Taylorpolynomet av grad n til f om pkt. a:

$$T_n f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$
$$+ \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Kalkulus 11.1

(1) Finn $T_4 f(x)$ når $f(x) = e^{x^2}$ og $a=0$.

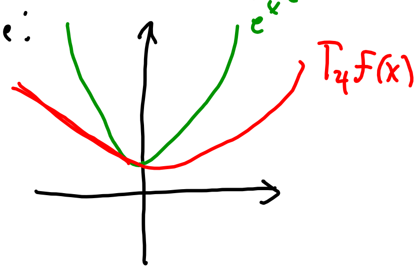
$$f(x) = e^{x^2} \qquad f(0) = e^0 = 1$$
$$f'(x) = (x^2)' \cdot e^{x^2} = \underbrace{2x}_{u}\underbrace{e^{x^2}}_{v} \qquad f'(0) = 0 \qquad (uv)' = u'v + uv'$$
$$f''(x) = 2e^{x^2} + \underbrace{4x^2}_{u}\underbrace{e^{x^2}}_{v} \qquad f''(0) = 2$$
$$f'''(x) = 4xe^{x^2} + 8xe^{x^2} + 8x^3e^{x^2}$$
$$= \underbrace{12x}_{u}\underbrace{e^{x^2}}_{v} + \underbrace{8x^3}_{u}\underbrace{e^{x^2}}_{v} \qquad f'''(0) = 0$$

$$f^{(4)}(x) = 12e^{x^2} + 24x^2e^{x^2} + 24x^2e^{x^2}$$
$$+ 16x^4e^{x^2}$$
$$f^{(4)}(0) = 12$$

$$f(0) = 1,\ f'(0) = 0,\ f''(0) = 2$$
$$f'''(0) = 0,\ f^{(4)}(0) = 12$$

$$T_4 f(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}x^2$$
$$+ \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4$$
$$= 1 + x^2 + \frac{1}{2}x^4.$$

skisse:



⑦ Finn Taylor-polynomet av grad 3 til $f(x) = \arctan x$ i punktet 0.

$f(x) = \arctan x \qquad f(0) = 0$

$f'(x) = \frac{1}{1+x^2} \begin{matrix} \leftarrow u \\ \leftarrow v \end{matrix} \qquad f'(0) = \frac{1}{1+0} = 1$

Kvotientregelen: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

$f''(x) = \frac{0 - 2x}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2} \begin{matrix} \leftarrow u \\ \leftarrow v \end{matrix} \qquad f''(0) = \frac{0}{1} = 0$

$$f'''(x) = \frac{-2(1+x^2)^2 + 2x\left((1+x^2)^2\right)'}{(1+x^2)^4}$$

$$\left((1+x^2)^2\right)' = 2(1+x^2)\cdot(1+x^2)' = 4x(1+x^2)$$

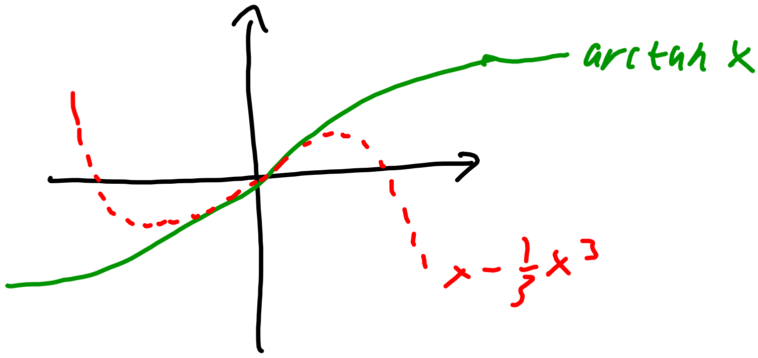
$$f'''(x) = \frac{-2(1+x^2)^2 + 8x(1+x^2)}{(1+x^2)^4}$$

$$= \frac{8x - 2(1+x^2)}{(1+x^2)^3} \qquad f'''(0) = -2$$

$$T_3 f(x) = 0 + x + \frac{0}{2}x^2 - \frac{2}{6}x^3$$

$$= x - \frac{1}{3}x^3.$$

skisse:



(10) Finn Taylor-polynomet av grad 3 til

$f(x) = x^4 - 3x^2 + 2x - 7$ i punktet 1.

$f(1) = 1 - 3 + 2 - 7 = -7$

$f'(x) = 4x^3 - 6x + 2$ $\qquad f'(1) = 4 - 6 + 2 = 0$

$f''(x) = 12x^2 - 6$ $\qquad f''(1) = 6$

$f'''(x) = 24x$ $\qquad f'''(1) = 24$

$$T_3 f(x) = -7 + 0(x-1) + \frac{6}{2}(x-1)^2 + \frac{24}{6}(x-1)^3$$

$$= \cdots$$

$$= 4x^3 - 9x^2 + 6x - 8.$$

Lagranges restleddformel:

f med sine $n+1$ første deriverte er kont. på $[a,x]$.

