^{k+1}\left(\frac{k(k+1)}{\underline{2}} - \frac{2(k+1)^2}{\underline{2}}\right)$$

$$= (-1)^{k+1}\frac{(k+1)}{2}\left(k - 2(k+1)\right)$$

$$= (-1)^{k+1}\frac{(k+1)}{2}(-k-2)$$

$$= (-1)^{k+1}\frac{(k+1)}{2}(-1)(k+2)$$

$$= (-1)^{k+2}\frac{(k+1)(k+2)}{2}.$$

P_{k+1} er sann. $\square$