Taylorpolynomer

Minner om

$$f(x) = T_n f(x) + R_n f(x)$$

$$T_n f(x) = f(a) + (x-a) f'(a) + \frac{1}{2}(x-a)^2 f''(a) \quad , a \in \mathbb{R}$$

$$+ \cdots + \frac{1}{n!}(x-a)^n f^{(n)}(a)$$

$$R_n f(x) = \frac{1}{(n+1)!}(x-a)^{n+1} f^{(n+1)}(c) \quad , \quad c \in (a, x)$$

Ex.
Finn grenseverdien

$$L = \lim_{x \to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4}$$

Bruk L'Hopital så får kee $L = 1/24$

Vi har $\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{6!}$ for små x.

Innsatt i grensen

$$L = \lim_{x \to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4} \approx \lim_{x \to 0} \frac{1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{6!} - 1 + \frac{x^2}{2}}{x^4}$$

$$= \lim_{x \to 0} \frac{\frac{x^4}{24} - \frac{x^6}{6!}}{x^4} = \lim_{x \to 0} \left(\frac{1}{24} - \frac{x^2}{6!} \right) = \frac{1}{24}$$

Ex. 11.2.4.

Beregn $\int_0^1 \frac{\sin x}{x}\,dx$ med feil mindre enn 10^{-4}

Strategi: Erstatt $\sin x$ med sitt Taylor-polynom og hold styr på feilen.

$f(x) = \sin x$.

$$\sin x = T_{2n-1}\sin x + R_{2n-1}\sin x$$

$$= x - \frac{x^3}{6} + \frac{x^5}{5!} + \dots + (-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!} + \underbrace{(-1)^{n+2}\frac{x^{2n}}{(2n)!}f^{(2n)}(c)}_{R_{2n-1}(x)}$$

$$\sin x = T_{2n}\sin x + (-1)^{n+3}\frac{x^{2n+1}f^{(2n+1)}(c)}{(2n+1)!}$$

$$T_n f(x) = f(a) + (x-a)f'(a) + \frac{1}{2}(x-a)^2 f''(a) + \frac{1}{6}(x-a)^3 f'''(a)$$

$a = 0$

$=$

$f(0)=0$	$f(x) = \sin x$
$f'(0)=1$	$f'(x) = \cos x$
$f''(0)=0$	$f''(x) = -\sin x$
$f'''(0)=-1$	$f'''(x) = -\cos x$

Skal regne ut $\int_0^1 \frac{\sin x}{x}\,dx$

ved hjelp av

$$\sin x = T_{2n}\sin x + R_{2n}\sin x.$$

$$\int_0^1 \frac{\sin x}{x}\,dx = \int_0^1 \left(\frac{T_{2n}\sin x + R_{2n}\sin x}{x}\right)dx$$

$$= \int_0^1 \frac{T_{2n}\sin x}{x}\,dx + \int_0^1 \frac{R_{2n}\sin x}{x}\,dx$$

Tilnærming til $\int_0^1 \frac{\sin x}{x}\,dx$ feil.

La oss se på $\int_0^1 \frac{T_{2n}\sin x}{x}\,dx$, for eks $n=3$

Da er $T_{2n}\sin x = x - \frac{x^3}{6} + \frac{x^5}{5!}$

$$\int_0^1 \frac{T_6\sin x}{x}\,dx = \int_0^1 \frac{x - \frac{x^3}{6} + \frac{x^5}{120}}{x}\,dx = \int_0^1 \left(1 - \frac{x^2}{6} + \frac{x^4}{120}\right)dx$$

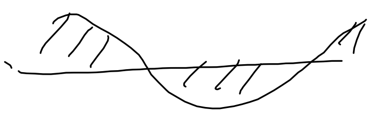
$$= \left[x - \frac{x^3}{3\cdot 6} + \frac{x^5}{5\cdot 120}\right]_0^1 = 1 - \frac{1}{18} + \frac{1}{600}$$

Feilen:

$$\left|\int_0^1 \frac{R_{2n}\sin x}{x}\,dx\right| = \left|\int_0^1 \left(\frac{x^{2n+1}}{(2n+1)!} f^{(2n+1)}(c)\right)/x \; dx\right|$$

$c \in (0, x)$

$f(x) = \sin x.$



$$\leq \int_0^1 \left|\frac{x^{2n+1}}{(2n+1)!} f^{(2n+1)}(c) / x\right| dx$$

$$= \int_0^1 \left|\frac{x^{2n}}{(2n+1)!} f^{(2n+1)}(c)\right| dx$$

$$= \int_0^1 \frac{x^{2n}}{(2n+1)!} \left|f^{(2n+1)}(c)\right| dx$$

$$\leq \int_0^1 \frac{x^{2n}}{(2n+1)!}\, dx$$

$$= \left[\frac{x^{2n+1}}{(2n+1)!\,(2n+1)}\right]_0^1 = \frac{1}{(2n+1)!\,(2n+1)} \leq 10^{-4}$$

Hvis dette er mindre enn 10^{-4}.

Det skjer første gang når $n = 3$.