Da er

$$R_n f(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \qquad c \in (a,x)$$

Altså er $f(x) = T_n f(x) + R_n f(x)$.

Kalkulus 11.2

1) Finn Taylor-polynomet av grad 4 om punktet 0 for $f(x) = e^x$ og vis at

$|R_4 f(b)| \leq \frac{e^b}{120} b^5$ for $b \geq 0$.

Vi vet at $f^{(k)}(x) = e^x$ for alle $k \geq 0$, så

$f^{(k)}(0) = 1$ for alle $k \geq 0$.

$$T_4 f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}.$$

Fra restleddformelen er

$$|R_4 f(b)| = \left| \frac{f^{(5)}(c)}{5!} b^5 \right| \quad \text{der } c \in (0,b)$$

$$\leq \frac{b^5}{5!} |e^c|$$

$e^c = f^{(5)}(c)$

og e^c er voksende, så for $c \in (0,b)$ er $e^c \leq e^b$

$$\leq \frac{b^5}{120} e^b,$$

som var det vi skulle vise.

(5) Bruk T.p. til $f(x)=\sqrt{x}$ av grad 2 om punktet 100 til å finne en tilnærmet verdi for $\sqrt{101}$. Anslå nøyaktighet.

$f(x)=\sqrt{x}$ $\qquad f(100)=10$

$f'(x)=\frac{1}{2\sqrt{x}}$ $\qquad f'(100)=\frac{1}{20}$

$f''(x)=-\frac{1}{4x\sqrt{x}}$ $\qquad f''(100)=-\frac{1}{4000}$.

Vi ser at $T_2f(x)=10+\frac{(x-100)}{20}-\frac{(x-100)^2}{8\,000}$

Overslag:

$\sqrt{101}\approx T_2f(101)=10+\frac{1}{20}-\frac{1}{8000}\approx 10.04986$

Hvor nøyaktig? Vi trenger $R_2f(x)$.

$f'''(x)=\frac{3}{8}x^{-5/2}$

Denne funksjonen er avtagende på $[100,101]$.

$|R_2f(101)|=\left|\frac{f'''(c)}{\underbrace{3!}_{6}}\underbrace{(101-100)^3}_{1}\right|$ der $c\in(100,101)$

$=\left|\frac{\frac{3}{8}c^{-5/2}}{6}\right|\leq\frac{1}{16}|100^{-5/2}|$

$=\frac{1}{16}\cdot10^{-5}=6.25\cdot10^{-7}$.

Så feilen er på størrelse med 10^{-6}.

(9) Bruk T.p. til $f(x) = e^x$ til å finne

$$\int_0^1 \frac{1-e^{-t}}{t}\,dt$$

med nøyaktighet på 10^{-3}.

Vi vet at

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^c}{(n+1)!}x^{n+1} \text{ der } c \in (0,x).$$

Hvis $x = -t$ får vi

$$e^{-t} = 1 + (-t) + \frac{t^2}{2!} + \cdots + (-1)^n \frac{t^n}{n!} + (-1)^{n+1}\frac{e^c}{(n+1)!}t^{n+1}$$

$$c \in (0,-t)$$

$$\int_0^1 \frac{1-e^{-t}}{t}\,dt = \int_0^1 \frac{1-\left(1+(-t)+\cdots+(-1)^n\frac{t^n}{n!}+(-1)^{n+1}\frac{e^{c(t)}}{(n+1)!}t^{n+1}\right)}{t}$$

$$= \int_0^1 \frac{t+\cdots+(-1)^{n+1}\frac{t^n}{n!}+(-1)^{n+2}\frac{e^{c(t)}}{(n+1)!}t^{n+1}}{t}\,dt$$

$$= \int_0^1 \underbrace{1-\frac{t}{2!}+\cdots+(-1)^{n+1}\frac{t^{n-1}}{n!}}_{\text{Tilnærming}} + \underbrace{(-1)^{n+2}\frac{e^{c(t)}}{(n+1)!}t^n}_{\text{Feilen}}\,dt.$$

Feilen vi gjør er

$$\left|\int_0^1 (-1)^{n+2}\frac{e^{c(t)}}{(n+1)!}t^n\,dt\right| \leq \int_0^1 \left|\cancel{(-1)^{n+2}}\frac{e^{c(t)}}{(n+1)!}t^n\right|dt$$

Observer at $c(t) \in (0,-t)$ og $t \geq 0$, og dermed er $e^{c(t)} \leq e^0 = 1$.

Feilen er da

$$\left|\int_0^1 (-1)^{n+2}\frac{e^{c(t)}}{(n+1)!}t^n\,dt\right| \leq \int_0^1 \frac{1}{(n+1)!}t^n\,dt = \frac{1}{(n+1)!}\int_0^1 t^n\,dt$$

$$\int_0^1 t^n\,dt = \left[\frac{t^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1} - \frac{0}{n+1} = \frac{1}{n+1}$$

$$= \frac{1}{(n+1)(n+1)!}.$$

Oppgaven ber om $\frac{1}{(n+1)(n+1)!} \leq 10^{-3}$, og da får vi $n=5$. (ved utprøving)

Vi setter $n=5$ i tilnærmingen:

$$\int_0^1 \frac{1-e^{-t}}{t}\,dt \approx \int_0^1 1-\frac{t}{2}+\frac{t^2}{6}-\frac{t^3}{24}+\frac{t^4}{120}\,dt$$

$$= \left[t-\frac{t^2}{4}+\frac{t^3}{3\cdot 6}-\frac{t^4}{4\cdot 24}+\frac{t^5}{5\cdot 120}\right]_0^1$$

$$= 1-\frac{1}{4}+\frac{1}{18}-\frac{1}{96}+\frac{1}{600} = \frac{5737}{7200} \approx 0.7968$$

15 a) $g(x) = (1+x)^{1/3}$. $\quad (u^{\alpha})' = \alpha u^{\alpha-1}$

Finn T.p. av orden 2 om origo.

$$g'(x) = \frac{1}{3}(1+x)^{-2/3} \cdot \Big(\underbrace{(1+x)'}_{=1}\Big)$$

$$g''(x) = \frac{-2}{9}(1+x)^{-5/3}$$

Vi kan regne ut

$$g(0) = (1+0)^{1/3} = 1,\ g'(0) = \frac{1}{3},\ g''(0) = \frac{-2}{9}$$

Dermed er

$$T_2 g(x) = 1 + \frac{1}{3}x - \frac{2}{9 \cdot 2}x^2 = 1 + \frac{1}{3}x - \frac{1}{9}x^2.$$

b) Vis at for $x \geq 0$ er $|R_2 g(x)| \leq \frac{5}{81}x^3$.

$$g'''(x) = \frac{10}{27}(1+x)^{-8/3}.$$

$$R_2 g(x) = \frac{\frac{10}{27}(1+c)^{-8/3}}{3!}x^3 = \frac{5}{81}x^3 \cdot (1+c)^{-8/3}$$

der $c \in (0,x)$.

Men $(1+y)^{-8/3}$ er avtagende på $(0,x)$! Så $(1+c)^{-8/3} \leq 1^{-8/3} = 1$.

$$|R_2 g(x)| = \left|\frac{5}{81}x^3(1+c)^{-8/3}\right| \leq \frac{5}{81}|x|^3 \overset{x \geq 0}{=} \frac{5}{81}x^3.$$

c) Feil mindre enn 10^{-7} for tilnærming av $\sqrt[3]{1003}$.

La $f(x) = \sqrt[3]{x}$, og vi utvikler $T_n f(x)$ om $a = 1000$. Da er $T_n f(1003)$ tilnærmingen vår.

$$f(x) = T_n f(x) + R_n f(x)$$

$$|f(1003) - T_n f(1003)| = |R_n f(1003)|.$$

$$f'(x) = \frac{1}{3}x^{-2/3}$$

$$f''(x) = -\frac{2}{9}x^{-5/3}$$

$$f'''(x) = \frac{10}{27}x^{-8/3},$$

så hvis vi prøver med f.eks. $n=2$:

$$|R_2 f(1003)| = \left|\frac{\frac{10}{27}c^{-8/3}}{3!}(1003-1000)^3\right|$$

der $c \in (1000, 1003)$ som over er $f'''(x)$ avtagende her

$$\leq \frac{10 \cdot 3^3}{3! \cdot 27} \cdot \left|\frac{1}{(\sqrt[3]{c})^8}\right| \leq \frac{5}{3} \cdot \frac{1}{(10)^8} = \frac{5}{3} \cdot 10^{-8}.$$

Fra det over er

$$f(1000) = 10,\ f'(1000) = \frac{1}{300},\ f''(1000) = -\frac{2}{9}\frac{1}{100000}$$

Da er

$$T_2 f(x) = 10 + \frac{1}{300}(x-1000) - \frac{1}{2!}\frac{2}{9}\frac{1}{100000}(x-1000)^2$$

så tilnærmingen vår er at

$$\sqrt[3]{1003} \approx T_2 f(1003) = 10 + \frac{3}{300} - \frac{9}{900000}$$

$$= 10 + \frac{1}{100} - \frac{1}{100000}$$

$$= \underline{\underline{10.0099900}}.$$